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Note

Packing 16, 17 or 18 circles in an equilateral triangle

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Abstract

We present new, efficient packings for 16, 17 and 18 congruent circles in an equilateral triangle. The results have been found by the use of simulated annealing and a quasi-Newton optimization technique, supplemented with some human intelligence.

1. Introduction

It is a classical problem to determine the smallest circle or, alternatively, the smallest square that can contain n equal nonoverlapping circles [3, 5, 22]. Conjectured optimal solutions to these types of packing problems have been found mainly by trial and error. Optimality proofs exist only for a few cases. The computational complexity can be measured by the fact that the standard discretizations of similar problems are $\mathcal{NP}$-complete [6].

The packing problem in a circle has been solved in [2] for $n \leqslant 7$ and by Pirl [25] for $n \leqslant 10$. Optimal circle packings have been conjectured by Kravitz [15] and Pirl [25] for $11 \leqslant n \leqslant 20$. Improvements for $n = 14, 16, 17$ and 20 were given by Goldberg [10]. In 1975, Reis [26] found some new packings for $n = 17$ and $21 \leqslant n \leqslant 25$ by employing a mechanical diaphragm and plastic cylinders. The configurations that he found are very good and we found that they are extremely difficult to equal using computer simulations. Recently, optimality of the conjectured packings for $n = 11$ was proved by Melissen [19].

For the case of a square, the circle packing problem has been solved for $n = 6$ by Graham, for $n = 7$ by Schaer (both unpublished, see [11, 17, 28, 30, 33]), for $n = 8$ by

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Schaer and Meir [30], for $n = 9$ by Schaer [28], for $n = 14, 16, 25$ by Wengerodt [35–37] and for $n = 36$ by Kirchner and Wengerodt [13]. The case $n = 10$ has successively been improved in [9, 12, 20, 21, 29, 34]. The best arrangement for $n = 10$, however, was already given in 1979 by Schlüter [31]. It is not symmetric and better than the best symmetric configuration [20]. Mollard and Payan [21] also presented efficient arrangements for $n = 11, 13$ and, apparently unaware of the results of Schlüter [31] and Wengerodt [36], rediscovered the arrangements for $n = 10$ and $n = 14$. For other values of $n \leqslant 27$, conjectured optimal packings have been given in [9, 31, 32]. Recently, computer-assisted proofs have been described for $10 \leqslant n \leqslant 20$ by Peikert et al. [24].

Optimal packings in an equilateral triangle were first determined by Oler [23] for the triangular numbers $n = k(k+1)/2$, $k = 2, 3, \ldots$, for $n \leqslant 6$ by Milano [20], and by the first author for $n \leqslant 12$ [16, 18]. For the numbers $n = k(k+1)/2 - 1$ it is conjectured by several authors (Oler [23], Fejes Tóth [4], Newmann [7]) that the best packing can be constructed from the case $n = k(k+1)/2$ by removing one arbitrary circle. This would mean that the hexagonal packing cannot be squeezed closer if only one circle is removed from it. For $n = k(k+1)/2 - 2$ we could do the same and remove two circles. It was shown in [16], however, that this situation can always be improved slightly. To our knowledge, no circle packings for $n = 16, 17$ and 18 are described in the literature. In this paper we will explore the terra incognita that exists between 15 and 19 by providing efficient packings for these cases.

The problem of finding the largest common radius r_n of n congruent circles that can be packed inside a unilateral triangle is equivalent to maximizing the minimum pairwise distance t_n of n points in a unilateral triangle. It is easily seen that there is a homothety between the unilateral triangle enclosing the circles and the smallest equilateral triangle containing their centers. This consideration leads to the following simple relation between t_n and r_n:

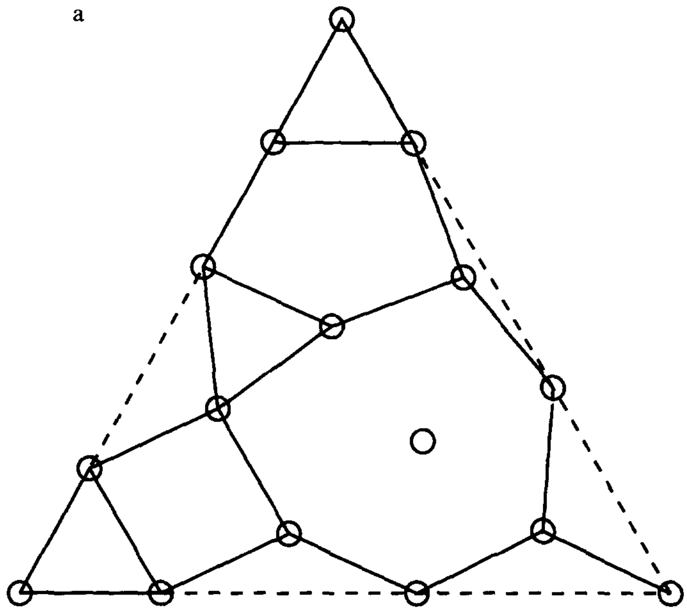
$$t_n = \frac{2r_n}{1 - 2\sqrt{3}r_n}.$$

In the sequel, the latter problem will be called the *maximum separation* problem.

In Section 5 we describe two numerical methods that generate near-optimal solutions to the maximum separation problem. The configurations emerging from these two methods may be further improved manually [21].

2. Sixteen points

In Fig. 1 we show two equivalent configurations of sixteen points in an equilateral triangle that we conjecture to be optimal. Both arrangements are nonsymmetric, so if the conjectures for $n = 13, 14$ and 15 in [16] are correct, then $n = 16$ would be the first a



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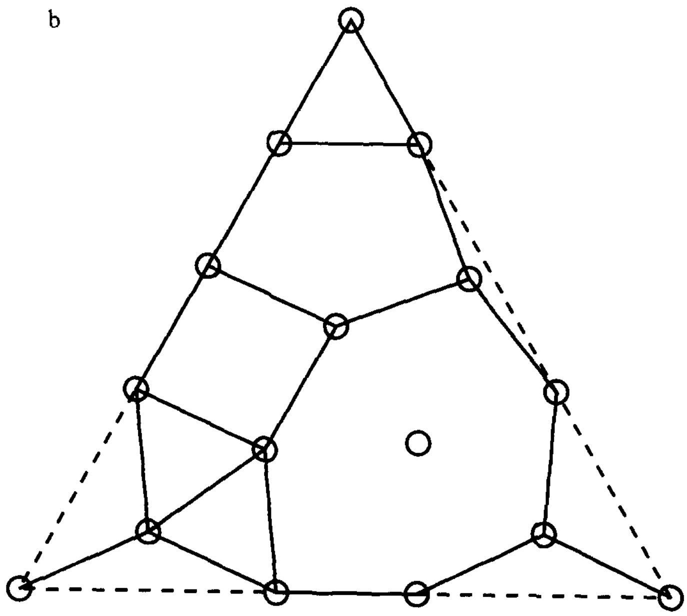


Fig. 1. Conjectured optimal arrangements of sixteen points in an equilateral triangle. The solid lines connect points at minimal distance.

case where the best solution, is not symmetric and cannot be chosen symmetric. The first optimal packing of circles in a square that is essentially nonsymmetric occurs for $n = 10$ [20, 24, 31], whereas in the circle the first optimal packing that cannot be chosen symmetric appears to occur for $n = 22$ [26].

The separation distance in the configurations in Fig. 1 is $t_{16} = 0.216227269\cdots$. This value can be obtained by numerically solving an equation of higher degree in the variable t_{16}, based on the equi-distance graphs displayed in Fig. 1. For explicit examples of such an approach we refer to [9, 24]. In both configurations there is one point that can move around a little without decreasing the minimum distance. The two arrangements can be obtained from each other by rearranging four points.

3. Seventeen points

Fig. 2(a) gives an efficient configuration of 17 points in an equilateral triangle. The minimum distance is equal to $t_{17} = (3 - \sqrt{3})/6 = 0.211324865\cdots$. Since one of the points in Fig. 2(a) has some freedom of movement, this packing is not rigid. Remarkably enough, there are also two equivalent, rigid packings, which are shown in Fig. 2(b) and (c). In fact, the arrangement in Fig. 2(b) emerges from Fig. 2(a) by moving the free point and repositioning one other point in Fig. 2(a). Similarly, Fig. 2(c) can be obtained from Fig. 2(b) by moving just one point. The packing shown in Fig. 2(b) is clearly not symmetric, whereas Fig. 2(a) can be chosen symmetric, and Fig. 2(c) is symmetric.

