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Defining relations for Lie algebras of vector fields

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ABSTRACT

We calculated defining relations of the graded nilpotent positive part of Lie algebras of vector fields. These calculations suggest that for $\mathcal{W}_n$, $\mathcal{S}_n$ and $\mathcal{K}_n$ (n sufficiently large) there are only trivial relations, i.e. relations of degree 2. For $\mathcal{H}_n$ however, we prove that there are non-trivial relations of degree 3, which form a standard module.

1. SURVEY OF RESULTS

In this paper we discuss some results concerning defining relations for classical Lie algebras of vector fields, i.e. for $\mathcal{W}_n$, $\mathcal{S}_n$, $\mathcal{H}_n$ and $\mathcal{K}_n$. These algebras are described in section 2. Let $\mathcal{X}$ denote one of these algebras. Then $\mathcal{X}$ has a $\mathbb{Z}$-grading such that $\mathcal{X}=\bigoplus_{m\geq -1}\mathcal{X}^{(m)}$, where $\mathcal{X}^{(m)}$ denotes the subspace of homogeneous elements of degree m. Let $n_+(\mathcal{X})$ denote the subalgebra $\bigoplus_{m\geq 1}\mathcal{X}^{(m)}$. We computed a minimal set of defining relations for $n_+(\mathcal{X})$, which are naturally described by $H_2(n_+(\mathcal{X}))$, see section 3. These relations are graded: let $r_m(\mathcal{X})$ denote the number of defining relations for $n_+(\mathcal{X})$ of degree m, that is $r_m(\mathcal{X})=\dim H_2^{(m)}(n_+(\mathcal{X}))$. Obviously $r_m(\mathcal{X})=0$ for $m\leq 1$. The relations of degree 2 are called trivial.

Combining known results and our computations, we found the following for $r_m(\mathcal{X})$ (see tables). We refer to section 4 for more details. In the tables, + means $r_m(\mathcal{X})\geq 1$, 0 means $r_m(\mathcal{X})=0$, - means not known; $m\geq 5$.

	$\mathcal{W}_2$	$\mathcal{W}_n$
m	0	0
4	0	0
3	+	0
2	+	+

	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_n$
m	0	-	-
4	+	-	-
3	+	0	-
2	+	+	+

	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_n$
m	0	-	-	-
4	+	0	0	-
3	+	+	+	+
2	+	+	+	+

	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_n$
m	-	-	-
4	0	-	-
3	+	0	-
2	+	+	+

Let us make some remarks on these results. First we remark that we left out $\mathscr{W}_1$ and (the isomorphic) $\mathscr{K}_0$. This because $\mathscr{W}_1$ is special in the following sense: $n_+(\mathscr{W}_1)$ is *not* generated by the elements of degree 1; all others are. Because of this, it seems, $H_2(n_+(\mathscr{W}_1))$ has a special structure. It is well known (see [2]) that $r_m(\mathscr{W}_1)=0$ $(m\neq 5,7)$ and $r_5(\mathscr{W}_1)=r_7(\mathscr{W}_1)=1$. This means that there are no relations of degree 2, whereas in all other algebras there are.

Let $\mathscr{X}_n$ be one of the tabulated algebras. We believe that the following convexity hypothesis is true:

$$\text{if } r_m(\mathscr{X}_n)=0, \text{ then also } r_{m+1}(\mathscr{X}_n)=0 \text{ and } r_m(\mathscr{X}_{n+1})=0.$$

This hypothesis is known to be true for $\mathscr{W}_n$. In fact, according to [2] $r_m(\mathscr{W}_n)=0$ for $m\geq 3$ and $n\geq 3$. Under the convexity hypothesis, - in the table means $r_m(\mathscr{X}_n)=0$. For $r_3(\mathscr{H}_n)$ we can not conclude anything from the calculated results. With the help of Y. Kotchetkov, we proved that $r_3(\mathscr{H}_n)\geq 2n$, $n\geq 2$. For the proof and explicit representatives, see section 5. This is contrary to what one would a priori expect.

2. THE ALGEBRAS $\mathscr{W}_n$, $\mathscr{S}_n$, $\mathscr{H}_n$ AND $\mathscr{K}_n$

2.1. *The algebras* $\mathscr{W}_n$

We define $\mathscr{W}_n$ to be the infinite-dimensional Lie algebra of polynomial vector fields, i.e. $f\in\mathscr{W}_n$ can be uniquely written as

$$\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$$

where $f_i\in\mathbb{K}[x_1,\dots,x_n]$, $\mathbb{K}$ a field of characteristic 0. As is well known, the Lie algebra structure is given by

$$\left[\sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i}, \sum_{i=1}^{n} g_i\frac{\partial}{\partial x_i}\right] = \sum_{i,j=1}^{n}\left\{f_j\frac{\partial g_i}{\partial x_j}-g_j\frac{\partial f_i}{\partial x_j}\right\}\frac{\partial}{\partial x_i}.$$

The elements $x_1^{m_1}\cdots x_n^{m_n}(\partial/\partial x_i)$ with $\sum m_i=m+1$ are a basis for the subspace $\mathscr{W}_n^{(m)}$. This defines a $\mathbb{Z}$-grading on $\mathscr{W}_n$. The subalgebra $\mathscr{W}_n^{(0)}$ is isomorphic to $gl(n)$, an isomorphism being

$$x_i\frac{\partial}{\partial x_j}\to e_{i,j}\quad (1\leq i,j\leq n).$$

2.2. *The algebras* $\mathscr{S}_n$

The Lie algebra $\mathscr{S}_n$ is the subalgebra of $\mathscr{W}_n$ consisting of vector fields $\sum f_i(\partial/\partial x_i)$ with 0 divergence, i.e.

