Indag. Mathem., N.S., 2 (3), 341-358 September 30, 1991

Colombeau algebras on a C^∞-manifold

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Communicated by Prof. P.J. Zandbergen at the meeting of May 27, 1991

ABSTRACT

In this paper Colombeau's algebra of functions on $\mathbb{R}^n$, containing the distributions, is generalized to a sheaf on a C^∞-manifold. The well-known problem of restricting distributions to a submanifold is solved within this framework.

INTRODUCTION

In [3] Colombeau has defined an algebra of generalized functions on $\mathbb{R}^n$ containing the distributions. In [3], as well as in other presentations in [1] and [12], it is given as a presheaf on $\mathbb{R}^n$ of algebras of functions divided by an ideal. Such an algebra is a particular case of the ones treated by Rosinger, for example in [12], where also the advantage of having an ideal is discussed. Partial differential equations have been studied in this algebra, cf. for example [9], and there are several applications in shock waves and other aspects of non-linear partial differential equations, cf. [1], [4] and [12], or in theoretical physics, cf. [3].

The definition given by Colombeau is not suited for treating restrictions to submanifolds. Therefore, in [9] Oberguggenberger used a slightly different algebra in order to get restrictions to subspaces $\mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$ and Biagioni has yet another approach in [1] which enables her to obtain restrictions to all linear subspaces of $\mathbb{R}^n$. However, in [4] Colombeau introduced a subalgebra

* For the second author financial support was provided by the Netherlands Organization for Scientific Research (NWO) via the Stichting Mathematisch Centrum (project nr. 611-306-525).

which, apart from being more simple, had the advantage of allowing restrictions to any submanifold, although this aspect has not been discussed in [1] or [4]. It seems that such a simplified algebra will be just as good for applications, cf. [1].

In this paper we extend this theory of Colombeau to an arbitrary C^∞-manifold X. Our method of introducing it straight away as a quotient sheaf $\mathscr{G}$ of algebras differs somewhat from the one given in [1] or [4]. Besides of being more natural it has other advantages, one of which is that, contrarily to [1] and [3], the sheaf structure, of course, needs no proof, and another that our definition of sections over an open subset is as simple as over X, while in [1], [3] or [12] the definition for $\Omega \subsetneqq \mathbb{R}^n$ is much more complicated than for $\mathbb{R}^n$ itself. By means of techniques of de Rham from [10] we will construct sheafmorphisms embedding the distributions into $\mathscr{G}$. Moreover, the restriction of sections in $\mathscr{G}$ to an arbitrary submanifold always exists and we will show that it coincides with the one given for distributions by Hörmander in [5].

Finally, as up to now no name has been given to sections of $\mathscr{G}$ except generalized functions, which is too general, we use the name ultrafunctions and we briefly explain why.

1. THE SHEAF $\mathscr{G}$ ON A C^∞-MANIFOLD X

Let $\mathbb{R}_+^* = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$, $\mathbb{R}_+ = \{\varepsilon \in \mathbb{R} \mid \varepsilon \geq 0\}$ and let injections

$$i\colon\ \mathbb{R}_+^* \times X \to \mathbb{R}_+ \times X$$

$$\alpha\colon\ X \to \mathbb{R}_+ \times X$$

be given by

$$i(\varepsilon, x) = (\varepsilon, x) \quad \text{and} \quad \alpha(x) = (0, x).$$

Furthermore, let $\mathscr{F}_1$ be the sheaf of functions on $\mathbb{R}_+^* \times X$ which are C^∞ in $x \in X$ with a parameter $\varepsilon \in \mathbb{R}_+^*$. Then we define a sheaf $\mathscr{F}$ on X by

$$\mathscr{F} = \alpha^{-1}(i_* \mathscr{F}_1).$$

Here $i_*\mathscr{F}_1$ is the direct image sheaf of $\mathscr{F}_1$ under i and $\alpha^{-1}(i_*\mathscr{F}_1)$ the inverse image sheaf of $i_*\mathscr{F}_1$ under α, cf. for example [6].

We adopt the following notations and conventions. By I_0^+ we mean an open interval $(0, \varepsilon_0)$ for some positive $\varepsilon_0 < 1$ which might be smaller each time there is referred to. For an open set $\Omega \subset X$ the restriction $f|_K$ to a compact set K of a section $f \in \mathscr{F}(\Omega)$ is induced by a function in $\mathscr{F}_1(I_0^+ \times \omega)$, denoted by f_K, for some open neighborhood ω of K; then we denote $f|_K$ also by $(f_{K,\varepsilon})_{\varepsilon \in I_0^+}$ or just $(f_{K,\varepsilon})$. Note that two such functions f_K^1 and f_K^2 induce the same section $f|_K$ if they coincide on a common domain $I_0^+ \times \omega$. The value at a point $x_0 \in K$ of a section f will be denoted by $(f_{K,\varepsilon}(x_0))$ and the germ in x by $(f_{K,\varepsilon})|_x$. It may happen that a section $f \in \mathscr{F}(\Omega)$ arises everywhere from a single function defined in $I_0^+ \times \Omega$ (i.e. the domain I_0^+ of ε is the same for every $x \in \Omega$), in which case we omit the subscript K and just write $f = (f_\varepsilon)_{\varepsilon \in I_0^+} = (f_\varepsilon)$. Since only the values for $\varepsilon \downarrow 0$ matter, and in fact one is interested in the asymptotic behavior, one could have taken $\varepsilon = 1/n$ for $n \in \mathbb{N}^*$ or $n \geq n_0$ and so equivalence classes of tail-ends of sequences $(f_{K,n})_{n \geq n_0}$ of functions on K would have been obtained. However, it has become standard to work with a continuous parameter ε, which can have an advantage, too, cf. [11], and only the name *sequences* for sections in $\mathscr{F}$ is sometimes used, cf. [12].

$\mathscr{F}$ is a sheaf of algebras; the algebra structure of $\mathscr{F}(\Omega)$ follows from the one for functions where the algebraic operations are defined pointwise.

Clearly, C^∞ is a subsheaf of $\mathscr{F}$: it consists of the elements coming from functions independent of ε. At the same time the sheaf $\mathscr{P}$ of sheafmorphisms in C^∞ extends canonically to $\mathscr{F}$, namely for a sheafmorphism $P \in \mathscr{P}(\Omega)$ and a sequence $f \in \mathscr{F}(\Omega)$ $P\,|_K$ acts on $f_{K,\varepsilon}$ for each $\varepsilon \in I_0^+$ separately.

In $\mathscr{F}$ we define the subsheaves $\mathscr{M}$ (whose sections are called *moderate sequences*) and $\mathscr{N}$ (whose sections are called *null sequences*) by:

$$f \in \mathscr{M}(\Omega) \;\Leftrightarrow\; f \in \mathscr{F}(\Omega) \text{ and } \forall K \Subset \Omega,\ \forall P \in \mathscr{P}(\Omega),\ \exists p \in \mathbb{N}:$$

$$P\,|_K f_{K,\varepsilon} = 0(\varepsilon^{-p}, \varepsilon \downarrow 0) \text{ uniformly on } K.$$

$$f \in \mathscr{N}(\Omega) \;\Leftrightarrow\; f \in \mathscr{M}(\Omega) \text{ and } \forall K \Subset \Omega,\ \forall P \in \mathscr{P}(\Omega),\ \forall q \in \mathbb{N}:$$

$$P\,|_K f_{K,\varepsilon} = 0(\varepsilon^{q}, \varepsilon \downarrow 0) \text{ uniformly on } K.$$

It follows immediately from these definitions that sections of $\mathscr{P}$ are sheafmorphisms in $\mathscr{M}$ and in $\mathscr{N}$, too. Since according to Peetre's theorem every sheafmorphism P in C^∞ is such that, in local coordinates, $P\,|_K$ is a finite-order differential operator, Leibniz rule can be applied implying that $\mathscr{M}$ and $\mathscr{N}$ are sheaves of subalgebras of $\mathscr{F}$. Moreover, $\mathscr{N}(\Omega)$ is an ideal in $\mathscr{M}(\Omega)$, so that the quotient sheaf

$$\mathscr{G} \overset{\text{def}}{=} \mathscr{M}/\mathscr{N}$$

is a sheaf of algebras on X, whose sections we will call *ultrafunctions*. Clearly, $\mathscr{P}$ acts in $\mathscr{G}$, too, thus, in local coordinates, partial derivatives of ultrafunctions are defined.

