Hybrid analytic-numeric method for light through a bounded planar dielectric domain

University of Twente
The Netherlands

J. B. Nicolau and E. van Groesen
Dept. of Applied Mathematics, MESA$^+$ Research Institute, University of Twente,
P.O. Box 217, 7500 AE Enschede, The Netherlands.

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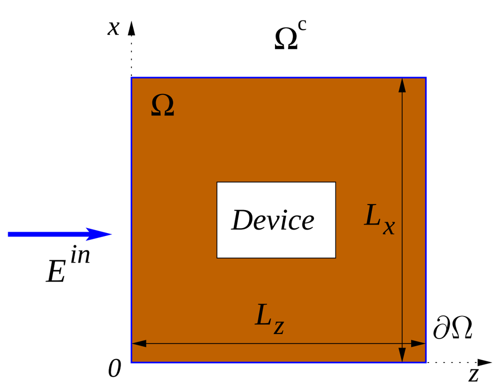
We present a hybrid analytic-numeric method to calculate the transmission and reflection of light that is fluxed into a bounded complicated optical structure surrounded by air. The solution is obtained by numerical calculations inside a square containing the structure and by analytical calculations outside the square. For solving the 2D Helmholtz equation we formulate Transparent-Influx Boundary Conditions (TIBC's) on the boundaries of the square; these are incorporated into a variational formulation of the Helmholtz equation to obtain a FEM-implementation for the interior calculations.

Modelling problem and general approach

On the plane, when restricting to TE-polarization, the propagation of time-harmonic light is described by the scalar Helmholtz Equation (HE) for the principal component E of the electric field perpendicular to the plane,

$$\Delta E + k^2 E = 0,$$

where $k(x,z) = k_0 n(x,z)$, with k_0 the free-space wavenumber, $n(x,z)$ is the refractive index of the media characterizing the geometry of the device, and $k_0 = \frac{2\pi}{\lambda}$, λ is the vacuum wavelength.
Our approach for solving the Helmholtz problem is as follows:



- given influx in exterior: E^{in}
- formulate TIBC's on $\partial\Omega$, which will include influx information $f = E^{in}|_{\partial\Omega}$
- calculate numerically the solution E_Ω in Ω with TIBC's on $\partial\Omega$
- determine the boundary value and calculate the E^{out} from $E_\Omega|_{\partial\Omega} - E^{in}|_{\partial\Omega}$

- E_Ω and $E^{ext} = E^{in} + E^{out}$ in Ω^c satisfy the correct continuity conditions on $\partial\Omega \Rightarrow$ solution on the whole domain.

Transparent-influx boundary conditions (TIBC's)

- An incoming field is assumed from the left $\Rightarrow$ $E^{in} = f$ and $\partial_n E^{in} = F$ at $\partial\Omega$
- Interface conditions $\Rightarrow$ $E_\Omega = E^{ext}$ and $\partial_n E_\Omega = \partial_n E^{ext}$ at $\partial\Omega$
- Solution in the exterior $\Rightarrow$ $E^{ext} = E^{in} + E^{out}$
- Dirichlet-to-Neumann operators (DtN): given a Dirichlet data g at $\partial\Omega$

$D^+(g) = \partial_n E|_{\partial\Omega}$ E outgoing solution (HE) with $E|_{\partial\Omega} = g$
$D^-(g) = \partial_n E|_{\partial\Omega}$ E incoming solution (HE) with $E|_{\partial\Omega} = g$

- Analytical expressions for the DtN operators can be obtained via plane wave decomposition
- For the square with influx though the western side, we find as boundary conditions:

$$\begin{aligned} \partial_n E_\Omega - D_W^+(E_\Omega|_{\partial\Omega}) &= D^-(f) - D^+(f) \quad \text{on } \partial\Omega_W \text{ with } f = E^{in}|_{\partial\Omega} \\ \partial_n E_\Omega - D_\gamma^+(E_\Omega|_{\partial\Omega}) &= 0 \quad \text{on } \partial\Omega_\gamma \text{ for } \gamma = N,\ E,\ S. \end{aligned}$$

Numerical implementation

We formulate a discretization of the TIBC's as follows:

- On the western side we have boundary function g on the interval $[0, L_x]$. We present this function by

$$g(x) = \sum_{m=-\infty}^{\infty} \widehat{g}(m) e^{i[mh_x x]}, \text{ with } h_x = \frac{2\pi}{\alpha L_x} \text{ and } \widehat{g}(m) = \frac{1}{\alpha L_x}\int_0^{L_x} g(x) e^{i[-mh_x x]}\, dx$$

- Taking $\alpha = 1 \Rightarrow g$ is represented by a periodic series, with period which is precisely the length of the side.
- Taking $\alpha > 1 \Rightarrow g$ is zero outside the basic interval.
- If the given function g at the western boundary corresponds to an influx field, this field is given by

$$E^{in}(x,z) = \sum_{m=-\infty}^{\infty} \widehat{g_{in}}(m) e^{i[mh_x x + \beta(mh_x) z]} \quad \text{for } z < 0, \text{ where } \{\beta(mh_x)\}^2 + \{mh_x\}^2 = k_0^2\,.$$

- The corresponding DtN operators : $D_W^+(E^{in}|_{\partial\Omega}) = -D_W^-(E^{in}|_{\partial\Omega}) = \sum_{m=-\infty}^{\infty} i\beta(mh_x)\widehat{g_{in}}(m) e^{i[mh_x x]}$.
Remark: The outgoing fields at the boundaries can be found in a similar manner.

