Maximum Throughput Gain of Compute-and-Forward for Multiple Unicast

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***Abstract*—Compute-and-Forward (CF), also known as reliable physical layer network coding, is a technique that provides the possibility of exploiting the features of broadcast and superposition in wireless networks. It has been shown that the throughput for multiple unicast traffic can be significantly boosted by CF. In this letter, the limit of such improvement is investigated by comparing the performance of CF with the traditional routing-based transmission schemes. For networks characterized by local interference and half-duplex constraints, it is proven that the throughput gain of CF over traditional routing, expressed by an improvement factor, is upper bounded by $3K$, where K is the number of unicast sessions. Furthermore, a class of networks is presented for which an improvement by a factor of $K/2$ is feasible by applying CF. Hence, the throughput gain of CF is at most on the order of K for any network, and a gain in that order is indeed achievable for some networks.**

***Index Terms*—Compute-and-forward, physical layer network coding, multiple unicast, wireless network.**

I. Introduction

BROADCAST and superposition are two characteristic physical layer features of wireless networks. In traditional physical and MAC layer schemes, broadcast is not well exploited and superposition is seen as an impediment: interfering signals cause an unrecoverable collision. Several techniques have recently been proposed to alleviate these issues. In this paper, we consider Compute-and-forward (CF), also known as reliable physical layer network coding. CF provides a way to exploit both broadcast and superposition and thus to significantly boost the throughput [1]. Some work considers uncoded versions of physical layer network coding, e.g. [2], [3]. However, their approaches suffer from noise accumulation along the stages of the network. On the contrary, CF allows nodes to efficiently and reliably recover a function of the messages from multiple senders. Its potential of improving the throughput compared to traditional approaches has already been shown in [1], [2], and [4]. In these studies, multiple unicast sessions are considered as pairs of sources and destinations, and the information is forwarded from the sources to the destinations with the help of some nodes in the network functioning as relays. With CF, instead of decoding the transmitted messages for each individual session, the relays decode linear combinations of multiple messages from different sessions upon receiving the superposition of their physical layer signals. The throughput of the network benefits from this alternation under several scenarios.

In the example of the two-way relay channel, CF already shows a doubled throughput in comparison to traditional routing schemes under the scenario of two unicast sessions transmitting in opposite directions [3]. This result has been extended to a line network, in which the throughput of CF is again essentially doubled as compared to traditional routing, see [4] for more precise results. Another example is the multi-way relay channel, where again, CF doubles the throughput [5]. Going beyond throughput, [6] studies the transport capacity (the maximum of the sum of the product of source-destination distance and rate over all possible placements of unicast sessions and transmission strategies). The improvement of CF over traditional routing is then a factor between 2.5 and 6 for nodes located on a two-dimensional hexagonal lattice. Some other related papers [7]–[9] show that the throughput gain of physical layer network coding depends on the distribution of the nodes and the allocation of the sessions in 2-D networks. It is an important challenge to find general bounds on the throughput gain of CF.

In this letter, we study the throughput gain of CF for multiple unicast for networks characterized by local interference and half-duplex constraints. By local interference, we mean that the network is characterized by a graph and that there is interference between any two vertices that are connected by an edge. All interference is assumed to be of the same strength. Half-duplex refers to the constraint that at any given point in time, any node can only either be transmitting or receiving, but not both. We assume that each unicast session is operated at the same rate, which we call the *common rate*. To capture the gain, we introduce the *improvement factor (IF)* as the ratio of the maximum common rate achieved by CF (exploiting both broadcast and superposition) and traditional routing (exploiting neither of these features). As our main contributions, we show that the IF is at most $3K$ for any network with K unicast sessions, and we introduce a class of networks with an IF of

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at least $K/2$, which is only a constant factor away from the upper bound. It should be noted that the derived upper bound on the improvement of CF holds for general networks, while many previous studies ([3]–[6]) only focus on some specific network structures such as the two-way relay channel, the hexagonal lattice, etc. Furthermore, while the gains of CF reported so far are rather limited, e.g., a factor of 2, our proposed class of networks achieves a gain that is growing linearly with the number of sessions, which shows that the potential benefit of CF over traditional routing is large. These bounds might be used as references for further studies on CF throughput gains, or for the design of network structures that enable significant benefits from applying CF.

The remainder of the letter is organized as follows. In Section II, we present the network model and define the IF. In Section III, we prove that the IF is upper bounded by $3K$ in any network. In Section IV, we propose a class of networks for which the IF is at least $K/2$. We conclude and give recommendations in Section V.

II. Model Set-Up

A. Network Features

The network studied in this letter is represented by a connected undirected graph $(\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Time is slotted, and in each time slot, each vertex can either transmit or receive. Moreover, we represent messages by symbols from $\mathbb{F}_q$, where q is a prime power. It will also turn out to be useful to introduce σ as the "silent symbol", to indicate that nothing is transmitted or received. Let $X_t(u) \in F_q \cup \{\sigma\}$ and $Y_t(u) \in F_q \cup \{\sigma\}$ denote the symbols transmitted and received, respectively, by vertex u in time slot t. Due to the half-duplex constraints, $Y_t(u) = \sigma$ when vertex u is transmitting in time slot t, while $X_t(u) = \sigma$ if it is receiving. A symbol transmitted by vertex u reaches another vertex v if and only if $(u, v) \in \mathcal{E}$, in which case u and v are called neighbors.