Since $C^\infty \subset \mathscr{M}$, it follows from the fact that $\mathscr{M}$ is a sheaf of algebras with $\mathscr{N}$ as a subsheaf of ideals, that $\mathscr{M}$ and $\mathscr{N}$ are C^∞-modules and hence they are fine sheaves. Therefore, the cohomology group $H^1(\Omega, \mathscr{N})$ vanishes for every open $\Omega \subset X$, so that

$$\text{(1)} \qquad \mathscr{G}(\Omega) = \mathscr{M}(\Omega)/\mathscr{N}(\Omega),$$

i.e. $\mathscr{G}$ could have been defined as the quotient presheaf, too, which is Colombeau's definition, cf. [1] and [4].

REMARK 1. For a function $\varrho \in C_0^\infty(\mathbb{R})$ we consider the following example of a moderate sequence f on $\mathbb{R}$:

$$(f_\varepsilon)_{\varepsilon \in I_0^+} = (x\varrho^\varepsilon)$$

where (ϱ^{ε}) is the collection of functions on $\mathbb{R}$ given by $\varrho^{\varepsilon}(x)=1/\varepsilon\,\varrho(x/\varepsilon)$ for $\varepsilon\in I_0^+$. Since the germs $f|_x$ vanish only for all $x\neq 0$, the support of f is $\{0\}$. However, at every point $x_0\in\mathbb{R}$ the value $(f_\varepsilon(x_0))_{\varepsilon\in I_0^+}$ (considered as a moderate sequence on the set $\{x_0\}$, or, in other words, as a generalized number, cf. [3]) vanishes. Thus, contrarily to ordinary functions sections of $\mathscr{G}$ are not yet determined by giving their values in all points. Therefore, the name *ultrafunctions* is proposed here.

REMARK 2. The more complicated sheaf of the original algebras of Colombeau in [3] can be obtained in our way by replacing $\mathscr{F}_1$ in the foregoing by the sheaf of maps from a certain fixed, filtered, set $\mathscr{A}$ of normalized test functions φ on $\mathbb{R}^n$ into our sheaf $\mathscr{F}_1$, where (in case $X=\mathbb{R}^n$) a partial derivative $\nabla^{\alpha}f$ of such a map f is obviously defined as $(\nabla^{\alpha}f)(\varphi)=\nabla^{\alpha}(f(\varphi))$. Furthermore, a more complicated definition of moderate and null sequences is given in which the filtration of $\mathscr{A}$ enters. The subsheaf of the so-called simplified algebras (treated in this paper and here denoted by $\mathscr{G}$, but in [1] and [4] denoted by $\mathscr{G}_s$) then consists of the sequences independent of the test functions in $\mathscr{A}$.

2. EMBEDDINGS OF $\mathscr{D}'$ INTO $\mathscr{G}$

The sheaf $\mathscr{D}'$ of distributions on X is in topological duality with the cosheaf $\mathscr{D}_1$ of C^{∞}-function densities with compact support in X.

In view of (1), for an ultrafunction $F\in\mathscr{G}(\Omega)$ on each compact set $K\subset\Omega$ there is always a moderate sequence $(f_{K,\varepsilon})$ representing $F|_K$. Let now F be such that for every density $\varphi\in\mathscr{D}_1(\Omega)$ the limit

$$(2)\qquad \lim_{\varepsilon\downarrow 0}\int f_{K,\varepsilon}\varphi$$

exists, when K is a compact set containing the support of φ.

Since this limit vanishes for any null sequence $(f_{K,\varepsilon})$, clearly the limit (2) does not depend on the set K, nor on the moderate sequence $(f_{K,\varepsilon})$ representing $F|_K$. By the Banach–Steinhaus theorem the limits (2), if they exist, define a distribution $u\in\mathscr{D}'(\Omega)$:

$$(3)\qquad \lim_{\varepsilon\downarrow 0}\int f_{K,\varepsilon}\varphi = \langle u,\varphi\rangle.$$

On the distributional level F and u are equal, so we will say that F is *weakly equal* to u, $F\sim u$. In [1] u is called the macroscopic aspect of F. As usual, cf. [1], [3] and [12], two ultrafunctions F_1 and F_2 are said to be *associated*, $F_1\approx F_2$, if $F_1-F_2\sim 0$. It follows immediately from (3) that if $F\sim u$ and $P\in\mathscr{P}(\Omega)$ also

$$(4)\qquad P(F)\sim P(u).$$

Obviously, each $F\in\mathscr{G}(\Omega)$ can be weakly equal to not more than one distribution and all ultrafunctions weakly equal to the same distribution are associated to each other, but there are ultrafunctions which are not weakly equal to any distribution. We will now show that for each distribution there is an ultrafunction weakly equal to it. In particular, we will show that this weak equality can be obtained by an injective sheaf morphism: $\mathscr{D}' \hookrightarrow \mathscr{G}$; hence each distribution is at least weakly equal to one ultrafunction with the same support.

Before embedding $\mathscr{D}'$ into $\mathscr{G}$ we give a summary of regularizations of distributions on a C^∞-manifold X given by de Rham in [10, § 15]. Such regularizations depend on a sequence of special local charts and on a C^∞-weight function ϱ with compact support in the translation group H in $\mathbb{R}^n$ with

$$\int_H \varrho(y)\,\mathrm{d}y = 1.$$

In particular, take a locally finite covering of X by relatively compact open coordinate patches U_j such that there are charts $\kappa_j : U_j \to B$ where B is the open unit ball in $\mathbb{R}^n$ with the property that they can be extended to open neighborhoods of $\bar{U}_j$ and $\bar{B}$. Furthermore, let h be a fixed diffeomorphism from B onto $\mathbb{R}^n$. Then the charts h_j are defined on U_j by

$$h_j = h \circ \kappa_j : U_j \to \mathbb{R}^n.$$

In [10] homomorphisms A^j of H into the diffeomorphism group of X are constructed such that for each $y \in H$

$$A^j_y(x) = x, \quad x \notin U_j.$$

For $\varepsilon \in I_0^+$ a partial regularization of a distribution $u \in \mathscr{D}'(X)$ is defined by

$$R^j_\varepsilon u = \int_H \varrho^\varepsilon(y)(A^j_{-y})^* u\,\mathrm{d}y, \tag{5}$$

where ϱ^ε is a δ-approaching sequence, for example

$$\varrho^\varepsilon(y) = \frac{1}{\varepsilon^n}\varrho\left(\frac{y}{\varepsilon}\right). \tag{6}$$

I.e. for any test function density φ (by de Rham called an n-form of the odd kind) we have

$$\left\{\begin{aligned} \langle R^j_\varepsilon u, \varphi\rangle &= \int_H \varrho^\varepsilon(y)\langle (A^j_{-y})^* u, \varphi\rangle\,\mathrm{d}y \\ &= \int_H \varrho^\varepsilon(y)\langle u, (A^j_y)^* \varphi\rangle\,\mathrm{d}y = \langle u, {}^t R^j_\varepsilon \varphi\rangle. \end{aligned}\right. \tag{7}$$

The linear operator $R^j_\varepsilon : \mathscr{D}'(X) \to \mathscr{D}'(X)$ has the following properties: for $u \in \mathscr{D}'(X)$

(8a) $\quad (R^j_\varepsilon u)\,|_{U_j} \in C^\infty(U_j)$

(8b) $\quad (R^j_\varepsilon u)\,|_{\bar{U}^c_j} = u\,|_{\bar{U}^c_j}$

(8c) $\quad \lim_{\varepsilon \downarrow 0} R^j_\varepsilon u = u$ weakly

(8d) $\quad$ if for some open $U \subset X$: $u\,|_U \in C^\infty(U)$ then also $(R^j_\varepsilon u)\,|_U \in C^\infty(U)$.