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} = 0.$$

Since the divergence operator $\sum_{i=1}^{n} \partial/\partial x_i$ is an homogeneous operator of degree 0, $\mathcal{S}_n$ inherits the $\mathbb{Z}$-grading of $\mathcal{W}_n$. The subalgebra $\mathcal{S}_n^{(0)}$ is isomorphic to $sl(n)$.

2.3. *The algebras $\mathcal{H}_n$*

We consider $\mathcal{W}_{2n}$, and denote x_i (resp. x_{i+n}) by p_i (resp. q_i), $i=1\cdots n$. The elements in $\mathcal{W}_{2n}$ that annihilate the 2-form $\alpha=\sum_{i=1}^{n} \mathrm{d}p_i\wedge \mathrm{d}q_i$ are called Hamiltonian; these elements form a subalgebra of $\mathcal{W}_{2n}$, denoted by $\mathcal{H}_n$. Since α can be seen as an operator of degree 2, $\mathcal{H}_n$ inherits the $\mathbb{Z}$-grading of $\mathcal{W}_{2n}$.

There is an equivalent description of $\mathcal{H}_n$, which is more practical for us, namely as Poisson algebra modulo center. In this description, $f\in\mathcal{H}_n$ is a polynomial in $p_1,\ldots,p_n,q_1,\ldots,q_n$ with coefficients in $\mathbb{K}$ and of degree ≥ 1. The Lie algebra structure is given by:

$$[f,g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) \bmod \mathbb{K}.1.$$

The isomorphism that relates these two descriptions is given by:

$$f\rightarrow \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right).$$

From this isomorphism, one sees that in the second description, the homogeneous space $\mathcal{H}_n^{(m)}$ consists of polynomials of degree $m+2$. In particular $\mathcal{H}_n^{(0)}$ consists of the polynomials with only quadratic terms; as Lie algebra it is isomorphic to $sp(2n)$. Let C_n be the abstract Lie algebra, which is isomorphic to $sp(2n)$. It has Chevalley generators e_i and f_i, which can be taken as follows:

$$e_i = p_i q_{i+1}, \quad i = 1\cdots n-1, \quad e_n = \tfrac{1}{2}p_n^2$$

and for f_i, the same with p and q interchanged for $i\leq n-1$ and $f_n = -\frac{1}{2}q_n^2$.

2.4. *The algebras $\mathcal{K}_n$*

To describe $\mathcal{K}_n$ we consider $\mathcal{W}_{2n+1}$, and denote x_i by p_i for $i=1\cdots n$, x_{i+n} by q_i for $i=1\cdots n$, and x_{2n+1} by z. The elements that send the 1-form $\alpha=\sum_{i=1}^{n}(p_i\,\mathrm{d}q_i - q_i\,\mathrm{d}p_i)+\mathrm{d}z$ into a $\mathbb{K}[p_1,\ldots,p_n,q_1,\ldots,q_n,z]$-multiple of α are called contact vector fields and constitute a subalgebra, which we denote by $\mathcal{K}_n$. Now α is not homogeneous; hence $\mathcal{K}_n$ does not inherit the $\mathbb{Z}$-grading of $\mathcal{W}_{2n+1}$. This can be solved by giving z degree 2 instead of 1. This is what is done in the following description, which relates $\mathcal{K}_n$ to the Poisson algebra described in subsection 2.3. Let $f\in\mathbb{K}[p_1,\ldots,p_n,q_1,\ldots,q_n,z]$. The Lie algebra structure is given by

$$[f,g] = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) + \frac{\partial f}{\partial z}\left(\sum_{i=1}^{n} p_i\frac{\partial g}{\partial p_i} + q_i\frac{\partial g}{\partial q_i} - 2g\right)$$
$$-\frac{\partial g}{\partial z}\left(\sum_{i=1}^{n} p_i\frac{\partial f}{\partial p_i} + q_i\frac{\partial f}{\partial q_i} - 2f\right).$$

The isomorphism that relates the descriptions above is given by:

$$f \to \sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) + \frac{\partial f}{\partial z}\sum_{i=1}^{n}\left(p_i\frac{\partial}{\partial p_i} + q_i\frac{\partial}{\partial q_i}\right)$$
$$-\left\{\sum_{i=1}^{n}\left(p_i\frac{\partial f}{\partial p_i} + q_i\frac{\partial f}{\partial q_i}\right) - 2f\right\}\frac{\partial}{\partial z}.$$

A $\mathbb{Z}$-grading in this description can be obtained by giving p_i and q_i degree 1 and z degree 2. Thus the homogeneous space $\mathcal{K}_n^{(m)}$ consists of polynomials with terms $\prod_{i=1}^{n}(p_i^{m_i}q_i^{n_i})z^k$ such that $\sum_{i=1}^{n}(m_i+n_i)+2k=m+2$. From this one sees that $\mathcal{K}_n^{(0)}$ is isomorphic to $sp(2n)\times\mathbb{C}$, the polynomial z being a central element in $\mathcal{K}_n^{(0)}$.

