Degree Sums for Edges and Cycle Lengths in Graphs

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Abstract: Let G be a graph of order n satisfying $d(u) + d(v) \geq n$ for every edge uv of G. We show that the circumference—the length of a longest cycle—of G can be expressed in terms of a certain graph parameter, and can be computed in polynomial time. Moreover, we show that G contains cycles of every length between 3 and the circumference, unless G is complete bipartite. If G is 1-tough then it is pancyclic or $G = K_{r,r}$ with $r = n/2$. © 1997 John Wiley & Sons, Inc. J Graph Theory **25:** 253–256, 1997

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We use [3] for terminology and notation not defined here, and consider finite, simple, undirected graphs only.

An early sufficient degree sum condition on a graph to be hamiltonian is the following due to Ore.

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Theorem 1 (Ore [7]). *Every graph G of order $n \geq 3$ satisfying $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices u, v is hamiltonian.*

We investigate the cycle lengths occurring in a graph where every pair of *adjacent* vertices u, v satisfies $d(u) + d(v) \geq n$. Certainly, we cannot hope any more that every such graph is hamiltonian, as the condition applies to every complete bipartite graph—hence also to the star $K_{1,n-1}$, which contains no cycle at all. But we will show that we can express the exact value of the circumference—the length of a longest cycle—of such a graph in terms of a certain parameter of the graph. Moreover, we can determine its circumference in polynomial time (while this problem is NP-hard within the class of graphs with minimum degree exceeding $n/2 - \sqrt{n}$, see [5]), and the graph contains cycles of every length between 3 and the circumference, unless it is complete bipartite.

Define the *independent deficiency* of a non-complete graph G as

$$\phi(G) = \max_S |S| - |N(S)| + 1,$$

where the maximum ranges over all non-empty sets S of independent vertices of G with $S \cup N(S) \neq V(G)$. Set

$$s(G) = \max\{0, \phi(G)\}.$$

The following simple observation gives an upper bound for the length of a longest cycle.

Lemma 1. *The circumference of any graph G of order n without isolated vertices is at most $n - s(G)$.*

In this note we will show that if the degree sum $d(u)+d(v) \geq n$ for every edge uv in G, then the circumference is precisely $n - s(G)$, and G contains cycles of all lengths l for $3 \leq l \leq n - s(G)$, unless G is complete bipartite. There are several necessary conditions known for a graph to be hamiltonian. The graph G must satisfy $\phi(G) \leq 0$, G must be 1-tough (i.e., the number of components of $G - S$ is at most $|S|$ for every vertex cut S), and G must be path-tough (i.e., every vertex-deleted subgraph has a spanning path). Note that the conditions in the sequence are increasing in strength.

Theorem 2. *Let $G \neq K_{1,n-1}$ be a graph of order n without isolated vertices. If the degree sum $d(u) + d(v) \geq n$ for every edge $uv \in E(G)$ then the circumference of G equals $n - s(G)$.*

In general, the circumference of a graph G of order n need not be equal to $n - s(G)$, as shown by any 1-tough nonhamiltonian graph.

Theorem 3. *Let G be a graph of order n. If every edge uv satisfies $d(u) + d(v) \geq n$ then G contains cycles of every length between 3 and the circumference of G, unless G is a complete bipartite graph.*

A graph is called *pancyclic* if it contains cycles of all lengths l for $3 \leq l \leq n$. Since every 1-tough graph satisfies $\phi \leq 0$ we obtain the following consequence.

Corollary 1. *Every 1-tough graph G of order $n \geq 2$ satisfying $d(u) + d(v) \geq n$ for every edge uv is pancyclic or $G = K_{n/2,n/2}$.*

As Theorem 1, Theorem 2 generalizes the well-known theorem of Dirac [4] that every graph G of order $n \geq 3$ with $\delta(G) \geq n/2$ is hamiltonian, since it is easily seen that such a graph has $\phi(G) \leq 0$. Corollary 1 resembles a theorem of Jung [6] stating that every 1-tough graph G of order $n \geq 11$ satisfying $d(u) + d(v) \geq n - 4$ for every pair of non-adjacent vertices u, v is hamiltonian. For odd $n \geq 11$ the join of $K_{(n-5)/2}$ and $\overline{K}_{(n-7)/2} \cup H$, where H is the unique graph with degree sequence $(1, 1, 1, 3, 3, 3)$, shows that the inequality in Jung's result cannot be relaxed [6]. The same graphs show that the inequality $d(u) + d(v) \geq n$ in Corollary 1 and Theorem 2 is sharp.

For proving Theorem 2 we need tools from hamiltonian graph theory. A powerful idea is the concept of the closure of a graph due to Bondy and Chvátal [2]. Define the *i-closure* $\mathrm{cl}_i(G)$ as the graph obtained from G by iteratively joining independent vertices x, y of degree sum $d(x) + d(y) \geq i$ by an edge until in the resulting graph every pair of independent vertices has degree sum less than i. Bondy and Chvátal observed that the i-closure is unique, and that a graph is hamiltonian if and only if its n-closure is hamiltonian. An easy generalization is that the circumference of G equals the circumference of $\mathrm{cl}_n(G)$ which we will show next using standard techniques.

Theorem 4. *If G is a graph of order n then the circumference of G equals the circumference of $\mathrm{cl}_n(G)$.*

Proof. It suffices to prove that adding an edge between two non-adjacent vertices u and v with $d(u) + d(v) \geq n$ cannot increase the circumference of the graph.

