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The Lie–Bäcklund algebra for the general underdetermined equation $u_r = f(x, u, \ldots, u_{r-1}, v, \ldots, v_k)$

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Abstract. The Lie–Bäcklund symmetry algebra for the equation $u_r = f(x, u, \ldots, u_{r-1}, v, \ldots, v_k)$ is established.

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1. Introduction and general

In a recent paper Anderson, Kamran and Olver [1] discussed internal, external and generalized symmetries and their interrelations. As an application they established the equivalence between first-order generalized symmetries and internal symmetries for the equation

$$u'' = F(x, u, u', v, v', v''). \tag{1.1}$$

As an example of such a type of equation they discussed the remarkable Hilber–Cartan equation [2, 3]

$$v_x = (u_{xx})^2 \tag{1.2}$$

and showed that for this equation there exist genuine internal symmetries, meaning those which cannot be obtained by restriction of an external symmetry to the solution manifold, cf. [1]. The Lie–Bäcklund algebra of (1.2) was not known. The algebra of generalized or Lie–Bäcklund symmetries associated to (1.2) was constructed in [4].

Here we shall establish the Lie–Bäcklund symmetry algebra for the general underdetermined equation

$$u_r = f(x, u, u_1, \ldots, u_{r-1}, v, \ldots, v_k) \tag{1.3a}$$

where in (1.3a)

$$u_r = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^r u(x)$$

$$v_k = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k v(x). \tag{1.3b}$$

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We assume $k > r$, otherwise we have to reverse the role of u and v. In section 2 we shall treat the case $r = 3$, $k = 5$. The general situation can be handled in a completely analogous way and is briefly discussed in section 3. From the results in this paper genuine internal symmetries can be obtained.

2. The Lie–Bäcklund symmetry algebra (case $r = 3$, $k = 5$)

We consider the general underdetermined equation

$$u_3 = f(x, u, u_1, u_2, v, v_1, \ldots, v_5) \tag{2.1}$$

and introduce the total derivative operator D defined on the infinite jet bundle $J(x; (u, v))$ [5], i.e.

$$D = \partial_x + u_1\,\partial_u + v_1\,\partial_v + u_2\,\partial_{u_1} + v_2\,\partial_v + \ldots \tag{2.2a}$$

and its restriction to $J^{n+1}(x; (u, v))$, i.e.

$$D^{(n+1)} = \partial_x + u_1\,\partial_u + v_1\,\partial_v + \ldots + u_{n+1}\,\partial_{u_n} + v_{n+1}\,\partial_{v_n}. \tag{2.2b}$$

In effect we use the restriction of $D^{(n+1)}$ (2.2b) to the submanifold $\mathscr{Y}$ of $J^{(n+1)}(x; (u, v))$ defined by (2.1) and its differential consequences up to and including $(n-4)$th order. Local coordinates on $\mathscr{Y}$ are given by

$$x, u, u_1, u_2, v, v_1, \ldots, v_{n+1}. \tag{2.3}$$

Now the vector field V, with defining functions F^0, H^0, i.e.

$$V = F^0\,\partial_u + H^0\,\partial_v + \ldots \tag{2.4}$$

is a generalized symmetry of (2.1) if and only if (cf. [6, 7])

$$D^3F^0 = \sum_{i=0}^{2} \frac{\partial f}{\partial u_i}(D^i)F^0 + \sum_{j=0}^{5} \frac{\partial f}{\partial v_j}(D^j)H^0 \tag{2.5}$$

where $D^i = D \circ D \circ \ldots \circ D$ (i times). It is an easy observation that (2.5) leads to the condition

$$F^0 = F^0[n] = F^0(x, u, u_1, u_2, v, \ldots, v_n) \tag{2.6a}$$

$$H^0 = H^0[n-2] = H^0(x, u, u_1, u_2, v, \ldots, v_{n-2}) \tag{2.6b}$$

where in (2.6) we introduced the notation '$[n]$' to indicate the dependency on the variables x, u, u_1, u_2, v, $\ldots$, v_n.