(8e) $\begin{cases} \text{for bounded sets } B \subset \mathscr{D}(X) \text{ and } B_1 \subset \mathscr{D}_1(X), \text{ the sets} \\ \{R_\varepsilon^j \psi \mid \psi \in B, \varepsilon \in I_0^+\} \text{ and } \{{}^t R_\varepsilon^j \varphi \mid \varphi \in B_1, \varepsilon \in I_0^+\} \\ \text{are bounded in } \mathscr{D}(X) \text{ or } \mathscr{D}_1(X), \text{ respectively.} \end{cases}$

In particular, expressed in the coordinates

$$x_j = h_j(x) \in \mathbb{R}^n, \quad x \in U_j \subset X$$

the operator R_ε^j is just regularization by convolution with ϱ^ε, i.e.

$$(h_j \circ A_y^j \mid_{U_j} \circ h_j^{-1})(x_j) = \tau_y(x_j) \overset{\text{def}}{=} x_j + y.$$

Then identifying H with $\mathbb{R}^n$ and applying

$$\int_H \varrho^\varepsilon(y)(\tau_y^* \varphi_j)(x_j)\,\mathrm{d}y = \int_{\mathbb{R}^n} (\tau_{-x_j}^* \varrho^\varepsilon)(y)\,\varphi_j(y)\,\mathrm{d}y$$

with $\varphi_j = \varphi \circ h_j^{-1}$ for a test function $\varphi \in \mathscr{D}(U_j)$, using (7) we can express (5) as

(9) $$(R_\varepsilon^j u) \mid_{U_j}(x) = \langle u_j, \tau_{-x_j}^* \check{\varrho}^\varepsilon \rangle, \; x_j = h_j(x), \; x \in U_j,$$

where

$$u_j = (h_j^{-1})^* u \mid_{U_j} \in \mathscr{D}'(\mathbb{R}^n)$$

and

$$\check{\varrho}(y) = \varrho(-y) \text{ for } y \in \mathbb{R}^n \text{ and } \varrho \in \mathscr{D}(\mathbb{R}^n).$$

Finally, for each $\psi \in C^\infty(X)$

(10) $$(A^j)^* \psi \in C^\infty(X \times H).$$

This can be verified just as in [10] by investigating derivatives with respect to $y \in H$ in $y = 0$ only, because of

$$A_{y_1+y_2}^j = A_{y_2}^j \circ A_{y_1}^j.$$

For each $\varepsilon \in I_0^+$ the global regularization operator is defined by

$$R_\varepsilon = \lim_{j \to \infty} R_\varepsilon^j R_\varepsilon^{j-1} \cdots R_\varepsilon^1,$$

which is a well-defined linear operator: $\mathscr{D}'(X) \to C^\infty(X)$.

In the following theorem $\mathscr{D}'$ and $\mathscr{G}$ are considered as sheaves of linear spaces on X, but not as C^∞-modules.

THEOREM 1. There exist injective sheaf morphisms

$$\sigma : \mathscr{D}' \to \mathscr{G}$$

on X with the following properties:

(i) for any section $u \in \mathscr{D}'(\Omega)$, $\Omega \subset X : \sigma(u) \sim u$.

(ii) σ is the identity on the subsheaves C^∞ of C^∞-functions in $\mathscr{D}'$ and $\mathscr{G}$.

REMARK 3. For a sheaf morphism σ satisfying the properties i) and ii) one cannot expect that it is compatible with the C^∞-module structure, nor that it commutes with differentiations in all coordinates.

For, by property ii) the C^∞-module structure in $\mathcal{G}$ is inherited from the algebra structure and then the impossibility of

$$\sigma(\psi u) = \psi\sigma(u), \ \psi \in C^\infty(X), \quad u \in \mathcal{D}'(X)$$

follows from the fact that the product on the right is associative, while the product on the left, due to the Schwartz example $x \cdot 1/x \cdot \delta$, is not. Furthermore, let Ξ be a derivation (vectorfield) on X which in one coordinate patch τ_μ: $V_\mu \to V'_\mu \subset \mathbb{R}^n$ can be expressed as $a \cdot \nabla_\mu$ for some $a \in \mathbb{R}^n$, i.e. as a homogeneous first order differential operator in V'_μ with constant coefficients. Then with the notation σ_μ: $\mathcal{D}'(V'_\mu) \to \mathcal{G}(V'_\mu)$ determined by $\sigma_\mu(u_\mu) = (\tau_\mu^{-1})^* \ \sigma(\tau_\mu^* u_\mu)$ for $u_\mu = (\tau_\mu^{-1})^* \ u\,|_{V_\mu}$, the assumption $\sigma_\mu(a \cdot \nabla_\mu u_\mu) = a \cdot \nabla_\mu \sigma_\mu(u_\mu)$ would lead to $\sigma(\Xi u)\,|_{V_\mu} = \Xi\sigma(u)\,|_{V_\mu}$, which expressed in another chart τ_ν again under the assumption that differentiation commutes, means that

$$\begin{aligned}\sigma_\nu(\psi_\nu \cdot \nabla_\nu u_\nu)\,|_{\tau_\nu(\upsilon_\mu \cap \upsilon_\nu)} &= \psi_\nu \cdot \nabla_\nu \sigma_\nu(u_\nu)\,|_{\tau_\nu(\upsilon_\mu \cap \upsilon_\nu)} \\ &= \psi_\nu \cdot \sigma_\nu(\nabla_\nu u_\nu)\,|_{\tau_\nu(\upsilon_\mu \cap \upsilon_\nu)}\end{aligned}$$

where the coefficients $\psi_\nu = (\psi_\nu^1, \ldots, \psi_\nu^n)$ in V'_ν are no longer constant. However we have just seen that in general this cannot be true. From (4) we only have that $\sigma(Pu)$ and $P\sigma(u)$ are equal in a weak sense, namely they are associated. However, equality is not necessary for applications: if, for example, one needs the product of a generalized function f with its derivative f', where f is somehow related to a certain distribution u, the obvious expression in $\mathcal{G}$ would be $\sigma(u)\sigma(u)'$, and not $\sigma(u)\sigma(u')$. So it does not matter whether $\sigma(u)'$ and $\sigma(u')$ are equal (as is the case for $X = \mathbb{R}^n$, cf. [3]), since there is a natural choice between them.

Before proving theorem 1 we will show in lemma 1 that the linear operator R_ε of [10], discussed above, provides a local operator from $\mathcal{D}'(X)$ into $\mathcal{G}(X)$. It turns out that in order to satisfy property ii) the weight function ϱ cannot have compact support. Therefore, an easier proof will be obtained if, instead of (6) the following collection $\{\varrho^\varepsilon \mid \varepsilon \in I_0^+\}$ of weight function is taken: let w be a C^∞-function in $H = \mathbb{R}^n$ equal to 1 for $|y| \le a$ and vanishing for $|y| \ge b$ for some $0 < a < b$ and let λ be a number with $0 < \lambda < 1$, then define

$$(11) \qquad \varrho^\varepsilon(y) = \frac{1}{\varepsilon^n}\varrho_\varepsilon\left(\frac{y}{\varepsilon}\right), \quad \varrho_\varepsilon(y) = w(\varepsilon^\lambda y)\varrho(y)$$

and let R_ε denote the associated linear map: $\mathcal{D}'(X) \to \mathcal{F}(X)$ determined by $u \mapsto (R_\varepsilon u)_{\varepsilon \in I_0^+}$. As a consequence, the sheaf morphism σ will depend on such a function w and on an exponent λ as well. However, it can be shown that another choice of w and λ would only change the induced sheaf morphism σ, to be constructed below, on the boundaries of the sets U_j, $j = 1, 2, \ldots$.

LEMMA 1. For every $x \in X$ we have $(R_\varepsilon u)|_x \in \mathcal{M}_x$ and if $x \notin \operatorname{supp}(u)$ then $(R_\varepsilon u)|_x \in \mathcal{N}_x$.

PROOF. For a C^∞ sheaf morphism P in X the restriction $P|_{\bar{U}_j}$ can be expressed as a differential operator in the local coordinates $\tilde{x}$ by the chart κ_j in a neighborhood of $\bar{B}$. Hence its coefficients pulled back by h^{-1} to $\mathbb{R}^n$ are bounded at infinity, while new factors containing the derivatives of h arise. So in the following estimate we have to stay away from the boundary of U_j.