Assume, otherwise, that the circumference of $G + uv$ exceeds the circumference of G. Certainly, every longest cycle C of $G + uv$ must contain the edge uv. Hence G contains a path P from u to v which contains all edges of C except uv. Define sets $A = \{x : x \in V(P), ux^+ \in E(G)\} \cup \{x : x \in V(G) - V(P), ux \in E(G)\}$, where x^+ is the successor of x on P, and $B = \{x : x \in V(G), vx \in E(G)\}$. As $|A| + |B| \geq n$ and $|A \cup B| \leq n - 1$ since $v \notin A \cup B$, we get that $A \cap B \neq \emptyset$, implying that there is a cycle in G which is at least as long as C. ∎

Note that the n-closure of a graph with degree sum $d(u) + d(v) \geq n$ for every edge uv is a graph with the same degree sum property, since the degree sums of the inserted edges exceed n and the degree sums of edges do not decrease in the closure process. In the proof of Theorem 2 we use a sufficient condition due to Pósa for a graph to be hamiltonian.

Theorem 5 (Pósa [8])**.** *Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the degree sequence of a graph G of order $n \geq 3$. If $d_i \geq i + 1$ for every $i < n/2$ then G is hamiltonian.*

We are now prepared to prove our main result.

Proof of Theorem 2. Consider the n-closure G' of G. We claim that for every pair of vertices u, v of G' with $d(u) \leq d(v)$ we have $N(u) \cup \{u, v\} \subseteq N(v) \cup \{u, v\}$. Indeed, assume that u has a neighbor $x \neq v$ which is not a neighbor of v. As $d(u) \leq d(v)$ and $d(u) + d(x) \geq n$ we get $d(v) + d(x) \geq n$, so $vx \in E(G')$, contradicting our assumption. Delete $s(G)(= s(G'))$ vertices with the smallest degrees in G' to obtain a graph G''. We will show that G'' satisfies the condition of Pósa (Theorem 5) and is therefore hamiltonian. By Lemma 1 and Theorem 4 this completes the proof.

Indeed, let $d_1 \leq d_2 \leq \cdots \leq d_{n-s(G)}$ be the degree sequence of G''. If $d_i \leq i$ for an index $i < (n - s(G))/2$ then for the set S of the $i + s(G)$ vertices of smallest degree in G', which is clearly independent, we have $|N(S)| \leq i$ by the initial observation, and therefore $|S| - |N(S)| + 1 \geq s(G) + 1$ and $|S| + |N(S)| < n$, a contradiction to the definition of $\phi(G)$. ∎

To compute the circumference of a graph G satisfying the hypothesis of Theorem 2, we can proceed along the lines of the proof. First compute the n-closure G' of G (this can be done in time $\mathcal{O}(n^4)$) and sort the vertices by degree. Determine $\phi\,(G')$ by taking the maximum of $|S_i| - |N(S_i)| + 1$, where S_i denotes the set of the i smallest degree vertices of G', and the maximum ranges over all S_i which are independent and satisfy $S_i \cup N(S_i) \neq V(G')$. With a slightly more careful reasoning we can actually find a longest cycle of G in polynomial time.

Zhang [9] defined the *critical independence number* of a graph G as

$$\alpha_c(G) = \max_S |S| - |N(S)|,$$

where the maximum ranges over all independent sets S (including $\emptyset$, so that $\alpha_c(G) \geq 0$). He proved that $\alpha_c(G)$ can be computed in polynomial time for any graph G. As $\alpha_c(G)$ is very much like $s(G)$, in fact $\alpha_c(G) \leq s(G) \leq \alpha_c(G) + 1$ holds, it seems likely that $s(G)$ is computable for any graph G in polynomial time. This would be another approach to computing the circumference of graphs satisfying our hypothesis.

To verify that the graphs satisfying the hypothesis of Theorem 2 contain cycles of every length between 3 and $n - s(G)$ we need the following famous result due to Bondy.

Theorem 6 (Bondy [1]). If G is a hamiltonian graph of order n with at least $n^2/4$ edges then G is pancyclic or $G = K_{n/2,n/2}$.

Proof of Theorem 3. Let C be a longest cycle of G having length k, and consider the edges of C. The endvertices of no such edge have a common neighbor in $V(G) - V(C)$, so the degree sum $d_{vC}(u) + d_{vC}(v) \geq k$ for every edge uv of C. This implies that the subgraph H of G induced by $V(C)$ has at least $k^2/4$ edges. By Theorem 6 the proof is complete unless $H = K_{k/2,k/2}$.

Now assume that H is complete balanced bipartite and let $A \cup B$ be the bipartition of H. Observe that no vertex in $V(G) - V(H)$ is adjacent to both ends of an edge in H. So, by the degree sum requirement applied to the edges in H, every vertex in $V(G) - V(H)$ must be adjacent to precisely one end of every edge in H, i.e., every vertex in $V(G) - V(H)$ is either adjacent to every vertex of A or to every vertex of B. If there are two vertices x, y in $V(G) - V(H)$ such that x is adjacent to the vertices in A and y to the vertices in B, then G has a $(k+2)$-cycle through x, y and the vertices in H, contradicting circumference k. So all vertices in $V(G) - V(H)$ are adjacent to all vertices on one side of the bipartition of H, say to A. If $G - V(H)$ contains an edge then we obtain a $(k+1)$-cycle through this edge and $k - 1$ vertices of H, which is again a contradiction. So, indeed, G is complete bipartite. ∎

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