In what follows, n (cf. (2.6)) is taken to be sufficiently large as to avoid interference of terms with their coefficients (cf. [2]). *At the end of this section we summarize the solution procedure.*

We introduce F^1, F^2, H^1, H^2, H^3, H^4 by

$$F^1[n+1] - D^{(n+1)}F^0[n] = 0$$

$$F^2[n+2] - D^{(n+2)}F^1[n+1] = 0 \tag{2.7}$$

$$H^1[n-1] - D^{(n-1)}H^0[n-2] = 0 \tag{2.8a}$$

$$H^2[n] - D^{(n)}H^1[n-1] = 0 \tag{2.8b}$$

$$H^3[n+1] - D^{(n+1)}H^2[n] = 0 \tag{2.8c}$$

$$H^4[n+2] - D^{(n+2)}H^3[n+1] = 0 \tag{2.8d}$$

and condition (2.5) is rewritten as

$$D^{(n+3)}F^2[n+2] - \sum_{i=0}^{2} g_i F^i[n+i] - \sum_{j=0}^{4} f_j H^j[n-2+j] - f_5 D^{(n+3)} H^4[n+2] = 0 \qquad (2.9)$$

and

$$g_i = \frac{\partial f}{\partial u_i} \qquad (i = 0, 1, 2)$$

$$f_j = \frac{\partial f}{\partial v_j} \qquad (j = 0, \ldots, 5).$$

We shall now construct the general solution of (2.5), i.e. (2.7), (2.8) and (2.9), step by step. First of all we put

$$F^2[n+2] = f_5 H^4[n+2] + F_1^2. \qquad (2.10a)$$

Substitution of (2.10*a*) into (2.9) and looking for highest-order terms in (2.9) we observe that

$$F_1^2 = F_1^2[n+1] \qquad (2.10b)$$

and (2.9) reduces to

$$D^{(n+2)}F_1^2[n+1] - g_0 F^0[n] - g_1 F^1[n+1] - g_2 F_1^2[n+1] - f_0 H^0[n-2]$$
$$- f_1 H^1[n-1] - f_2 H^2[n] - f_3 H^3[n+1] + ((D - g_2) f_5 - f_4) H^4[n+2] = 0. \qquad (2.11)$$

If we put

$$\alpha_0 = f_4 - (D - g_2) f_5 \qquad (2.12a)$$

$$B_0 = f_5. \qquad (2.12b)$$

Then (2.11) and (2.7) are rewritten as

$$D^{(n+2)}F_1^2[n+1] - \alpha_0 H^4[n+2] - g_0 F^0[n] - g_1 F^1[n+1] - g_2 F_1^2[n+1]$$
$$- f_0 H^0[n-2] - f_1 H^1[n-1] - f_2 H^2[n] - f_3 H^3[n+1] = 0 \qquad (2.13a)$$

$$- D^{(n+2)} F^1[n+1] + \beta_0 H^4[n+2] + F_1^2[n+1] = 0 \qquad (2.13b)$$

$$- D^{(n+1)} F^0[n] + F^1[n+1] = 0. \qquad (2.13c)$$

In (2.11) and (2.12*a*), Df_5 means 'total derivative of f_5'. We now solve (2.8*d*) for $H^4[n+2]$ in terms of a total derivative of $H^3[n+1]$, i.e.