Denote $R_\varepsilon^{(i)} = R_\varepsilon^i \ldots R_\varepsilon^1$ for $i = 1, 2, \ldots$ and $R_\varepsilon^{(i)} = \mathrm{Id}$ for $i = 0$. Then with $j = i + 1$ we will estimate

$$(12) \qquad |\nabla_j^\alpha((h_j^{-1})^*(R_\varepsilon^j R_\varepsilon^{(i)} u)|_{U_j})(x_j)|$$

where ∇_j^α is a partial differential operator with respect to the coordinates x_j in a compact set K' in $\mathbb{R}^n$, $i = 0, 1, 2, \ldots$. By (9) the expression (12) equals

$$(13) \qquad |\langle (h_j^{-1})^*(R_\varepsilon^{(i)} u)|_{U_j}, \nabla_j^\alpha \tau_{-x_j}^* \check{\varrho}^\varepsilon \rangle|.$$

It follows from (8.e) that the set

$$\{(h_j^{-1})^*(R_\varepsilon^{(i)} u)|_{U_j} \mid \varepsilon \in I_0^+\}$$

is weakly bounded in $\mathcal{D}'(\mathbb{R}^n)$, so, by the Banach–Steinhaus theorem, it is strongly bounded. Furthermore, since

$$(14) \qquad y \in \operatorname{supp}(\varrho^\varepsilon) \Rightarrow |y| \le b\varepsilon^{1-\lambda},$$

using L^1-norms of test functions of the variable $y \in \mathbb{R}^n$ we can majorate (13) by

$$C \sup_{|\beta| \le k} \|(\nabla^{\alpha+\beta} \varrho^\varepsilon)(x_j - y)\|_1 = 0(\varepsilon^{-|\alpha|-k}, \varepsilon \downarrow 0)$$

for some constants C and k, uniformly for $x_j \in K' \Subset \mathbb{R}^n$, thus for $x \in K = h_j^{-1}(K') \Subset U_j$ for any compact set K in U_j. This shows that $(R_\varepsilon^{(j)} u)|_{U_j} \in \mathcal{M}(U_j)$ for $j = 1, 2, \ldots$.

In view of (8.b), with the notation $U^{(i)} = U_1 \cup \ldots \cup U_i \subset X$, we still have to consider points $x_0 \in U^{(i)} \cap \partial U_j$ for $i = j - 1 = 1, 2, \ldots$. Assume that $(R_\varepsilon^{(i)} u)|_v \in \mathcal{M}(V)$ for a neighborhood $V \Subset U^{(i)}$ of x_0, which we have shown to be true for $i = 1$. With V sufficiently small we can express a moderate sequence $f \in \mathcal{M}(V)$ in the coordinates $\tilde{x} = \kappa_j(x)$, which live in a neighborhood of $\bar{B}$ in $\mathbb{R}^n$, as $\tilde{f} = (\kappa_j^{-1})^* f$. For a partial differential operator $\tilde{\nabla}^\alpha$ in $\tilde{x}$ we have to estimate $\tilde{\nabla}^\alpha R_\varepsilon^j f_{W,\varepsilon}(\kappa_j^{-1}(\tilde{x}))$ for $\varepsilon \in I_0^+$ and for $\tilde{x} \in W'$, where $W' = \kappa_j(W)$ for a neighborhood $W \Subset V$ of x_0. By (5) we have

$$(15) \qquad \begin{cases} \tilde{\nabla}^\alpha R_\varepsilon^j f_{W,\varepsilon}(\kappa_j^{-1}(\tilde{x})) = \int \varrho^\varepsilon(y) \tilde{\nabla}^\alpha (A_{-y}^j)^* f_{W,\varepsilon}(\kappa_j^{-1}(\tilde{x}))\, dy \\ \qquad\qquad = \int \varrho^\varepsilon(y) \tilde{\nabla}^\alpha (\kappa_j \circ A_{-y}^j|_{\bar{U}_j} \circ \kappa_j^{-1})^* \tilde{f}_{W,\varepsilon}(\tilde{x})\, dy. \end{cases}$$

For any $y \in \mathbb{R}^n$ denote $\tilde{A}_{-y} = \kappa_j \circ A_{-y}^j|_{\bar{U}_j} \circ \kappa_j^{-1} : \bar{B} \to \bar{B}$. In a neighborhood of $\bar{B}$ outside B the transformation $\tilde{A}_{-y}$ is the identity and for $\tilde{x} \in B$ it is given by

$$\tilde{A}_{-y}(\tilde{x}) = h^{-1}(h(\tilde{x}) - y).$$

We now express $\tilde{\nabla}^\alpha \tilde{A}^*_{-y}\tilde{\psi}$ for a function $\tilde{\psi}\in C^\infty(\bar{B})$ in the partial derivatives of $\tilde{\psi}$:

$$(16)\qquad (\tilde{\nabla}^\alpha \tilde{A}^*_{-y}\tilde{\psi})(\tilde{x}) = \sum_{\beta\leq\alpha} g^\alpha_\beta(\tilde{x},y)(\tilde{A}^*_{-y}\tilde{\nabla}^\beta\tilde{\psi})(\tilde{x})$$

where by (10) the coefficients g^α_β are C^∞-functions in a neighborhood of $\bar{B}\times\mathbb{R}^n\subset\mathbb{R}^{2n}$, in $B\times\mathbb{R}^n$ consisting of products of factors $\tilde{\nabla}^\gamma h^{-1}(h(\tilde{x})-y)$ and outside $B\times\mathbb{R}^n$ equal to 1 if $\beta=\alpha$ and vanishing for other β. Because of (14) integration of a function g^α_β with the weight function ϱ^ε yields a function uniformly bounded in a neighborhood of $\bar{B}$ for $\varepsilon\in I_0^+$.

Finally, there is a compact set $W_0\Subset V$ such that for all $y\in\mathbb{R}^n$ with $|y|$ sufficiently small we have $A^j_{-y}(W)\subset W_0$, for example for $y\in\bigcup_{\varepsilon\in I_0^+}\operatorname{supp}(\varrho^\varepsilon)$. Combination of (15) and (16) gives the following estimate for (15)

$$\sup\{\sum_{\beta\leq\alpha} |\tilde{\nabla}^\beta\tilde{f}_{W_0,\varepsilon}(\tilde{z})| \mid \tilde{z}\in\kappa_j(W_0)\}$$

which is $0(\varepsilon^{-p},\varepsilon\downarrow 0)$ for some p, because f is a moderate sequence in V.

The second statement of the lemma is more trivial. In fact, since the covering by the sets U_j, $j=1,2,\dots$, is locally finite and since by (14) $\varrho^\varepsilon(y)=0$ for $|y|\geq b\varepsilon^{1-\lambda}$ we even have $(R_\varepsilon u)|_x=0$ for $x\notin\operatorname{supp}(u)$. □

PROOF OF THEOREM 1. Since the sheaves $\mathscr{D}'$ and $\mathscr{G}$ are soft their sections on X with compact supports form flabby cosheaves $\mathscr{D}'_c$ and $\mathscr{G}_c$ and in lemma 1 we have given a cosheaf morphism (= local operator) between them. This can be turned canonically into a sheaf morphism: $\mathscr{D}'\to\mathscr{G}$, cf. [2], as follows: for any $u\in\mathscr{D}'(\Omega)$, $\Omega\subset X$, decompose $u=\sum u_k$ as a locally finite sum of distributions with compact support in Ω, and define

$$\sigma_\Omega(u) = \sum_k [(R_\varepsilon u_k)_{\varepsilon\in I_0^+}]$$

where $[f]$ denotes the class in $\mathscr{G}(\Omega)$ of a moderate sequence f on Ω. This yields a locally finite sum in $\mathscr{G}(\Omega)$, hence a section in $\mathscr{G}(\Omega)$, which is independent of the way u is decomposed as a locally finite sum, cf. also [7, lemma 2.3] or [8]. Since the operator: $\mathscr{D}'_c\to\mathscr{M}$ given in lemma 1 is linear, the local maps σ_Ω: $\mathscr{D}'(\Omega)\to\mathscr{G}(\Omega)$ determine a sheaf morphism σ: $\mathscr{D}'\to\mathscr{G}$ on X.