4. Eighteen points

For $n = 18$ we have a configuration which is probably optimal (Fig. 3(a)). The minimum distance in this arrangement is $t_{18} = 0.203465240\cdots$. A subtlety of this packing is that the first point from the right in the third row from the top in Fig. 3(a) does not lie on the boundary of the triangle, although it is very close. Numerically, this packing is very difficult to find, because there is a very stable symmetric packing (Fig. 3(b)) with several nonsymmetric equivalent configurations that have a minimal distance which is equal to $(9 - \sqrt{33})/16 = 0.203464834\cdots$, which is extremely close to the best value that we have found. This also indicates that the computer-assisted proof methods as employed by Peikert et al. would experience severe problems, because the numerical resolution should be high enough to distinguish between these cases, resulting in exorbitant computing times.

5. The numerical methods

In this section we briefly sketch how we obtained the configurations given above. For simplicity we illustrate our approaches for a circular circle packing. Apart from using software tools that facilitate the manipulation of geometrical objects (as Mollard and Payan did [21]), an obvious way to attack the maximum separation problem is to formulate it as an optimization problem and to apply numerical techniques for its solution. In doing so we used two essentially different approaches. The first is based on simulated annealing [1, 14], a technique, which has already proven its usefulness for packing problems (e.g. [38]). The second uses a quasi-Newton method for nonlinear optimization starting from random initial configurations.

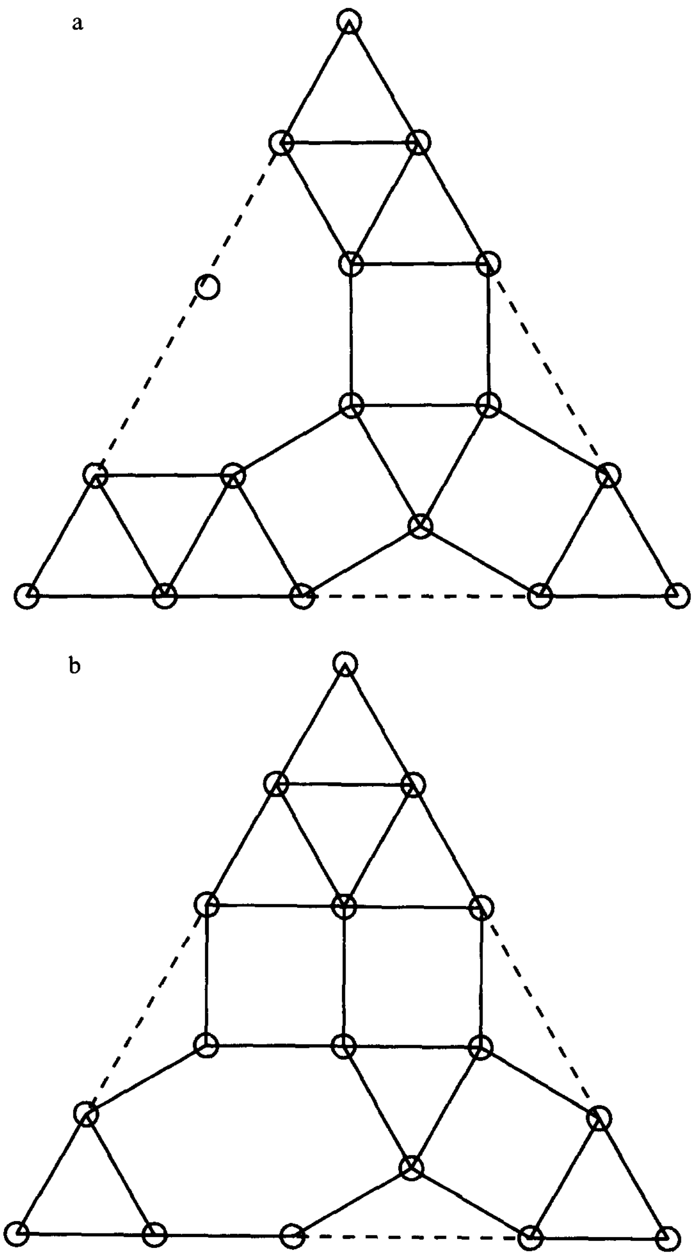


Fig. 2. Conjectured optimal arrangements of seventeen points in an equilateral triangle.