3. RELATIONS AND HOMOLOGY

3.1. *Relations and homology for nilpotent Lie algebras*

Let $\mathscr{X}$ be one of the algebras $\mathscr{W}_n$, $\mathscr{S}_n$ $(n\geq 2)$ or $\mathscr{H}_n$, $\mathscr{K}_n$ $(n\geq 1)$. Then $\mathscr{X}$ has a $\mathbb{Z}$-grading $\mathscr{X}^{(m)}$ as described in section 2. Define $n_+(\mathscr{X}) = \bigoplus_{m\geq 1}\mathscr{X}^{(m)}$. Let $C_q(n_+(\mathscr{X}))$ denote the q-forms (or q-chains), and $\partial : C_q(n_+(\mathscr{X}))\to C_{q-1}(n_+(\mathscr{X}))$ the boundary operator. As usual let $H_q(n_+(\mathscr{X}))$ denote the corresponding homology. $C_q(n_+(\mathscr{X}))$ and $H_q(n_+(\mathscr{X}))$ inherit the grading of $n_+(\mathscr{X})$: hence we have

$$H_q(n_+(\mathscr{X})) = \bigoplus_{m\geq q} H_q^{(m)}(n_+(\mathscr{X})).$$

The spaces $H_1(n_+(\mathscr{X}))$ and $H_2(n_+(\mathscr{X}))$ have the following interpretation (see [2], p. 335): a basis in $H_1(n_+(\mathscr{X}))$ yields a minimal set of generators of $n_+(\mathscr{X})$ and a basis of $H_2(n_+(\mathscr{X}))$ is a minimal set of defining relations of $n_+(\mathscr{X})$. One can prove in our case that $H_1^{(m)}(n_+(\mathscr{X}))=0$ for $m\neq 1$, i.e. that $n_+(\mathscr{X})$ is generated by $\mathscr{X}^{(1)}$. Since we are interested in defining relations, we need to compute $H_2^{(m)}(n_+(\mathscr{X}))$ $(m=2,3,\ldots)$. Here the case $m=2$ is trivial in the following sense: since $C_3^{(2)}(n_+(\mathscr{X}))=0$, we have $H_2^{(2)}(n_+(\mathscr{X}))=\ker(\partial\,|_{\Lambda^2(\mathscr{X}^{(1)})})$. Note that

$$\partial\,|_{\Lambda^2(\mathscr{X}^{(1)})}\left(\sum a_i x_i\wedge y_i\right) = \sum a_i[x_i,y_i] \quad (x_i,y_i\in\mathscr{X}^{(1)}).$$

Hence $H_2^{(2)}(n_+(\mathscr{X}))$ is easily described, and $\dim(H_2^{(2)}(n_+(\mathscr{X})))$ can easily be computed: $\partial : \Lambda^2(\mathscr{X}^{(1)}))\to\mathscr{X}^{(2)}$ is surjective in our cases, hence

$$\dim(H_2^{(2)}(n_+(\mathscr{X}))) = \tfrac{1}{2}\dim(\mathscr{X}^{(1)})\cdot(\dim(\mathscr{X}^{(1)})-1)-\dim(\mathscr{X}^{(2)}).$$

The study of $H_2^{(m)}(n_+(\mathscr{X}))$ $(m\geq 3)$ is less trivial, but is simplified by the following observation

$H_2^{(m)}(n_+(\mathscr{X}))$ is a module over $\mathscr{X}^{(0)}$

the action being the summed componentwise action. We study $H_2^{(m)}(n_+(\mathscr{X}))$ as a module over a suitable subalgebra of $\mathscr{X}^{(0)}$. Let us exemplify how this helps the computation. Suppose we choose a subalgebra of $\mathscr{X}^{(0)}$ isomorphic to $sl(2) = \langle e, f, h\rangle$. Since $H_2^{(m)}(n_+(\mathscr{X}))$ is finite-dimensional, it is completely reducible over $sl(2)$. As is well-known, (see i.e. [7]) the number of irreducible components is equal to $\dim(V_1)+\dim(V_0)$, where V_j is the space $\{x \in H_2^{(m)}(n_+(\mathscr{X})) \mid h \cdot x = j \cdot x\}$. Hence to find all homology $H_2^{(m)}(n_+(\mathscr{X}))$, we only need to consider homology of weight 0 and 1. This fact is exploited in the sections 4 and 6.

3.2. *Connection with the original algebra*

As said before, any algebra of vector fields $\mathscr{X}$ has a decomposition $\mathscr{X} = \bigoplus_{m \geq -1} \mathscr{X}^{(m)}$. In this paper, we mainly consider $n_+(\mathscr{X})$. This, however, is not a severe restriction. In fact, if we know the defining relations for $n_+(\mathscr{X})$, one can easily write down a set of defining relations for $\mathscr{X}$. Namely, $\mathscr{X}^{(0)}$ is a finite-dimensional subalgebra of $gl(n)$, and $\mathscr{X}^{(-1)}$ is submodule in $\mathbb{K}^n$. Therefore one can find (non-canonical) generators for these parts, and find a (non-canonical) set of defining relations. Since $\mathscr{X}^{(0)} \oplus \mathscr{X}^{(-1)}$ is finite-dimensional, these sets are finite. To define all commutators in $\mathscr{X}$ it is sufficient to fix the commutators of the generators of $\mathscr{X}^{(0)} \oplus \mathscr{X}^{(-1)}$ with the generators of $n_+(\mathscr{X})$. We remark that this strategy of finding defining relations is similar to the case of finite-dimensional semi-simple Lie algebras or Kac–Moody algebras (Chevalley generators). Probably, the main difference is that we do not take the maximal nilpotent subalgebra, but the maximal *graded* nilpotent subalgebra. This has great technical advantages.