Property i) follows from what has been shown in [10]: let $\varphi\in\mathscr{D}_1(\Omega)$ with support contained in a relatively compact set Ω' and for $u\in\mathscr{D}'(\Omega)$ let u' be a distribution with compact support in X such that $u'|_{\Omega'}=u|_{\Omega'}$. Then

$$\int(R_\varepsilon u)\varphi = \int(R_\varepsilon u')\varphi = \int(R_\varepsilon^{(m)}u')\varphi = \langle u',{}^tR_\varepsilon^{(m)}\varphi\rangle$$

with m so large that $\bar{U}_j\cap\operatorname{supp}(u')=\phi$ for $j\geq m$. Just as (8.c) and (8.e) have been derived in [10] it follows that if $\varepsilon\downarrow 0$

$${}^tR_\varepsilon^{(m)}\varphi\to\varphi \text{ in } \mathscr{D}_1(X), \text{ hence } \int(R_\varepsilon u)\varphi\to\langle u',\varphi\rangle = \langle u,\varphi\rangle.$$

If for some open set $V\subset X$ and $u\in\mathscr{D}'_c(X)$ we have $((R_\varepsilon u)|_V)_{\varepsilon\in I_0^+}\in\mathscr{N}(V)$, then

$$\langle u,\varphi\rangle = \lim_{\varepsilon\downarrow 0}\int(R_\varepsilon u)\varphi = 0, \quad \varphi\in\mathscr{D}_1(V)$$

follows from the estimates of a null sequence (with $q=1$) on the support of φ. Hence $u|_V=0$. This shows that the local map: $\mathscr{D}'_c(X)\to\mathscr{G}_c(X)$ derived from R_ε does not shrink supports, so that the induced sheaf morphism σ is injective.

Finally, we will prove property ii) of the theorem. Of course, this property for the maps σ_Ω would have implied the second statement in lemma 1, but that property is more fundamental in the sense that it is valid under the conditions we have shown it, while for the more general property ii) we have to impose further conditions on the weight function ϱ. Namely, besides $\int \varrho(y)\,\mathrm{d}y=1$ we also require that

$$\int y^\alpha \varrho(y)\,\mathrm{d}y=0$$

for all multi-indices α with $|\alpha|\geq 1$. Such functions ϱ exist in $S(\mathbb{R}^n)$, but they cannot have compact support. From the definition of $S(\mathbb{R}^n)$ and from $\varrho_\varepsilon=\varrho$ for $|y|\leq a\varepsilon^{-\lambda}$ it follows that

$$\text{(17)}\qquad \int y^\alpha \varrho_\varepsilon(y)\,\mathrm{d}y=0(\varepsilon^q,\varepsilon\downarrow 0) \text{ and } \int \varrho_\varepsilon(y)\,\mathrm{d}y=1+0(\varepsilon^q,\varepsilon\downarrow 0)$$

for any $q\in\mathbb{N}$, $|\alpha|\geq 1$.

Let $\psi\in C^\infty(\Omega)$ and let m be so large that for a certain point $x_0\in\Omega$ $x_0\notin\bar{U}_j$ for $j\geq m$. We may assume that ψ has compact support and we have to show that $(R_\varepsilon^{(m)}\psi-\psi)|_{x_0}\in\mathscr{N}_{x_0}$. Since

$$R_\varepsilon^{(m)}\psi-\psi=R_\varepsilon^m(R_\varepsilon^{(m-1)}\psi)-R_\varepsilon^{(m-1)}(\psi)+R_\varepsilon^{m-1}(R_\varepsilon^{(m-2)}\psi)-\ldots+R_\varepsilon^1\psi-\psi,$$

in view of (8.e) it suffices to show that $(R_\varepsilon^j\chi_\varepsilon-\chi_\varepsilon)|_{x_0}\in\mathscr{N}_{x_0}$ for any bounded family $\{\chi_\varepsilon\mid\varepsilon\in I_0^+\}$ of functions in $\mathscr{D}(X)$, $j=1,\ldots,m$.

In the coordinates $\tilde{x}=\kappa_j(x)\in B$ for $x\in U_j$, we have

$$\tilde{\nabla}^\alpha(R_\varepsilon^j\chi_\varepsilon)(\kappa_j^{-1}(\tilde{x}))=\tilde{\nabla}^\alpha\int\tilde{\chi}_\varepsilon(h^{-1}(h(\tilde{x})-\varepsilon y)\varrho_\varepsilon(y)\,\mathrm{d}y$$

where $\tilde{\chi}_\varepsilon=(\kappa_j^{-1})^*\chi_\varepsilon$ denotes χ_ε in the coordinates $\tilde{x}$. For $\varepsilon y=0$ this equals $\tilde{\nabla}^\alpha\tilde{\chi}_\varepsilon(\tilde{x})\int\varrho_\varepsilon(y)\,\mathrm{d}y$. By (10) we can expand the function $\tilde{\nabla}^\alpha(\tilde{\chi}_\varepsilon(h^{-1}(h(\tilde{x})-\varepsilon y))-\tilde{\nabla}^\alpha\tilde{\chi}_\varepsilon(\tilde{x})$ with respect to εy in a Taylor series and as in (16) we then have to expand

$$\sum_{\beta\leq\alpha} g_\beta^\alpha(\tilde{x},\varepsilon y)\tilde{\nabla}^\beta\tilde{\chi}_\varepsilon(h^{-1}(h(\tilde{x})-\varepsilon y))-\tilde{\nabla}^\alpha\tilde{\chi}_\varepsilon(\tilde{x}).$$

Because of (17) after integrating the first term in this formula with ϱ_ε and the second term with ϱ we find that all terms in the expansion up to the q^{th} order are $0(\varepsilon^q,\varepsilon\downarrow 0)$ uniformly for $\tilde{x}\in B$. Hence the remainder term is left, which can be estimated by

$$C_q\sup\{\nabla_{y'}^\beta\tilde{\nabla}^\alpha\tilde{\chi}_\eta(h^{-1}(h(\tilde{z})-y'))\mid|\beta|=q+1,\tilde{z}\in B,|y'|\leq b\varepsilon^{1-\lambda},\eta\in I_0^+\}$$
$$\cdot\varepsilon^{q+1}\int|y|^{q+1}|\varrho_\varepsilon(y)|\,\mathrm{d}y=C_{\alpha,q}\varepsilon^{q+1}$$

uniformly for $x=\kappa_j^{-1}(\tilde{x})\in U_j$. Since q can be as large as desired and since $R_\varepsilon^j\chi_\varepsilon(x)-\chi_\varepsilon(x)=0$ for all $x\notin U_j$ by (8.b) and (8.c), $\varepsilon\in I_0^+$, we have $(R_\varepsilon^j\chi_\varepsilon-\chi_\varepsilon)|_{x_0}\in\mathscr{N}_{x_0}$ for any $x_0\in\Omega\subset X$. □

REMARK 4. For applications it is convenient to have some freedom in the construction of a sheaf morphism σ: $\mathscr{D}' \to \mathscr{G}$. In our construction the weight functions ϱ^ε in the collection (11) have compact support, but this is not necessary. For example, in [11] another possibility will be given, which for $n=1$ yields

$$\varrho^\varepsilon(x) = \frac{e^{-\gamma(x^2/\varepsilon)} \sin x/\varepsilon}{\pi x}$$

for some arbitrary number $\gamma > 0$, and in [3] an example has been given which is needed in quantum field theory.

REMARK 5. In the construction, similar to the one given here, of embeddings of $\mathscr{D}'$ into the sheaf of the original (non-simplified) Colombeau algebras on a C^∞-manifold (cf. remark 2) the role of the weight function ϱ is taken by the test functions $\varphi \in \mathscr{A}$. Hence such an embedding does no longer depend on the function ϱ, but the dependence on the covering $\{U_j\}$ with the sequence of associated charts (h_j) remains. So, also in that case, there is no canonical embedding. However, as is remarked in [1] there is no need for a canonical embedding, since in applications it matters to find a suitable embedding adapted to the problem considered.