FEM implementation

- Variational form of the Helmholtz equation:

$$\mathcal{L}(V,E) := \underbrace{\iint_\Omega \nabla V \cdot \nabla E_\Omega - k_\Omega^2 V E_\Omega \, ds}_{\text{Interior}} - \underbrace{\int_{\partial\Omega} V \partial_n E^{ext}\, dl}_{\text{Boundary}} = 0\,, \forall V \in H^1(\Omega)$$

with $\int_{\partial\Omega} V \partial_n E^{ext}\, dl = \int_{\partial\Omega} V\{D^+(E_\Omega|_{\partial\Omega}) + D^-(E^{in}|_{\partial\Omega}) - D^+(E^{in}|_{\partial\Omega})\}\, dl.$

Challenge: Boundary operator discretization

- The obtained analytical expressions for the DtN operators can be incorporated into the variational form of the HE.

$\Rightarrow$ Discrete form of the DtN operator: $\int_0^{L_x} V D_W^+(g_W)\, dl \cong \frac{1}{\alpha L_x}\sum_{p,q}^N v_p g_q (\sum_m i\beta(mh_x)\widehat{\tau}_{qm}\tau_{pm})$

with $\widehat{\tau}_{qm} = \int_0^{L_x} \phi_q(x,0) e^{i[-mh_x x]}\, dx$ and $\tau_{pm} = \int_0^{L_x} \phi_p(x,0) e^{i[mh_x x]}\, dx$.

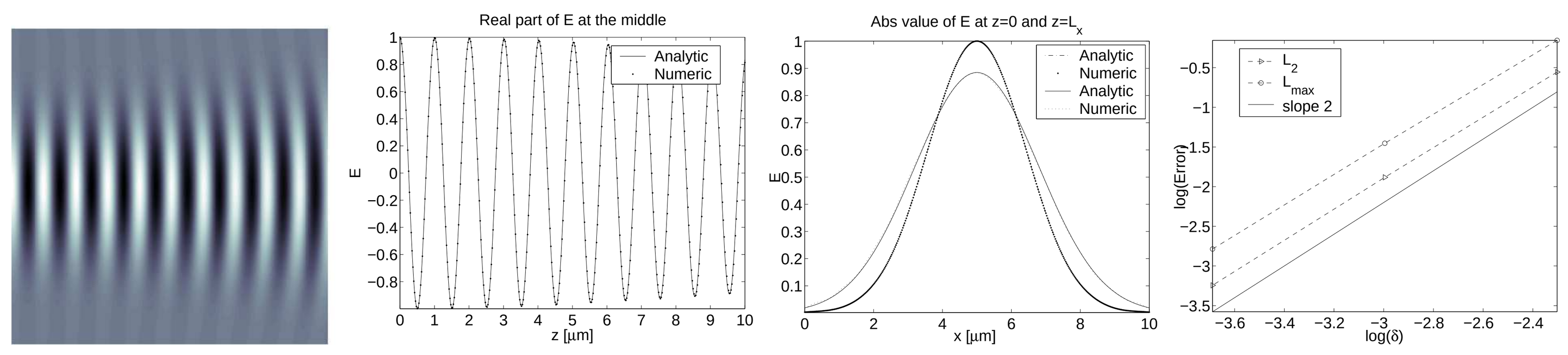
Note that the range of m is now finite for numerical purposes and ϕ_j are the piecewise linear basis functions.