4. THE COMPUTER RESULTS

Here we describe the results which we found by symbolic computations using REDUCE (see [3], [4]).

4.1. *The algebras $\mathscr{W}_n$*

Thanks to Fuks, we know that for $n \geq 3$, $n_+(\mathscr{W}_n)$ has only trivial defining relations; moreover for $n=2$ there are 25 relations, distributed as follows: $r_2(\mathscr{W}_2) = 7$ $(= \frac{1}{2} \cdot 6 \cdot 5 - 8)$ and $r_3(\mathscr{W}_2) = 18$. See [2], Appendix. For completeness, we describe $H_2^{(3)}(n_+(\mathscr{W}_2))$ as $sl(2)$-module. It has 4 irreducible components: 2 of highest weight 1, 1 of highest weight 5 and 1 of highest weight 7. The highest weight vectors, denoted by v_1, v_1', v_5 and v_7 can be chosen to be:

$$\begin{aligned} v_1 = {} & x_1^2 \frac{\partial}{\partial x_1} \wedge x_1^2 x_2 \frac{\partial}{\partial x_1} + x_1^2 \frac{\partial}{\partial x_1} \wedge x_1 x_2^2 \frac{\partial}{\partial x_2} + x_1 x_2 \frac{\partial}{\partial x_2} \wedge x_1^2 x_2 \frac{\partial}{\partial x_1} \\ & + x_1 x_2 \frac{\partial}{\partial x_2} \wedge x_1 x_2^2 \frac{\partial}{\partial x_2} - x_1 x_2 \frac{\partial}{\partial x_1} \wedge x_1^3 \frac{\partial}{\partial x_1} - x_1 x_2 \frac{\partial}{\partial x_1} \wedge x_1^2 x_2 \frac{\partial}{\partial x_2} \\ & - x_2^2 \frac{\partial}{\partial x_2} \wedge x_1^3 \frac{\partial}{\partial x_1} - x_2^2 \frac{\partial}{\partial x_2} \wedge x_1^2 x_2 \frac{\partial}{\partial x_2} \end{aligned}$$

$$v_1' = x_1^2 \frac{\partial}{\partial x_2} \wedge x_1 x_2^2 \frac{\partial}{\partial x_1} - x_1^2 \frac{\partial}{\partial x_1} \wedge x_1 x_2^2 \frac{\partial}{\partial x_2} - 2x_1x_2 \frac{\partial}{\partial x_2} \wedge x_1^2 x_2 \frac{\partial}{\partial x_1}$$
$$+ 2x_1x_2 \frac{\partial}{\partial x_1} \wedge x_1^2 x_2 \frac{\partial}{\partial x_2} + x_2^2 \frac{\partial}{\partial x_2} \wedge x_1^3 \frac{\partial}{\partial x_1} - x_2^2 \frac{\partial}{\partial x_1} \wedge x_1^3 \frac{\partial}{\partial x_2}$$
$$v_5 = 2x_1^2 \frac{\partial}{\partial x_2} \wedge x_1^3 \frac{\partial}{\partial x_1} + 3x_1^2 \frac{\partial}{\partial x_2} \wedge x_1^2 x_2 \frac{\partial}{\partial x_2} - x_1 x_2 \frac{\partial}{\partial x_2} \wedge x_1^3 \frac{\partial}{\partial x_2}$$
$$v_7 = x_1^2 \frac{\partial}{\partial x_2} \wedge x_1^3 \frac{\partial}{\partial x_2}.$$

4.2. *The algebras $\mathcal{S}_n$*

We can be brief about this case: $\mathcal{S}_2$ is isomorphic to $\mathcal{H}_1$, and we refer to subsection 4.3 and section 6 for a complete description. Further we calculated $r_3(\mathcal{S}_3)$, and it turned out to be 0. Assuming the convexity hypothesis, $\mathcal{S}_n$ $(n \geq 3)$ has only trivial relations.

4.3. *The algebras $\mathcal{H}_n$*

This case turned out to be the most interesting case. The algebra $\mathcal{H}_n^{(0)}$ consists of polynomials in $p_1, \ldots, p_n, q_1, \ldots, q_n$ with terms of degree 2 only. One easily determines n mutually commuting subalgebras $sl(2)_i$ isomorphic to $sl(2)$, namely $sl(2)_i = \langle \frac{1}{2}p_i^2, -\frac{1}{2}q_i^2, -p_i q_i \rangle$. Hence we can restrict our search for homology $H_2^{(m)}(n_+(\mathcal{H}_n))$ to the elements with weight 0 or 1 with respect to all $sl(2)_i$. We denote these weights as a vector: by the weight $(j_1, \ldots, j_n)$ we mean weight j_i with respect to $sl(2)_i$. Let $r_m(j_1, \ldots, j_n)$ denote the dimension of the subpace of $H_2^{(m)}(n_+(\mathcal{H}_n))$ of weight $(j_1, \ldots, j_n)$. Due to the symmetry of $\mathcal{H}_n$ it is clear that $r_m(j_1, \ldots, j_n) = r_m(j_{\sigma(1)}, \ldots, j_{\sigma(n)})$ for $\sigma \in Sym(n)$. Hence to find out whether $r_m(\mathcal{H}_n) = 0$, we only need to compute $r_m(j_1, \ldots, j_n)$ with $1 \geq j_1 \geq j_2 \geq \cdots \geq j_n \geq 0$. Moreover, one easily checks that the parity of m and $\sum_i j_i$ is the same. All this reduces our problem considerably. For example to conclude that $r_4(\mathcal{H}_3) = 0$, we only need to compute that $r_4(0,0,0) = 0$ and $r_4(1,1,0) = 0$.