3. RESTRICTIONS TO A SUBMANIFOLD

Just as functions, under a C^∞-map $\Phi: Y \to X$ between two C^∞-manifolds ultrafunctions on X can be pulled back to ultrafunctions on Y. In this respect they behave more like functions than distributions. Indeed, the pull back $\Phi^* f$ of a sequence $f \in \mathscr{F}(X)$ is a sequence in $\mathscr{F}(Y)$:

LEMMA 2. If $f \in \mathscr{M}(X)$ or $\mathscr{N}(X)$ then also $\Phi^* f \in \mathscr{M}(Y)$ or $\mathscr{N}(Y)$, respectively.

PROOF. Let there be local charts $\tau_\nu: V_\nu \subset Y \to V'_\nu \subset \mathbb{R}^m$ and $\kappa_\mu: U_\mu \subset X \to U'_\mu \subset \mathbb{R}^n$ such that Φ is expressed as

$$\Phi_{\mu\nu} = \kappa_\mu \circ \Phi \,|_{V_\nu} \circ \tau_\nu^{-1} : V'_\nu \to U'_\mu.$$

For a compact set $K \Subset V'_\nu$ denote $S = \kappa_\mu^{-1} \circ \Phi_{\mu\nu}(K) \Subset U_\mu$; then for a partial differential operator ∇^α in V'_ν, for $\varepsilon \in I_0^+$ one can express $\nabla^\alpha((f_{S,\varepsilon} \circ \kappa_\mu^{-1}) \circ \Phi_{\mu\nu})$ in sums of products of partial derivatives of the function $f_{S,\varepsilon} \circ \kappa_\mu^{-1}$ in U'_μ and partial derivatives of $\Phi_{\mu\nu}$.

Hence if f is a moderate or a null sequence in X, then $\Phi^* f$ is a moderate or a null sequence in Y, respectively. □

COROLLARY. If Y is a regular submanifold of X, then the restriction $F\,|_Y$ to Y of an ultrafunction F in X is defined by $i^* F$, where $i: Y \to X$ is the injection of Y into X.

This corollary is in fact a generalization of the pointvalues of F, noted in remark 1, and it should be handled with the same care as the following examples illustrate.

EXAMPLE 1. It is in general not true that, if $F \approx G$, then also $F|_Y \approx G|_Y$. For take $X = \mathbb{R}^2$, $Y = \mathbb{R} = \{(x_1, x_2) \in X \mid x_2 = 0\}$, let for $\varepsilon \in I_0^+$

$$f_\varepsilon(x_1, x_2) = -x_2 \frac{1}{\varepsilon^3} \varrho'\left(\frac{x_2}{\varepsilon}\right) \varrho\left(\frac{x_1}{\varepsilon}\right)$$

for a function $\varrho \in \mathscr{D}(\mathbb{R})$ with $\varrho(0) = 1$ and $\int \varrho = 1$, and let

$$g_\varepsilon(x_1, x_2) = \frac{1}{\varepsilon^2} \varrho\left(\frac{x_1}{\varepsilon}\right) \varrho\left(\frac{x_2}{\varepsilon}\right).$$

Both F and G, determined by the sequences (f_ε) and (g_ε), respectively, are weakly equal to δ, so they are associated. Nevertheless $F|_Y = 0$ while $G|_Y$ is determined by the sequence $(1/\varepsilon^2 \varrho(x_1/\varepsilon))$ on Y which is not weakly equal to any distribution.

EXAMPLE 2. If $F \sim u$ and $F|_Y \sim v$ for some distributions u in X and v in Y, then it is in general not true that $u|_Y$ exists and equals v. Namely, the ultrafunction F of example 1 satisfies $F \sim \delta$, $F|_Y = 0$, but $\delta|_Y$ does not exist.

EXAMPLE 3. Also it need not be true that "$F \sim u$ and $u|_Y$ exists" implies $F|_Y \sim u|_Y$. With

$$f_\varepsilon(x_1, x_2) = \frac{1}{\varepsilon\sqrt{\varepsilon}} \varrho\left(\frac{x_1}{\varepsilon}\right) \varrho\left(\frac{x_2}{\varepsilon}\right)$$

we have $F \sim 0$ but $F|_Y$, determined by

$$\left(\frac{1}{\varepsilon\sqrt{\varepsilon}} \varrho\left(\frac{x_1}{\varepsilon}\right)\right)_{\varepsilon \in I_0^+},$$

is not weakly equal to any distribution.

We will now show that the conclusion of example 3 is true if $F = \sigma(u)$ for any of the sheaf morphisms σ constructed in the proof of theorem 1 of section 2. For a distribution $u \in \mathscr{D}'(X)$ the restriction $u|_Y \in \mathscr{D}'(Y)$ to a regular submanifold Y is defined in [5, § 8.2], provided that u satisfies the condition: the wave front set $WF(u)$ of u and the conormal bundle $N(Y)$ of Y are disjoint (cf. also lemma 3 below).

THEOREM 2. Let σ be a sheaf morphism: $\mathscr{D}' \to \mathscr{G}$ on X as constructed in the proof of theorem 1, and let $u \in \mathscr{D}'(X)$ be such that

$$WF(u) \cap N(Y) = \phi \tag{18}$$

for a regular submanifold $Y \subset X$. Then

$$\sigma(u)|_Y \sim u|_Y.$$

Before proving this theorem, we shall give a property of a distribution u satisfying (18) and a characterization of $u|_Y$.

A local chart $\tau_\mu : U_\mu \subset X \to U'_\mu \subset \mathbb{R}^n$ induces a trivialization over U_μ of the line bundle whose sections are densities. This will be expressed as follows:

$$(\tau_\mu^{-1})^* \varphi \,|_{U_\mu} = \varphi_\mu |\mathrm{d}x_\mu|, \quad \varphi \in C_1^\infty(X),$$

where $|\mathrm{d}x_\mu|$ denotes the basis vector in the fibre if $x_\mu = \tau_\mu(x) \in U'_\mu$ for $x \in U_\mu$ and where now φ_μ is a C^∞-function in U'_μ.

LEMMA 3. Let $u \in \mathscr{D}'(X)$ be a distribution satisfying (18) and let $\tau_\mu : V_\mu \subset X \to V'_\mu \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ be any chart such that

$$Y_\mu \overset{\mathrm{def}}{=} \tau_\mu(Y \cap V_\mu) = \{(x_1, x_2) \in V'_\mu \mid x_2 = 0 \in \mathbb{R}^{n-m}\}.$$

Then for any compact set in Y_μ there is an open neighborhood W_μ in V'_μ such that $u_\mu \overset{\mathrm{def}}{=} (\tau_\mu^{-1})^* u \,|_{V_\mu} \in \mathscr{D}'(V'_\mu)$ restricted to W_μ can be represented as

(19) $$u_\mu \,|_{W_\mu} = P_1 g,$$

for a continuous function g in W_μ and for a differential operator P_1 in W_μ with differentiations only with respect to x_1, and such that for any $\varphi_\mu \in \mathscr{D}(Y_\mu \cap W_\mu)$ the function

$$x_2 \mapsto \int g(x_1, x_2)\, {}^t P_1 \varphi_\mu(x_1)\, \mathrm{d}x_1$$

is C^∞ in a neighborhood of $0 \in \mathbb{R}^{n-m}$.