c

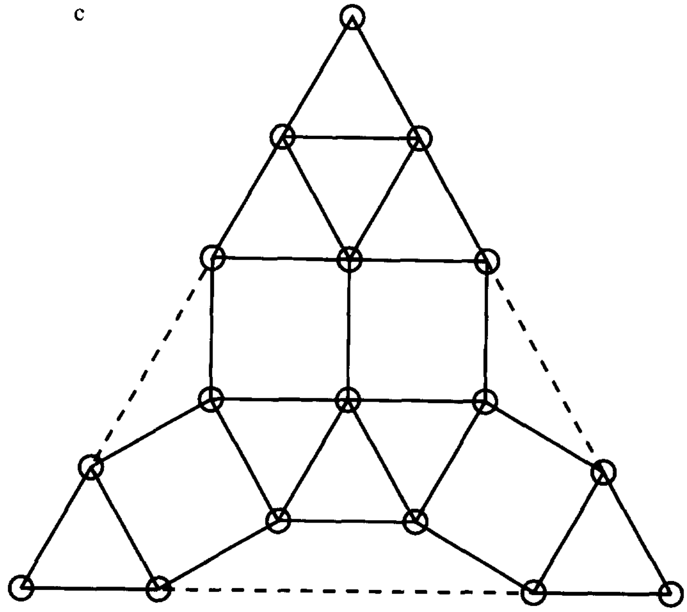


Fig. 2. (Continued).

5.1. *The annealing approach*

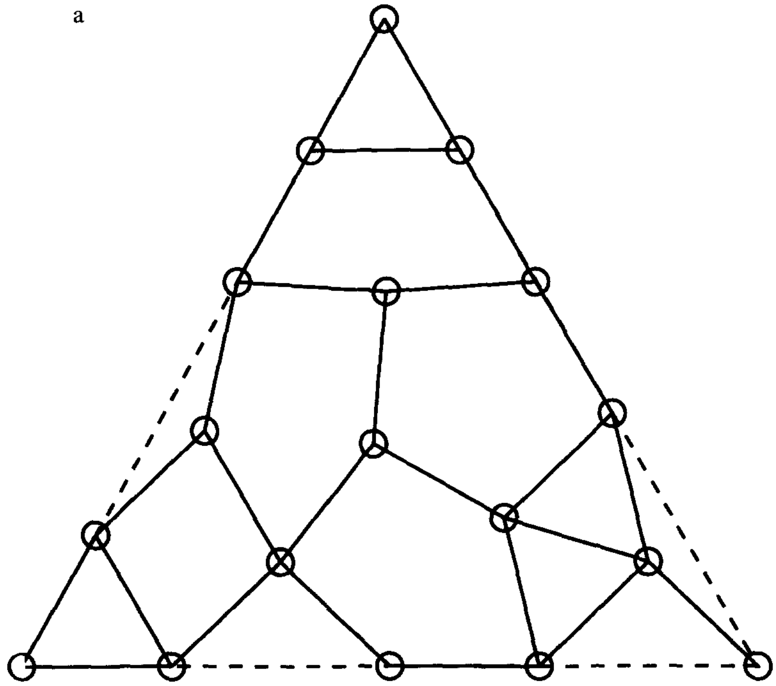
The packing problem in a circle is equivalent with the following:

Place n points in the unit disk such that the minimal pairwise distance is maximal, i.e.

$$\begin{aligned} &\text{maximize} \quad z \\ &\text{subject to} \quad (x_i - x_j)^2 + (y_i - y_j)^2 \geqslant z \quad i,j = 1,2,\ldots,n,\ i > j, \\ &\qquad x_i^2 + y_i^2 \leqslant 1 \quad i = 1,2,\ldots,n. \end{aligned}$$

To obtain an approximate solution to the above maximum separation problem via simulated annealing we place a grid over the unit disk. During the optimization process this grid is gradually refined. As configurations we take all the assignments of the n points to grid points. The cost function is chosen as the negative of the minimal pairwise distance between the n points. The algorithm starts off from an arbitrary initial configuration. In each iteration a new configuration is generated by slightly perturbing the current configuration. The difference in cost is compared with an acceptance criterion which accepts all improvements but also admits, in a limited way, deteriorations in cost.