Using this technique, combining it with the $\mathcal{H}_n^{(0)}$-action, we computed $r_m(\mathcal{H}_1)$, $(m = 3, 4, \ldots, 8)$, $r_m(\mathcal{H}_2)$, $(m = 3, 4)$, $r_m(\mathcal{H}_3)$, $(m = 3, 4)$ and $r_3(\mathcal{H}_4)$. The results are tabulated below.

m				
m	0	-	-	-
4	3	0	0	-
3	10	4	6	8
	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$

Under the convexity hypothesis, the places with - are also 0. The case $r_m(\mathcal{H}_1) = 0$, for $m \geq 9$ is described in section 6. For $m \geq 5$, we also found that $r_m(\mathcal{H}_1) = 0$. Moreover it turned out that $r_3(\mathcal{H}_1) = 10$, and $r_4(\mathcal{H}_1) = 3$. Considered as $sl(2)$-module, $H_2^{(4)}(\mathcal{H}_1)$ is irreducible with highest weight vector w_2, and $H_2^{(3)}(\mathcal{H}_1)$ has two irreducible components with highest weights 1 and 7; we denote the highest weight vectors by v_1 and w_7, respectively. We give them below.

The cases $\mathcal{H}_n$, for $n \geq 2$ seem quite similar in result: $r_m(\mathcal{H}_n) = 0$, $m \geq 4$ and $r_3(\mathcal{H}_n) = 2n$. We investigated the $\mathcal{H}_n^{(0)}$-module structure of $H_2^{(3)}(n_+(\mathcal{H}_n))$, $n = 2, 3, 4$. Reminding that $\mathcal{H}_n^{(0)}$ is isomorphic to $sp(2n)$, see subsection 2.3, it is clear that it has a standard $2n$-dimensional representation. It turned out that $H_2^{(3)}(n_+(\mathcal{H}_n))$, $n = 2, 3, 4$ is isomorphic to the standard module $\mathbb{K}^{2n}$. This result suggests that $r_3(\mathcal{H}_n) = 2n$ for $n \geq 2$. We prove that $r_3(\mathcal{H}_n) \geq 2n$ in the next section. The highest weight vector of $H_2^{(3)}(n_+(\mathcal{H}_2))$, denoted by v_2, is given below.

$$
\begin{aligned}
w_2 &= 3p_1^2q_1^2 \wedge p_1^3q_1 - p_1q_1^3 \wedge p_1^4 \\
w_7 &= p_1^3 \wedge p_1^4 \\
v_1 &= -q_1^3 \wedge p_1^4 + 3p_1q_1^2 \wedge p_1^3q_1 - 3p_1^2q_1 \wedge p_1^2q_1^2 + p_1^3 \wedge p_1q_1^3 \\
v_2 &= p_1^3 \wedge p_1q_1^3 - 3p_1^2q_1 \wedge p_1^2q_1^2 + 3p_1q_1^2 \wedge p_1^3q_1 - q_1^3 \wedge p_1^4 + 3p_1^2p_2 \wedge p_1q_1^2q_2 \\
&\quad - 6p_1q_1p_2 \wedge p_1^2q_1q_2 + 3q_1^2p_2 \wedge p_1^3q_2 + 3p_1p_2^2 \wedge p_1q_1q_2^2 - 3q_1p_2^2 \wedge p_1^2q_2^2 \\
&\quad + p_2^3 \wedge p_1q_2^3 - 3p_1^2q_2 \wedge p_1q_1^2p_2 + 6p_1q_1q_2 \wedge p_1^2q_1p_2 - 3q_1^2q_2 \wedge p_1^3p_2 \\
&\quad - 6p_1p_2q_2 \wedge p_1q_1p_2q_2 + 6q_1p_2q_2 \wedge p_1^2p_2q_2 - 3p_2^2q_2 \wedge p_1p_2q_2^2 \\
&\quad + 3p_1q_2^2 \wedge p_1q_1p_2^2 - 3q_1q_2^2 \wedge p_1^2p_2^2 + 3p_2q_2^2 \wedge p_1p_2^2q_2 - q_2^3 \wedge p_1p_2^3.
\end{aligned}
$$