Furthermore, for any density $\chi |\mathrm{d}x_2| \in \mathscr{D}_1(\mathbb{R}^{n-m})$ with $\int \chi(x_2)\, \mathrm{d}x_2 = 1$ we have

(20) $$\lim_{\delta \downarrow 0} \langle u_\mu, \varphi_\mu \chi_\delta |\mathrm{d}x_1\, \mathrm{d}x_2| \rangle = \langle u \,|_Y, \varphi \rangle, \quad \varphi \in \mathscr{D}_1(Y \cap V_\mu),$$

where

$$\chi_\delta(x_2) = \frac{1}{\delta^{n-m}} \chi\left(\frac{x_2}{\delta}\right) \quad \text{and} \quad \varphi_\mu |\mathrm{d}x_1| = (\tau_\mu^{-1})^* \varphi.$$

PROOF. For a compact set in Y_μ with neighborhood W_μ in V'_μ let ψ be a function in $C_0^\infty(V'_\mu)$ which is equal to 1 in W_μ. The Fourier transform of ψu_μ satisfies

(21) $$|\widehat{\psi u_\mu}(\xi)| \leq C(1 + \xi^2)^M$$

for some M, and, by the definition of wave front set, if W_μ is sufficiently small

(22) $$|\widehat{\psi u_\mu}(\xi)| \leq C_N (1 + \xi^2)^{-N}$$

for any N and ξ in a conic neighborhood of the conormals $(0, \pm 1) \in \mathbb{R}_m \times \mathbb{R}_{n-m}$ to Y_μ in the points of Y_μ. Therefore, the function

$$g(x) = \frac{1}{(2\pi)^n} \int \frac{\widehat{\psi u_\mu}(\xi) e^{i\xi \cdot x}}{(1 + \xi_1^2)^{M+n+1}}\, \mathrm{d}\xi$$

is continuous, and, with $P_1 = (1 - \Delta_1)^{M+n+1}$ where Δ_1 is the Laplace operator in the variables $x_1 \in \mathbb{R}^m$, it satisfies

$$P_1 g \,|_{W_\mu} = (\psi u_\mu) \,|_{W_\mu} = u_\mu \,|_{W_\mu}.$$

Moreover, denoting for any $\varphi_\mu \in \mathscr{D}(Y_\mu \cap W_\mu)$ by Φ the function in $S(\mathbb{R}_m)$

with $\hat{\Phi} = \varphi_\mu$, we have

$$\int g(x_1, x_2)\,{}^tP_1\,\varphi_\mu(x_1)\,\mathrm{d}x_1 = \frac{1}{(2\pi)^{n-m}} \int \widehat{\psi u}(\xi)\, e^{i\xi_2 \cdot x_2}\, \Phi(\xi_1)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2,$$

where according to (21) and (22) the right-hand side is a convergent integral, also after differentiation with respect to x_2.

To prove the second statement of the lemma, note first that now the limit in the left-hand side of (20) obviously exists. Second, if $u \in C^\infty(V_\mu)$ this clearly coincides with the restriction of u to $Y \cap V_\mu$. For a general distribution u satisfying (18) the same conclusion can be derived from a continuity argument, cf. [5, 8.2.2 and 8.2.4], which can be applied to formula (20), too. Instead of proving this, we will just show that (20) can serve as a definition of $u\,|_Y$, because the method (namely formula (30) below) is needed later on anyway.

Let A be a coordinate transformation

$$A : V \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \to V' \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$$

with

(23) $\qquad A_2(x_1, 0) = 0,$

where for $x = (x_1, x_2) \in V$ we denote

$$A(x) = (A_1(x), A_2(x)) = (x_1', x_2') = x' \in V',$$

and let $\chi'|\mathrm{d}x_2'|$ be an arbitrary C^∞-density with compact support and with $\int \chi'(x_2')\,\mathrm{d}x_2' = 1$.

Bearing in mind that u_μ can be represented as (19) for $W_\mu \subset V$ we have to show that for $\varphi \in C_0^\infty(V\,|_{x_2=0})$

(24) $\qquad \lim\limits_{\delta \downarrow 0} \langle A^{-1}{}^*(P_1 g), \varphi' \chi_\delta' |\mathrm{d}x'| \rangle = \int g(x_1, 0)\,{}^tP_1(x_1, 0)\,\varphi(x_1)\,\mathrm{d}x_1$

with $\varphi' \in C_0^\infty(V'\,|_{x_2'=0})$ satisfying

(25) $\qquad |\det D_1 A_1(x_1, 0)|\,\varphi'(A_1(x_1, 0)) = \varphi(x_1),$

where $D_1 A_1$ denotes the derivative of A_1 with respect to the variables x_1 and where the meaning of $P_1(x_1, 0)$ follows from the notation

$$P_1(x) = \sum_\alpha a_\alpha(x)\,\nabla_1^\alpha$$

for C^∞-coefficients a_α with m-dimensional multi indices α.

Because of (23)

(26) $\qquad DA(x_1, 0) = \left(\begin{array}{c:cc} \bullet & \bullet & \bullet \\ \hdashline & \bullet & \bullet \\ 0 & & \\ & \bullet & \bullet \end{array}\right),$

both $\det D_1 A_1(x_1, 0)$ and $\det D_2 A_2(x_1, 0)$ do not vanish.

Therefore, according to the implicit function theorem from the equation $A_2(x_1, x_2) = x_2'$ the variables x_2 can be solved for $|x_2'|$ sufficiently small and any $x_1 \in V\,|_{x_2=0}$:

$$x_2 = B(x_1, x_2') \text{ with } B(x_1, 0) = 0$$

satisfying

$$D_2' B(x_1, x_2') = D_2 A_2(x_1, B(x_1, x_2'))^{-1}. \tag{27}$$

Finally, denote by $\tilde{B}$ the transformation $\tilde{B}(x_1, x_2') = (x_1, B(x_1, x_2'))$, which is non-singular for small $|x_2'|$. Of course, the inverse $\tilde{B}^{-1}$ is the transformation $\tilde{B}^{-1}(x_1, x_2) = (x_1, A_2(x_1, x_2))$.

The reason for introducing the transformation $\tilde{B}$ is that we want to express the left hand side of (24) before taking the limit for $\delta \downarrow 0$ as an integral of a continuous function in which no derivatives of g with respect to x_1 and no derivatives of χ_δ' occur.

We now perform the following operations on the left hand side of (24)

$$\begin{aligned}\langle A^{-1*}(P_1 g), \varphi' \chi_\delta' |\mathrm{d}x'|\rangle &= \langle P_1 g, A^*(\varphi' \chi_\delta' |\mathrm{d}x'|)\rangle \\ &= \langle \tilde{B}^*(P_1 g), \tilde{B}^* A^*(\varphi' \chi_\delta' |\mathrm{d}x'|)\rangle.\end{aligned}$$

Let us express $\tilde{B}^*(P_1 g)$ in the coordinates (x_1, x_2'). First, for ∂_1 denoting a partial derivative with respect to one of the variables x_1 we have symbolically

$$\left\{\begin{aligned}(\tilde{B}^*(\partial_1 g))(x_1, x_2') &= \partial_1 \big|_{x_2 = B(x_1, x_2')} g(x_1, x_2) \\ &= \partial_1 g(x_1, B(x_1, x_2')) - \partial_1 B(x_1, x_2') \cdot \nabla_2 \big|_{x_2 = B(x_1, x_2')} g(x_1, x_2).\end{aligned}\right. \tag{28}$$

In the same way the distribution $\tilde{B}^*(P_1 g)$ can be expressed as follows

$$\left\{\begin{aligned}(\tilde{B}^*(P_1 g))(x_1, x_2') = \sum_\alpha a_\alpha(x_1, B(x_1, x_2')) \{\nabla_1^\alpha g(x_1, B(x_1, x_2')) \\ + \sum_\beta b_\beta^\alpha(x_1, x_2') \nabla_2^\beta \big|_{x_2 = B(x_1, x_2')} g(x_1, x_2)\},\end{aligned}\right. \tag{29}$$

or in short

$$\tilde{B}^*(P_1 g) = P_B(\tilde{B}^* g) + \tilde{B}^*(Q_2 g)$$

where $P_B \overset{\text{def}}{=} \sum_\alpha (\tilde{B}^* a_\alpha) \nabla_1^\alpha$ and where Q_2 is a differential operator with differentiations with respect to x_2 only, whose coefficients contain factors $(\tilde{B}^{-1})^* b_\beta^\alpha$ for certain C^∞-functions b_β^α consisting of sums of products of factors $\nabla_1^\gamma B(x_1, x_2')$ which all vanish for $x_2' = 0$ (here β is an $n-m$ dimensional multi index).