$$H^4[n+2] = D^{(n+2)} H^3[n+1] \qquad (2.14a)$$

and put

$$F^1[n+1] = \beta_0 H^3[n+1] + F_1^1[n] \qquad (2.14b)$$

$$F_1^2[n+1] = \alpha_0 H^3[n+1] + F_2^2[n] \qquad (2.14c)$$

where it is easily observed that $F_1^1 = F_1^1[n]$ and $F_2^2 = F_2^2[n]$ due to the vanishing of highest-order terms in (2.13*a*) and (2.13*b*). Equations (2.13) reduce to

$$D^{(n+1)}F_2^2[n] - \alpha_1 H^3[n+1] - g_0 F^0[n] - g_1 F_1^1[n] - g_2 F_2^2[n] - f_0 H^0[n-2]$$
$$- f_1 H^1[n-1] - f_2 H^2[n] = 0 \qquad (2.15a)$$

$$- D^{(n+1)} F_1^1[n] + \beta_1 H^3[n+1] + F_2^2[n] = 0 \qquad (2.15b)$$

$$- D^{(n+1)} F^0[n] + \gamma_1 H^3[n+1] + F_1^1[n] = 0 \qquad (2.15c)$$

where α_1, β_1, γ_1 are defined by

$$\alpha_1 = -D\alpha_0 + g_1\beta_0 + g_2\alpha_0 + f_3 \qquad (2.16a)$$

$$\beta_1 = -D\beta_0 + \alpha_0 \qquad (2.16b)$$

$$\gamma_1 = \beta_0. \qquad (2.16c)$$

Now by solving (2.8c) for $H^3[n+1]$, i.e.

$$H^3[n+1] = D^{(n+1)}H^2[n] \qquad (2.17a)$$

and putting

$$F^0[n] = \gamma_1 H^2[n] + F_1^0[n-1] \qquad (2.17b)$$

$$F_1^1[n] = \beta_1 H^2[n] + F_2^1[n-1] \qquad (2.17c)$$

$$F_2^2[n] = \alpha_1 H^2[n] + F_3^2[n-1] \qquad (2.17d)$$

whereas the dependencies of F_1^0, F_2^1, f_3^2 arise from highest-order terms (2.15) reduces to

$$+D^{(n)}F_3^2[n-1] - \alpha_2 H^2[n] - g_0 F_1^0[n] - g_1 F_2^1[n-1] - g_2 F_3^2[n-1]$$
$$-f_0 H^0[n-2] - f_1 H^1[n-1] = 0 \qquad (2.18a)$$

$$-D^{(n)}F_2^1[n-1] + \beta_2 H^3[n] + F_3^2[n-1] = 0 \qquad (2.18b)$$

$$-D^{(n)}F_1^0[n-1] + \gamma_2 H^2[n] + F_2^1[n-1] = 0 \qquad (2.18c)$$

where α_2, β_2, γ_2 are given by

$$\alpha_2 = -D\alpha_1 + g_0\gamma_1 + g_1\beta_1 + g_2\alpha_1 + f_2 \qquad (2.19a)$$

$$\beta_2 = -D\beta_1 + \alpha_1 \qquad (2.19b)$$

$$\gamma_2 = -D\gamma_1 + \beta_1. \qquad (2.19c)$$

By solving $H^2[n]$, $H^1[n-1]$ in terms of $H^0[n-2]$ we finally arrive at the system

$$D^{(n-2)}F_5^2[n-3] - \alpha_4 H^0[n-2] - g_0 F_3^0[n-3] - g_1 F_4^1[n-3] - g_2 F_5^2[n-3] = 0 \qquad (2.20a)$$

$$-D^{(n-2)}F_4^1[n-3] + \beta_4 H^0[n-2] + F_5^2[n-3] = 0 \qquad (2.20b)$$

$$-D^{(n-2)}F_3^0[n-3] + \gamma_4 H^0[n-2] + F_4^1[n-3] = 0 \qquad (2.20c)$$

where α_3, β_3, γ_3 and α_4, β_4, γ_4 are defined by

$$\begin{aligned} \alpha_3 &= -D\alpha_2 + g_0\gamma_2 + g_1\beta_2 + g_2\alpha_2 + f_1 \\ \beta_3 &= -D\beta_2 + \alpha_2 \\ \gamma_3 &= -D\gamma_2 + \beta_2 \end{aligned} \qquad (2.21)$$

and

$$\begin{aligned} \alpha_4 &= -D\alpha_3 + g_0\gamma_3 + g_1\beta_3 + g_2\alpha_3 + f_0 \\ \beta_4 &= -D\beta_3 + \alpha_3 \\ \gamma_4 &= -D\gamma_3 + \beta_3 \end{aligned} \qquad (2.22)$$

respectively.