4.4. *The algebras $\mathcal{K}_n$*

For the algebras $\mathcal{K}_n$, we found that $r_3(\mathcal{K}_2) = 0$ and $r_4(\mathcal{K}_1) = 0$. Hence, from the convexity hypothesis, we conclude that there are only non-trivial relations for $\mathcal{K}_1$ at degree 3. As $sp(2) \times \mathbb{K}$ (or $sl(2)$)-module, it has 5 irreducible submodules: 2 of highest weight 1, and 1 of highest weight 3, 5 and 7. As highest weight vectors, we can take respectively

$$
\begin{aligned}
v_1 &= p_1z \wedge z^2 \\
v_1' &= p_1^3 \wedge q_1^2z - 2p_1^2q_1 \wedge p_1q_1z + p_1q_1^2 \wedge p_1^2z \\
v_3 &= 2p_1z \wedge p_1^2z + p_1^3 \wedge z^2 \\
v_5 &= p_1z \wedge p_1^4 + 2p_1^3 \wedge p_1^2z \\
v_7 &= p_1^3 \wedge p_1^4.
\end{aligned}
$$

5. RELATIONS OF DEGREE 3 OF $\mathcal{H}_n$

In this section, we will describe natural representatives for $H_2^{(3)}(n_+(\mathcal{H}_n))$ and $H_{(3)}^2(n_+(\mathcal{H}_n))$. Moreover we prove that these are indeed representatives for the homology and cohomology and form a standard representation of $sp(2n)$, see subsection 5.3.

In some of the proofs presented in this section, we use that the following actions induce automorphisms on $\mathcal{H}_n$ of degree zero:

$$p_i \leftrightarrow p_j,\ q_i \leftrightarrow q_j \quad \text{and} \quad p_i \to q_i,\ q_i \to -p_i.$$

These morphisms reduce the cases to be investigated considerably.

5.1. $H_2^{(3)}(n_+(\mathcal{H}_n))$

The representatives of $H_2^{(3)}(n_+(\mathcal{H}_n))$ are most easily described in the following way. Consider the formal product

$$c = (\sum_1^n (p_i \wedge q_i - q_i \wedge p_i))^3 \cdot (1 \wedge p_1)$$

in $C_2^{(3)}(n_+(\mathcal{H}_n))$ of weight $(1, 0, \ldots, 0)$. The product $\cdot$ is defined as:

$$(a \wedge b) \cdot (c \wedge d) = (ac) \wedge (bd).$$

In the cases $n=1$ and $n=2$ we obtain for c the elements υ_1 and υ_2 of subsection 4.3.

We state that c is a cycle. This can be seen in the following way: $\partial(c) \in C_1^{(3)}(n_+(\mathcal{H}_n))$ is a linear combination of the basiselements $(i, j \neq 1,\ i \neq j)$:

$$p_1^3 q_1^2 \quad p_1^2 q_1 p_i q_i \quad p_1 q_i^2 p_i^2 \quad p_1 q_i p_i q_j p_j.$$

For each of these elements we write out from which terms in c they arise. These elements, x, are listed in the first column below. In the second column, the coefficient of the considered basiselement in $\partial(x)$ is given. Their sum is 0. Here we consider only the basiselement $p_1^3 q_1^2$. The other cases can be handled similarly, see [5].

$q_1^3 \wedge p_1^4$	$-1 \cdot -12 = 12$	due to p_1, q_1 action
$q_1^2 p_1 \wedge q_1 p_1^3$	$3 \cdot (1-6) = -15$	
$q_1 p_1^2 \wedge q_1^2 p_1^2$	$-3 \cdot (4-2) = -6$	
$p_1^3 \wedge q_1^3 p_1$	$1 \cdot 9 = 9$	
$q_1^2 q_i \wedge p_i p_1^3$	$-3 \cdot -1 = 3$	due to p_i, q_i action
$q_1 q_i p_1 \wedge q_1 p_i p_1^2$	$6 \cdot -1 = -6$	
$q_i p_1^2 \wedge q_1^2 p_i p_1$	$-3 \cdot -1 = 3$	

Thus we can prove that c is a cycle. On account of the morphisms given before, it is obvious that

$$(\sum_1^n (p_i \wedge q_i - q_i \wedge p_i))^3 \cdot (1 \wedge p_j) \quad \text{and} \quad (\sum_1^n (p_i \wedge q_i - q_i \wedge p_i))^3 \cdot (1 \wedge q_j)$$

are cycles for all j $(1 \leq j \leq n)$. So we have constructed $2n$ linearly independent cycles.

5.2. *The non-triviality of the cycles*

In order to prove that the given cycle c is non-trivial, it is sufficient to show that for a certain cocycle c^* holds $\langle c^* \mid c \rangle \neq 0$. For if $c = \partial(b)$, then

$$\langle c^* \mid c \rangle = \langle c^* \mid \partial(b) \rangle = \langle d(c^*) \mid b \rangle = 0.$$

We construct c^* in the following way: write out c:

$$c = \sum_i c_i f_i \wedge g_i, \qquad \text{where } f_i = p^{\alpha^i} q^{\beta^i} \quad \text{and} \quad g_i = p^{\gamma^i} q^{\delta^i}$$

where α^i, β^i, γ^i and δ^i are multi-indices of length n with $|\alpha^i| + |\beta^i| = 3$ and $|\gamma^i| + |\delta^i| = 4$. Then c^* is given by

$$c^* = \sum_i c_i^* f_i^* \wedge g_i^* \quad \text{with } c_i^* = (-1)^{|\beta^i|} \gamma^i! \, \delta^i!$$

Then clearly $\langle c^* \mid c \rangle = \sum c_i^* \cdot c_i > 0$.