Continuing we get

$$
(30)\quad\left\{\begin{aligned}
&\langle A^{-1}*(P_1 g), \varphi'\chi'_\delta |\mathrm{d}x'|\rangle\\
&=\langle \tilde{B}^*g, {}^tP_B(\tilde{B}^*A^*(\varphi'\chi'_\delta|\mathrm{d}x'|))\rangle + \langle Q_2 g, A^*(\varphi'\chi'_\delta|\mathrm{d}x'|)\rangle\\
&=\int\int \chi'(x'_2)\, g(x_1, B(x_1, \delta x'_2)) \sum_\alpha (-\nabla_1)^\alpha(\tilde{B}^* a_\alpha A^*\varphi'|\mathrm{d}x'|)(x_1,\delta x'_2)\\
&+\sum_\alpha\sum_\beta \int\Big\{\nabla_2^\beta\,|_{x_2=\delta x''_2} \int a_\alpha(x_1,\delta x''_2)\, b^\alpha_\beta(x_1, A_2(x_1,\delta x''_2))\\
&\times g(x_1,x_2)\varphi'(A_1(x_1,\delta x''_2))\chi'\Big(\frac{A_2(x_1,\delta x''_2)}{\delta}\Big)\,|\det DA(x_1,\delta x''_2)|\mathrm{d}x_1\Big\}\,\mathrm{d}x''_2.
\end{aligned}\right.
$$

Here the integrants are continuous, compactly supported, functions whose supports and majorants remain uniformly bounded for $\delta\in I_0^+$, because $(A_2(x_1,\delta x''_2))/\delta = x''_2\cdot \nabla_2 A_2(x_1,0)+o(\delta)$. Hence by Lebesgue's theorem, if $\delta\downarrow 0$ in the right hand side of (30) the second term vanishes since all $b^\alpha_\beta(x_1,0)=0$. Concerning the first term, as in (28), in a derivative ∂_1 to one of the x_1's of a function of (x_1,x'_2) of the form $\psi(x_1,x_2)\,|_{x_2=B(x_1,\delta x'_2)}$ terms with derivatives to x_2 are accompanied by $\partial_1 B(x_1,\delta x'_2)$ which vanishes for $\delta=0$. Similarly, as in (29) also higher order partial derivatives $\nabla_1{}^\alpha$ to x_1 of such a function only yield an, in general, nonvanishing contribution due to the term $\nabla_1{}^\alpha\psi(x_1,0)$ for $\delta\downarrow 0$. Therefore,

$$
\lim_{\delta\downarrow 0}\langle A^{-1}*(P_1 g),\varphi'\chi'_\delta|\mathrm{d}x'|\rangle = \int g(x_1,0)\,{}^tP_1(x_1,0)\{|\det DA(x_1,0)\times\det D'_2B(x_1,0)|\varphi'(A_1(x_1,0))\}\,\mathrm{d}x_1.
$$

According to (26) and (27) $\det DA(x_1,0)\cdot\det D'_2B(x_1,0)=\det D_1A_1(x_1,0)$ and by (25) formula (24) follows. □

PROOF OF THEOREM 2. Let τ_μ be a chart as in lemma 3, with a relatively compact coordinate patch V_μ, let $\varphi\in\mathscr{D}_1(Y\cap V_\mu)$ with $\varphi_\mu|\mathrm{d}x'|=(\tau_\mu^{-1})^*\varphi$ and let for $\varepsilon\in I_0^+$ and $\delta\in I_0^+$

$$
H_u(\varepsilon,\delta)=\int((\tau_\mu^{-1})^* f_{K,\varepsilon})(x'_1,x'_2)\,\varphi_\mu(x'_1)\,\chi_\delta(x'_2)\,\mathrm{d}x'_1\,\mathrm{d}x'_2
$$

for some moderate sequence $(f_{K,\varepsilon})_{\varepsilon\in I_0^+}$ in the class of $\sigma(u)\,|_K$, where K is a compact set in V_μ containing all sets $\tau_\mu^{-1}(\mathrm{supp}(\varphi_\mu)\times\mathrm{supp}(\chi_\delta))$ in its interior for $0<\delta<\delta_0$ where δ_0 can still be chosen arbitrarily small. If the function $H_u(\varepsilon,\delta)$ is continuously extendible to $\overline{I_0^+}\times\overline{I_0^+}$, the limits for $\varepsilon\downarrow 0$ and $\delta\downarrow 0$ can be interchanged. Since on the one hand by property i) of theorem 1 and by (20) we have

$$
\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} H_u(\varepsilon,\delta)=\lim_{\delta\downarrow 0}\langle u_\mu,\varphi_\mu\chi_\delta|\mathrm{d}x'|\rangle=\langle u\,|_Y,\varphi\rangle
$$

and on the other hand

$$
(\lim_{\delta\downarrow 0} H_u(\varepsilon,\delta))_{\varepsilon\in I_0^+}=(\int f_{K,\varepsilon}\,|_Y\,\varphi)_{\varepsilon\in I_0^+}=\int\sigma(u)\,|_Y\,\varphi,
$$

where the last equality is in the sense of generalized numbers, cf. remark 1, we would then get the desired result.

The transformation $A^1_{\varepsilon y_1}\dots A^k_{\varepsilon y_k}: X \to X$, discussed in section 2, with k so large that $A^1_{\varepsilon y_1}\dots A^l_{\varepsilon y_l}|_{V_\mu} = A^1_{\varepsilon y_1}\dots A^k_{\varepsilon y_k}|_{V_\mu}$ for $l \geq k$, $y_j \in \mathbb{R}^n$ and $\varepsilon \in I_0^+$, transforms K into V_μ if $|\varepsilon y_j|$ is sufficiently small for all $j = 1, \dots, k$. For such εy_j, $j = 1, \dots, k$, denote

$$A_{\varepsilon y} = \tau_\mu \circ (A^1_{\varepsilon y_1}\dots A^k_{\varepsilon y_k})|_K \circ \tau_\mu^{-1} : \tau_\mu(K) \to S_{\varepsilon y} \stackrel{\text{def}}{=} A_{\varepsilon y}(\tau_\mu(K)) \subset V'_\mu,$$

where

$$y = (y_1, \dots, y_k) \in \mathbb{R}^{nk}.$$

Represent u_μ as in (19), take δ_0 so small that $\tau_\mu(K) \subset W_\mu$ and choose the representant of $\sigma(u)|_K$ obtained in the proof of theorem 1 from the collection weight functions (11).

Then for $\varepsilon \in I_0^+$, $\delta \in I_0^+$

$$H_u(\varepsilon, \delta)$$
$$= \int w(\varepsilon^\lambda y_1)\varrho(y_1)\dots w(\varepsilon^\lambda y_k)\varrho(y_k)\langle (A_{\varepsilon y}^{-1})^*(P_1 g)|_{S_{\varepsilon y}}, \varphi_\mu \chi_\delta |\mathrm{d}x'|\rangle \,\mathrm{d}y,$$

because in this formula $|\varepsilon y_j| \leq b\varepsilon^{1-\lambda}$ for some positive λ with $\lambda < 1$, $j = 1, \dots, k$.

If $\varepsilon = 0$ the transformation $A_{\varepsilon y}$ is the identity, hence for $|\varepsilon y_j|$ sufficiently small, $j = 1, \dots, k$, as in (26), in

$$DA_{\varepsilon y} = \left(\begin{array}{c:cc} I & \bullet & \bullet \\ \hdashline \bullet & & \\ & & II \\ \bullet & & \end{array}\right)$$

the determinants of the submatrices I and II do not vanish.

Thus for $\varepsilon \in I_0^+$, $\delta \in I_0^+$, $|\varepsilon y_j| \leq b\varepsilon^{1-\lambda}$, $j = 1, \dots, k$ we can write $\langle (A_{\varepsilon y}^{-1})^*(P_1 g)|_{S_{\varepsilon y}}, \varphi_\mu \chi_\delta |\mathrm{d}x'|\rangle$ as in (30) with transformations $A = A_{\varepsilon y}$ and $B = B_{\varepsilon y}$ depending continuously on $(x_1, x_2, \varepsilon y)$ and $(x_1, x_2', \varepsilon y)$, respectively, due to (10). Moreover, the partial derivatives of $A_{\varepsilon y}$ and $B_{\varepsilon y}$ with respect to (x_1, x_2) or x_1, respectively, occurring in the integrant of (30) are computed as in (16) and they remain uniformly bounded for $|\varepsilon y_j| \leq b\varepsilon^{1-\lambda}$, $\varepsilon \in I_0^+$.

Hence by Lebesgue's theorem $H_u(\varepsilon, \delta)$ is continuous up to $\varepsilon = 0$ and $\delta = 0$. □

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