Numerical results

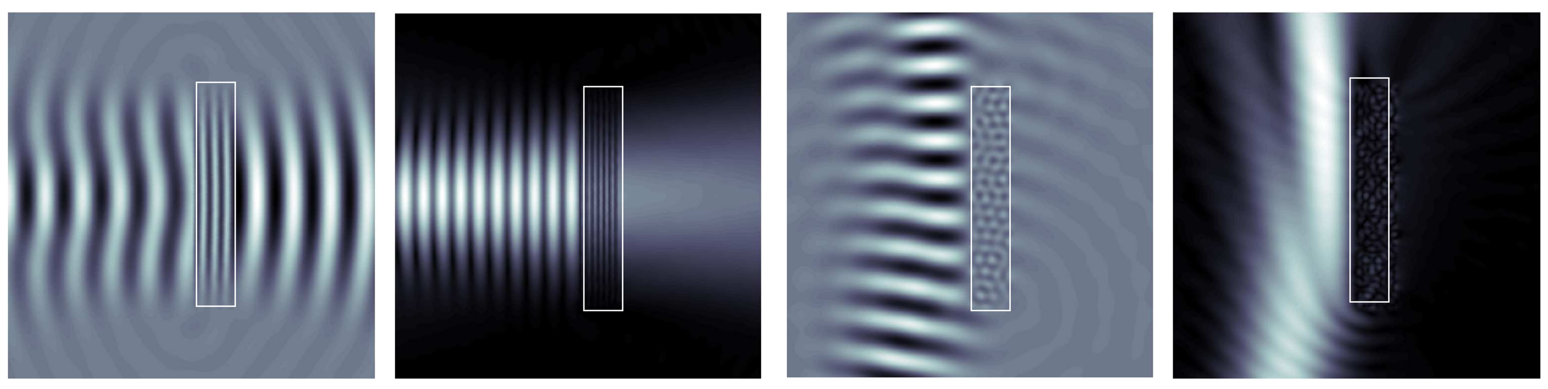
Interior domain

Parameters: computational window of $[0,\ 10]\ \mu m \times [0,\ 10]\ \mu m$, $\lambda = 1\ \mu m$.

- Error analysis: propagation of a Gaussian beam in free space

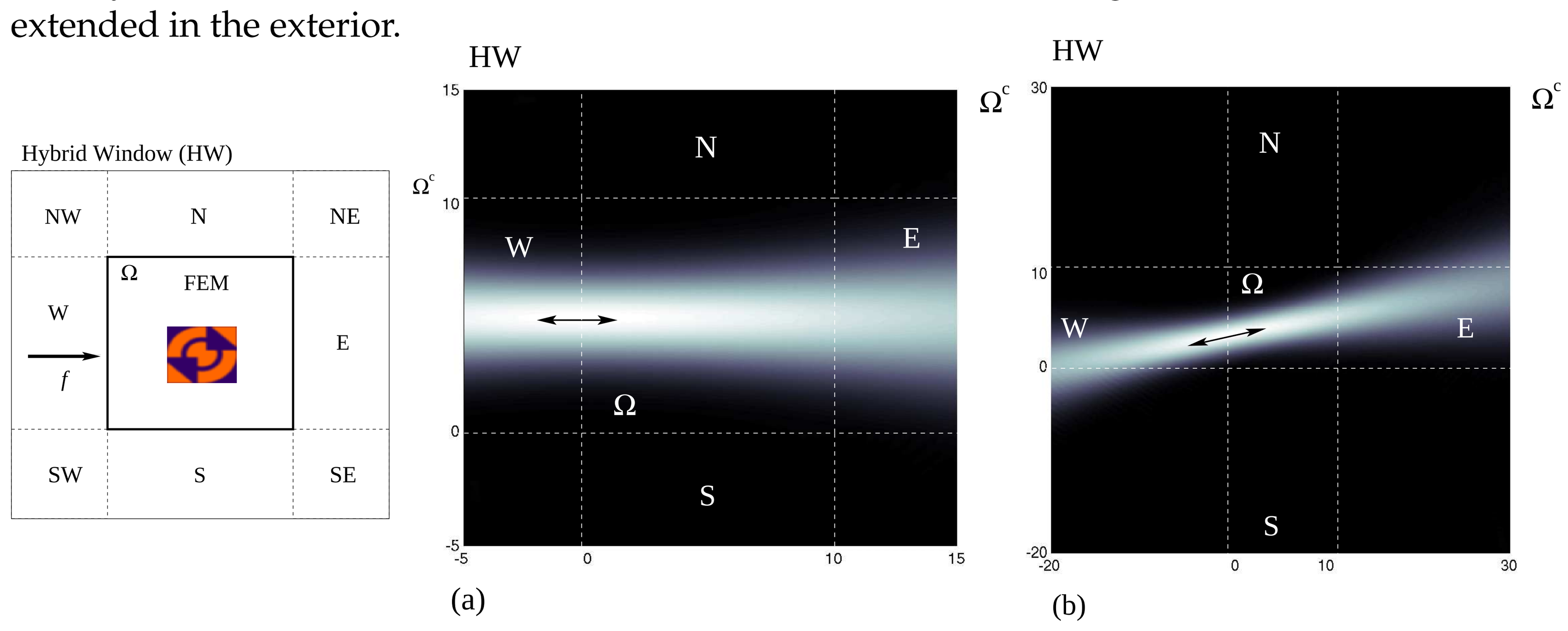


- Inhomogeneous interior: $n_d = 3.25$



Exterior: hybrid analytic-numeric method

The Hybrid Window (HW): Interior domain Ω (FEM solution) + a region on which the solution is extended in the exterior.



(a) A horizontally centered Gaussian beam with HW= $[-5,\ 15]\ \mu m \times [-5,\ 15]\ \mu m$.
(b) An obliquely Gaussian beam with HW= $[-20,\ 30]\ \mu m \times [-20,\ 30]\ \mu m$.

Conclusions

- We have developed well posed boundary conditions (TIBC's) for 2D Helmholtz problems in optics
- The boundary conditions depend only on the behavior of the solution in the exterior domain
- Implementation of those boundary conditions can be done in any numerical method
- The boundary conditions are formulated here for harmonic problems (TE-modes). Extension to TM-case is immediate.

References

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