To prove that c^* is a cocycle, we study the terms that can arise in $d(c^*) \in C^3_{(3)}(n_+(\mathscr{H}_n))$ of weight $(1,0,\dots,0)$. In these terms, the total power of p's and q's equals 9, so there can be maximal 5 different p's and q's. From this observation and the morphisms given at the beginning of this section, we conclude:

LEMMA 5.1. *If c^* is a cocycle in $C^2_{(3)}(n_+(\mathscr{H}_n))$ for $n \leq 5$, then it is a cocycle for all $n \in \mathbb{N}$.*

With help of the computer, we have checked that c^* is a cocycle for $n \leq 4$. (The space $C^3_{(3)}(n_+(\mathscr{H}_5))$ of weight $(1,0,\dots,0)$ is 4237-dimensional!) The only difference between the case $n=4$ and $n=5$ is the existence of terms $q_i q_j q_k \wedge q_l p_i p_j \wedge p_k p_l p_1$. These terms can arise from:

$$\left.\begin{array}{ll} q_l p_i p_j \wedge q_i q_j p_l p_1 & -1 \cdot -1 = 1 \\ q_i q_j q_k \wedge p_i p_j p_k p_1 & -1 \cdot 1 = -1 \end{array}\right\} \quad \begin{array}{l} \text{due to } p_k, q_k\text{-action} \\ \text{due to } p_l, q_l\text{-action.} \end{array}$$

Obviously the same reasoning holds for all c's $(2n)$ given in the previous subsection and thus we have proven that the space $H_2^{(3)}(n_+(\mathscr{H}_n))$ is at least $2n$-dimensional.

5.3. *The space $H_2^{(3)}(n_+(\mathscr{H}_n))$ as $sp(2n)$-representation*

In section 2.3 we described that $\mathscr{H}_n^{(0)}$ is isomorphic to $sp(2n)$. From this it follows that we can view $H_2^{(3)}(n_+(\mathscr{H}_n))$ as an $sp(2n)$-representation.

THEOREM 5.2. *The subspace generated by*

$$c_p^1, \dots, c_p^n, c_q^1, \dots, c_q^n; \qquad c_y^i = c^3 \cdot (1 \wedge y_i),\ c = \sum (p_j \wedge q_j - q_j \wedge p_j)$$

is an irreducible module, isomorphic to the standardrepresentation of $sp(2n)$.

PROOF. If v_i denotes the standardbasis of $\mathbb{C}^{2n}$, then the standard $sp(2n)$-representation can be schematically given by:

$$v_{n+1} \xrightarrow{e_1} v_{n+2} \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} v_{2n} \xrightarrow{e_n} v_n \xrightarrow{e_{n-1}} v_{n-1} \cdots \xrightarrow{e_1} v_1.$$

We shall show that $v_i \to (-1)^{i+1} c_p^i$, $(1 \leq i \leq n)$ and $v_i \to (-1)^{n+1} c_q^{i-n}$, $(n+1 \leq i \leq 2n)$ defines an isomorphism of representations.

We remember the Leibniz-rule in $\mathscr{H}_n$:

$$\{f, gh\} = \{f, g\} h + g\{f, h\}.$$

From this we find for the action $*$ of $\mathscr{H}_n$ on $C_2(n_+(\mathscr{H}_n))$

$$f*((g_1\wedge g_2)\cdot(h_1\wedge h_2)) = (f*(g_1\wedge g_2))\cdot(h_1\wedge h_2)+(g_1\wedge g_2)\cdot f*(h_1\wedge h_2)$$

and hence for all $f\in\mathscr{H}_n$

$$f*c_y^i = f*(c^3\cdot(1\wedge y_i)) = 3c^2\cdot(f*c)\cdot(1\wedge y_i)+c^3\cdot(1\wedge\{f,y_i\}).$$

LEMMA 5.3. $e_i*c=f_i*c=0.$

PROOF.

$$e_i*\sum(p_j\wedge q_j-q_j\wedge p_j) = -p_i\wedge q_{i+1}+p_i\wedge q_{i+1}+q_{i+1}\wedge p_i-q_{i+1}\wedge p_i=0$$
$$(1\le i<n).$$

Similar for e_n and f_i.

From this lemma we conclude that $e_j*c_y^i=c^3\cdot(1\wedge\{e_j,y_i\})$ and so a direct calculation verifies the isomorphism given above. This proves the theorem.

6. FINITE-DIMENSIONALITY OF $H_2(n_+(\mathscr{H}_1))$

In this section, we will prove that $r_m(\mathscr{H}_1)=0$ for $m\ge 9$. Once the proof is written down, one notices that a similar reasoning works for a class of subalgebras as well. The simplest subalgebra of this class is the set of even polynomials, which is of interest in connection with $sl(\lambda)$, see [1].

6.1. *The class* $\mathfrak{g}_k$

We consider the following class $\mathfrak{g}_k$ of subalgebras of $n_+(\mathscr{H}_1)$: let $\mathfrak{g}_k^{(m)}$ be the span of elements of degree $m\cdot k$, $m=1,2,\dots$, and

$$\mathfrak{g}_k=\bigoplus_{m\ge 1}\mathfrak{g}_k^{(m)}=\bigoplus_{m\ge 1}\mathscr{H}_1^{(mk)}.$$

We have that $\mathfrak{g}_1=n_+(\mathscr{H}_1)$. As degree on $\mathfrak{g}_k$ we take the old degree divided by k, and call it k-degree. We denote the ideal spanned by all elements of k-degree ≥ 2 by $\mathfrak{h}_k$.