So conditions (2.8a)–(2.8d) are solved and we are left with the underdetermined system (2.20a)–(2.20c); a system of three equations for which $H^0[n-2]$, $F_3^0[n-3]$, $F_4^1[n-3]$, $F_5^2[n-3]$, are to be determined.

Solving (2.20*a*) for $H^0[n-2]$ in terms of F_3^0, F_4^1, F_5^2 and by substitution into (2.20*b*) and (2.20*c*) we arrive at

$$-D^{(n-2)}F_4^1[n-3] + \frac{\beta_4}{\alpha_4} D^{(n-2)}F_5^2[n-3] + \left(1 - \frac{\beta_4}{\alpha_4} g_2\right) F_5^2[n-3] - \frac{\beta_4}{\alpha_4} g_1 F_4^1[n-3] - \frac{\beta_4}{\alpha_4} g_0 F_3^0[n-3] = 0 \tag{2.23a}$$

$$-D^{(n-2)}F_3^0[n-3] + \frac{\gamma_4}{\alpha_4} D^{(n-2)}F_5^2[n-3] - \frac{\gamma_4}{\alpha_4} g_2 F_5^2[n-3] + \left(1 - \frac{\gamma_4}{\alpha_4} g_1\right) F_4^1[n-3] - \frac{\gamma_4}{\alpha_4} g_0 F_3^0[n-3] = 0. \tag{2.23b}$$

From conditions (2.23*a*) and (2.23*b*) we obtain by

$$\begin{aligned} F_4^1[n-3] &= \frac{\beta_4}{\alpha_4} F_5^2[n-3] + F_5^1 \\ F_3^0[n-3] &= \frac{\gamma_4}{\alpha_4} F_5^2[n-3] + F_4^0 \end{aligned} \tag{2.24a}$$

conditions on F_5^1, F_4^0. First of all note that

$$F_5^1 = F_5^1[n-4] \qquad F_4^0 = F_4^0[n-4] \tag{2.24b}$$

and (2.23) reduces to

$$-D^{(n-3)}F_5^1[n-4] + \beta_5 F_5^2[n-3] - \frac{\beta_4}{\alpha_4} g_1 F_5^1[n-3] - \frac{\beta_4}{\alpha_4} g_0 F_4^0[n-4] = 0 \tag{2.25a}$$

$$-D^{(n-3)}F_4^0[n-4] + \gamma_5 F_5^2[n-3] + \left(1 - \frac{g_1}{\alpha_4}\right) F_5^1[n-4] - \frac{g_0}{\alpha_4} F_4^0[n-4] = 0 \tag{2.25b}$$

where β_5, γ_5 are given by

$$\begin{aligned} \beta_5 &= -D\left(\frac{\beta_4}{\alpha_4}\right) + \left(1 - \frac{\beta_4}{\alpha_4} g_2\right) - g_1\left(\frac{\beta_4}{\alpha_4}\right)^2 - g_0\left(\frac{\beta_4}{\alpha_4}\right)\left(\frac{\gamma_4}{\alpha_4}\right) \\ \gamma_5 &= -D\left(\frac{\gamma_4}{\alpha_4}\right) - \frac{\gamma_4}{\alpha_4} + \left(1 - \frac{\gamma_1}{\alpha_4} g_1\right)\left(\frac{\beta_4}{\alpha_4}\right) - g_0\left(\frac{\gamma_4}{\alpha_4}\right)^2. \end{aligned} \tag{2.26}$$