Initially, the acceptance criterion is taken such that deteriorations are accepted with a high probability. As the optimization process proceeds, the acceptance criterion is a



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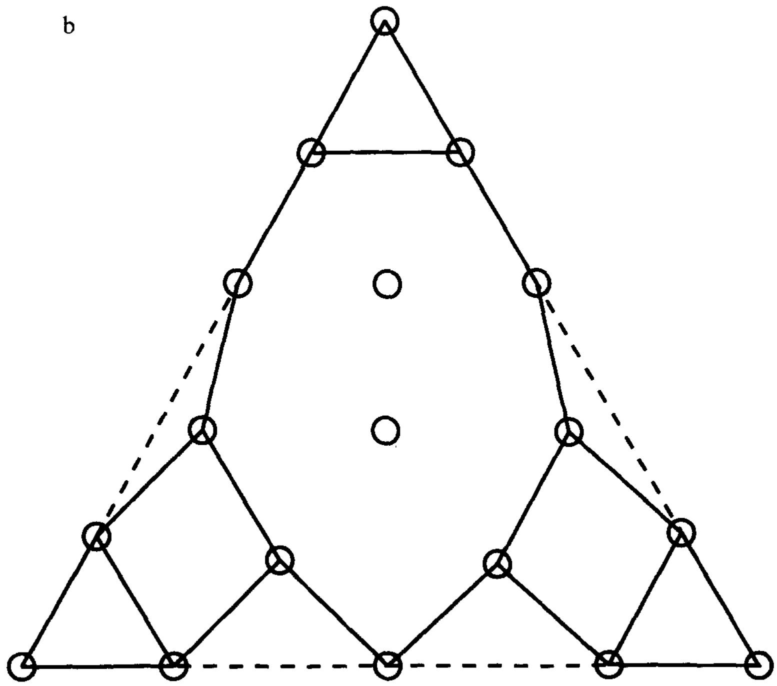

Fig. 3. Conjectured optimal arrangement of eighteen points in an equilateral triangle (a), and the putatively best symmetric configuration (b).

modified such that the probability for accepting deteriorations decreases. At the end of the process this probability tends to zero. In this way the optimization process may be prevented from getting stuck in a local optimum. The process comes to a halt when – during a prescribed number of iterations – no further improvement of the best value found so far occurs.

Of course, an essential element in this annealing algorithm is the perturbation mechanism. On the one hand this must be simple enough to be implemented incrementally; on the other hand the mechanism must contain routines to escape from time-consuming local obstructions. The way we handle this is as follows. The basic and most simple perturbation is the single-point perturbation, i.e., choose randomly one of the n points and displace it over a small distance. This perturbation is easily implemented and hardly time-consuming since the difference in cost can easily be found incrementally. Most of the perturbations we use are of this type. Occasionally, we perform multiple-point perturbations: take a random collection of points and perturb each of them separately in the above way. This can be implemented as a sequence of single-point perturbations. The third type of perturbation used is the one-in-two reflection: choose randomly two points and reflect a third random point in the axis through the first two. Since the one-in-two reflection is the least local of the three perturbation types, it is used only now and then as a means to escape from geometrical obstructions.

5.2. Nonlinear optimization

Our second method to obtain near-optimal configurations for the maximum separation problem is entirely different from the first one. We now consider the maximum separation problem in its obvious nonlinear optimization form. Of course, one could start with the formulation given in the previous subsection. Note that in that case the objective function is smooth. However, in view of the large number of constraints, we prefer a different approach, namely the form

$$\begin{array}{lll} \text{maximize} & \min\limits_{i,\, j=1,\, 2, \dots, n,\, i>j} & (x_i - x_j)^2 + (y_i - y_j)^2 \\ \text{subject to} & & x_i^2 + y_i^2 \leqslant 1, \quad i = 1, 2, \dots, n. \end{array}$$

In our implementation the triangle is parametrized with coordinates in a square where the constrains are simple bounds on the active variables. An appropriate correction to the random starting values in these coordinates is made to guarantee a uniform distribution over the triangle. The object function is almost everywhere differentiable, but not everywhere. Despite this nondifferentiability, the quasi-Newton algorithm we employed for the nonlinear optimization (NAG routine E04JAF, see [8]) performed quite well. Because of the excessive number of local optima, the algorithm had to be used for a large number of random starting points. Only the best configurations were inspected and improved manually.

6. Concluding remarks

By the use of two different optimization techniques we have been able to find efficient circle packings in an equilateral triangle, among which there are several that are equivalent. Both optimization methods proved to be useful tools in the quest for new efficient packings. However, in the long run, simulated annealing tended to out-perform the quasi-Newton approach, when it came to finding the right local optima within a reasonable computation time. Due to the large number of local optima, a considerable amount of computational effort was required to generate candidates for the most efficient arrangements. A touch of common sense remains indispensable in both approaches.

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