It is obvious that $\mathfrak{g}_k$ is a module over $\mathscr{H}_1^{(0)}\simeq sl(2)$. Note that the k-degree and the weight determine the basiselements of $\mathfrak{g}_k$ up to a factor. We will denote the element $p_1^iq_1^j$ of k-degree m and weight α by $\{m;\alpha\}$; hence $m=(i+j-2)/k$, and $\alpha=i-j$. We will use the following lemma

LEMMA 6.1. *Let* $x=\{m;\alpha\}\in\mathfrak{h}_k\subset\mathfrak{g}_k$.

1. *Then there exists an element g of k-degree* 1 *and a* $y=\{m-1;\beta\}$ *such that*

$$\partial(g\wedge y)=g\cdot y=[g,\{m-1;\beta\}]=\{m;\alpha\}.$$

2. *Suppose* $-k(m-1)-2\le\beta\le k(m-1)$ *and* $-2k\le\alpha-\beta\le 2k+2$. *Then either* $\partial(\{1;\alpha-\beta\}\wedge\{m-1;\beta\})=cx$ *or* $\partial(\{1;\alpha-\beta-2\}\wedge\{m-1;\beta+2\})=cx$, *with* $c\ne 0$.

In particular, $\mathfrak{g}_k$ is generated by $\mathfrak{g}_k^{(1)}$. Using $[\{m;\alpha\}, \{l;\beta\}]=0$ iff $\alpha(l+2)=\beta(m+2)$, this lemma is easily proved.

6.2. *The main theorem; sketch of the proof*

Our purpose is to prove that $\mathfrak{g}_k$ has only finitely many defining relations. In fact we get a bound on the k-degree of the minimal set of defining relations as follows:

THEOREM 6.2. *For $\mathfrak{g}_k$, defined above, we have*

$$H_2^{(m)}(\mathfrak{g}_k)=0 \quad \textit{for } m\geq 9.$$

We prove this theorem using the Hochshild-Serre spectral sequence (see [6] or [2]) with respect to the ideal $\mathfrak{h}_k\subset\mathfrak{g}_k$. From now on, we will write $\mathfrak{g}$ and $\mathfrak{h}$ instead of $\mathfrak{g}_k$ and $\mathfrak{h}_k$.

The second term in the Hochshild-Serre spectral sequence, E^2, consists of 3 parts:

$$E^2_{2,0}=H_2^{(m)}(\mathfrak{g}/\mathfrak{h};H_0(\mathfrak{h})),\quad E^2_{1,1}=H_1^{(m)}(\mathfrak{g}/\mathfrak{h};H_1(\mathfrak{h})),\quad E^2_{0,2}=H_0^{(m)}(\mathfrak{g}/\mathfrak{h};H_2(\mathfrak{h})).$$

Obviously, we have $E^2_{2,0}=H_2^{(m)}(\mathfrak{g}/\mathfrak{h})=0$ for $m\geq 3$. Further, since $H_1(\mathfrak{h})=\mathfrak{g}^{(2)}\oplus\mathfrak{g}^{(3)}$, we have $E^2_{1,1}=H_1^{(m)}(\mathfrak{g}/\mathfrak{h};H_1(\mathfrak{h}))=0$ for $m\geq 5$. The most difficult part is $E^2_{0,2}=H_0^{(m)}(\mathfrak{g}/\mathfrak{h};H_2(\mathfrak{h}))$. We can write this term in the following way:

$$E^2_{0,2}=H_2^{(m)}(\mathfrak{h})/\mathfrak{g}^{(1)}\cdot H_2^{(m-1)}(\mathfrak{h})=(Z_2^{(m)}(\mathfrak{h})/(B_2^{(m)}(\mathfrak{h})+\mathfrak{g}^{(1)}\cdot Z_2^{(m-1)}(\mathfrak{h}))$$

where $Z_2^{(m)}(\mathfrak{h})$ stands for the space of cycles of k-degree m, and $B_2^{(m)}(\mathfrak{h})$ for the boundaries of k-degree m. In the second simplification, we have used that $\mathfrak{g}^{(1)}\cdot B_2^{(m-1)}(\mathfrak{h})\subset B_2^{(m)}(\mathfrak{h})$.

The space $H_0^{(m)}(\mathfrak{g}/\mathfrak{h};H_2(\mathfrak{h}))$ can be considered as an $sl(2)$-module, see section 3. Hence we know that the homology $H_0^{(m)}(\mathfrak{g}/\mathfrak{h}))$ occurs in strings, in which weight 0 (for $k\cdot m$ even) or weight 1 (for $k\cdot m$ odd) certainly is present. Therefore we can restrict our attention to weight 0 and 1. We denote these spaces by $H_0^{(m,0)}(\mathfrak{g}/\mathfrak{h};h_2(\mathfrak{h}))$ and $H_0^{(m,1)}(\mathfrak{g}/\mathfrak{h};H_2(\mathfrak{h}))$, respectively. Thus in order to prove that $E^2_{2,0}=0$, it is sufficient to show that these spaces are 0. We do this by showing that

$$Z_2^{(m,i)}(\mathfrak{h})=\sum_j \mathfrak{g}^{(1,j)}\cdot Z_2^{(m-1,i-j)}(\mathfrak{h})\quad \text{for } i=m\cdot k \bmod 2.$$

We proved this by a case by case check. We refer to [5] for the proof.

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