System (2.25) is similar to the original system (2.20) but now there are two equations and three functions $F_5^2[n-3]$, $F_5^1[n-4]$, $F_4^0[n-4]$. So we solve (2.25) for $F_5^2[n-3]$ and substitute into (2.25*b*), leading to

$$-D^{(n-3)}F_4^0[n-4] + \frac{\gamma_5}{\beta_5} D^{(n-3)}F_5^1[n-4] + \delta_5 F_4^0[n-4] + \varepsilon_5 F_5^1[n-4] = 0 \tag{2.27}$$

where δ_5, ε_5 depend on α_4, β_4, γ_4, β_5, γ_5, g_0, g_1, g_2. Finally, by putting

$$F_4^0[n-4] = \frac{\gamma_5}{\beta_5} F_5^1[n-4] + F_5^0[n-5] \tag{2.28}$$

we arrive at

$$-D^{(n-4)}F_5^0[n-5] + \delta_5 F_5^0[n-5] + \left\{-D\left(\frac{\gamma_5}{\beta_5}\right) + \delta_5 \frac{\gamma_5}{\beta_5} + \varepsilon\right\} F_5^1[n-4] = 0. \tag{2.29}$$

From this equation we obtain $F_5^1[n-4]$ in terms of the *arbitrary function* $F_5^0[n-5]$.

This completes the construction of the Lie–Bäcklund algebra of (2.1), since the defining functions $F^0[n]$, $H^0[n-2]$ of the generalized symmetry

$$V = F^0[n]\,\partial_u + H^0[n-2]\,\partial_v + \ldots$$

are expressed by (2.10), (2.14), (2.17), (2.20*a*), (2.24*a*), (2.25*a*), (2.28) and (2.29) in terms of the arbitrary function $F_5^0[n-5]$.

We now summarize the solution procedure explained in detail before.

We start at the symmetry condition which can be written as a linear first-order (in D) system of differential equations (2.7)–(2.9) for the components of the vector field V, i.e. $F^0, \ldots, F^2, H^0, \ldots, H^4$.

(i) By *a recursive procedure* (2.10), (2.14), (2.17), ... the original system (2.7)–(2.9) is reduced to a much simpler linear first-order system (2.20); a system of three equations for four functions, i.e.

$$D^{(n-2)}F[n-3] - cH^0[n-2] - AF[n-3] = 0 \tag{2.30a}$$

where

$$F = \begin{bmatrix} F_3^0 \\ F_4^1 \\ F_5^2 \end{bmatrix} \qquad c = \begin{bmatrix} \gamma_4 \\ \beta_4 \\ \alpha_4 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -g_0 & -g_1 & -g_2 \end{bmatrix} \tag{2.30b}$$

while α_4, β_4, γ_4 are defined recursively in terms of g_i $(i = 0, \ldots, 2)$ f_j $(j = 0, \ldots, 5)$ and their total derivatives by (2.12), (2.16), (2.19), (2.21) and (2.22). *Note*: in effect $H^1, \ldots, H^4$ have been eliminated while $F^0, \ldots, F^2$ can be expressed by (2.10*a*), (2.14*b*), (2.14*c*) and (2.17*b*)–(2.17*d*) in terms of F_3^0, F_4^1, F_5^2.

(ii) By another recursive procedure the system (2.20) is reduced by (2.23), (2.24) and (2.27), (2.28) to a *single* 'differential' equation (2.29) which is of a similar form to (2.30), i.e.

$$D\bar{F} - \bar{c}\bar{H} - \bar{a}\bar{F} = 0 \tag{2.31}$$

and can be solved for $\bar{H}$ in terms of $\bar{F}$.

This now leads to the following theorem.

Theorem 2.1. For *every arbitrary* function

$$\bar{F}(x, u, u_1, u_2, v, \ldots, v_n) \tag{2.32}$$

there exists a generalized symmetry of (2.1) whose components can be expressed in term of $\bar{F}$, partial derivatives of f and their total derivatives. Moreover, every generalized symmetry can be obtained in this way.

3. The general case

Here we shall make a few comments concerning the construction of the Lie–Bäcklund algebra for the general case equation

$$u_r = f(x, u, u_1, \ldots, u_{r-1}, v, \ldots, v_l) \tag{3.1}$$

which is analogous to the case ($r = 3$, $l = 5$) handled in section 2. The symmetry condition on the vector field V,

$$V = F^0\,\partial_u + H^0\,\partial_v \tag{3.2}$$

is given by

$$D^rF^0=\sum_{i=0}^{r-1} g_iD^iF^0+\sum_{j=0}^{l} f_jD^jH^0 \tag{3.3a}$$

where

$$g_i=\frac{\partial f}{\partial u_i} \quad (i=0, \ldots, r-1) \qquad f_j=\frac{\partial f}{\partial v_j} \quad (j=0, \ldots, l). \tag{3.3b}$$

By introduction of $F^1, \ldots, F^{r-1}, H^1, \ldots, H^{l-1}$ by

$$\begin{aligned} &F^i-DF^{i-1}=0 \quad (i=1, \ldots, r-1)\\ &H^j-DH^{j-1}=0 \quad (j=1, \ldots, l-1) \end{aligned} \tag{3.4a}$$

(3.3*a*) is written in terms of *first-order* total derivatives as

$$DF^{r-1}-\sum_{i=0}^{r-1} g_iF^i-\sum_{j=0}^{l-1} f_jH^j-f_lDH^{l-1}=0. \tag{3.4b}$$

We can start the solution procedure of (3.4*a*) and (3.4*b*) in exactly the same way as we did in section 2 and arrive at a system of equations similar to (2.20), which look like

$$\begin{aligned} &-DF^0_r+F^1_{r+1}+\alpha_0H^0=0\\ &-DF^1_{r+1}+F^2_{r+2}+\alpha_1H^0=0\\ &\quad\vdots \qquad\qquad \vdots\\ &-DF^{r-2}_{r+l-1}+F^{r-1}_{r+l}+\alpha_{r-2}H^0=0\\ &DF^{r-1}_{r+l}+\alpha_{r-1}H^0-\sum_{i=0}^{r-1} g_iF^i_{r+i}=0 \end{aligned} \tag{3.5}$$

where $\alpha_0, \ldots, \alpha_{r-1}$ are constructed in the same way as α_4, β_4, γ_4 in section 2.

System (3.5) can be written as

$$DF-\alpha H^0+AF=0 \tag{3.6a}$$

where

$$F=\begin{bmatrix} F^0_r\\ \vdots\\ F^{r-1}_{r+l}\end{bmatrix} \qquad \alpha_4=\begin{bmatrix}\alpha_0\\ \vdots\\ \alpha_{r-1}\end{bmatrix} \qquad A=\begin{bmatrix} 0 & -1 & & & \\ & & -1 & & 0\\ & & & -1 & \\ & 0 & & & \\ & & & & -1\\ g_0 & g_1 & & & g_{r-1}\end{bmatrix} \tag{3.6b}$$

which is similar to (2.30).

The solution procedure to solve (2.20) recursively as demonstrated in section 2 where we reduced the system (2.20), i.e. three equations with four functions to the system (2.25), i.e. a system of two equations for three functions, and finally from (2.25) to (2.29) can be applied to the system (3.5) to complete the final result in $r-1$ steps, i.e. we have the following.

Theorem 3.1. For every arbitrary function

$$\bar{F}=\bar{F}(x, u, \ldots, u_{r-1}, v, \ldots, v_n)$$

there exists a generalized symmetry of (3.1) whose components can be expressed in terms $\bar{F}$, partial derivatives of f (3.1) and their total derivatives. Moreover, every generalized symmetry of (3.1) can be obtained in this way.

4. Conclusion

The construction of the Lie–Bäcklund algebra of the general underdetermined equation is established and turns out to be completely algorithmic in nature.

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