In order to study the improvement of CF in such a network, we transform the underlying physical and MAC layer behavior into two transmission modes, representing traditional routing and CF, respectively, which are described in the next two subsections.

B. Traditional Routing Mode

Traditional routing (TR), which is based on point-to-point communication, is modeled as follows, reflecting that broadcast and superposition are not exploited. In each time slot, a vertex u in the transmit state can communicate a symbol to at most one neighbor v. For any other neighbor $v' \neq v$ of u, the fact that u is transmitting to v implies that v' cannot receive any useful information in that time slot. Finally, successful transmission of a symbol from u to target vertex v in time slot t is possible if and only if u is in the transmit state, v is in the receive state, and all other neighbors of v are silent (in order to avoid collisions at v), i.e.,

$$X_t(u') = \sigma, \qquad \forall u' \neq u : (u', v) \in \mathcal{E}. \tag{1}$$

We introduce the notation $A_t(u, v)$, which is a variable that is set equal to 1 if u successfully transmits a symbol to v in time slot t, and set equal to 0 otherwise. If $A_t(u, v) = 1$, then successful communication is not possible in time slot t for many vertices close to u and v, as shown in the following lemma, which will be used in Section IV.

Lemma 1: In TR mode, $A_t(u, v) = 1$ implies that

$$A_t(v, u') = 0, \qquad \forall u', \tag{2}$$

$$A_t(v', u) = 0, \qquad \forall v', \tag{3}$$

$$A_t(u', v') = 0, \qquad \forall u', v' : v' \neq v \wedge (u, v') \in \mathcal{E}, \tag{4}$$

$$A_t(u', v') = 0, \qquad \forall u', v' : u' \neq u \wedge (u', v) \in \mathcal{E}, \tag{5}$$

for any neighbors u and v and time slot t.

Proof: Results (2) and (3) are due to the half-duplex constraint. Results (4) and (5) immediately follow from the TR characteristics as given in this subsection. ■

C. Compute-and-Forward Mode

In the CF mode, both the broadcast and superposition properties are exploited. CF techniques allow a receiving vertex to retrieve the sum of all symbols transmitted by its neighbors after the reception of the superposition of the physical layer signals of these symbols. Hence, for any vertex v in the receive state, it holds that

$$Y_t(v) = \sum_u X_t(u) \tag{6}$$

where the summation is over all non-silent vertices u for which $(u, v) \in \mathcal{E}$.

D. Multiple Unicast

We consider multiple unicast traffic. In each of K sessions, information needs to be transferred from a source to a destination, possibly via relays. Session i is denoted by $S_i = (s_i, d_i)$, where $s_i \in \mathcal{V}$ and $d_i \in \mathcal{V}$ are the source and destination, respectively. The set of sessions is denoted as $\mathcal{S} = \{S_1, S_2, \ldots, S_K\}$.

The *rate* of a session is the long-term ratio of the number of successfully retrieved source symbols at the destination for that session and the number of time slots used. We use R_i to denote the rate of session S_i. If all sessions communicate at the same rate R, we call R the *common rate*. In this letter, we focus on the maximum achievable common rate. In particular, we investigate the *improvement factor* I of a network with multiple unicast sessions, defined as

$$I = R^{\mathrm{CF}} / R^{\mathrm{TR}} \tag{7}$$

where R^{CF} and R^{TR} are the maximum achievable common rates in CF mode and TR mode, respectively.

III. Upper Bound on the Improvement Factor

In this section, we will show that the improvement factor is upper bounded by $3K$ in any network. In order to do so, we first present a lemma providing a lower bound on the achievable rate in the TR mode for any network with a single session.

Lemma 2: For any network with one session, a rate of 1/3 is achievable in TR mode, which implies

$$R^{\mathrm{TR}} \geq 1/3.$$

Proof: Let $\mathcal{L} = (r_1, r_2, \ldots, r_L)$ be the sequence of vertices on a shortest path from the session's source to its destination, i.e., $r_1 = s_1$, $(r_i, r_{i+1}) \in \mathcal{E}$ for all i, $r_L = d_1$, and there does not exist another path $\mathcal{L}'$ from s_1 to d_1 with $L' < L$. Since $\mathcal{L}$ is the shortest path from s_1 to d_1, it holds that $r_i \neq r_j$ if $i \neq j$.

We use the following simple scheduling scheme in TR mode. In time slot t, all vertices in the network are in the receive state, except the vertices r_i with $i \equiv t \pmod 3$ and $1 \leq i \leq L-1$, which transmit to r_{i+1}. The transmitted symbol by vertex r_i is the symbol received by that vertex in the previous time slot in case $2 \leq i \leq L-1$, while it is a new source symbol in case $i = 1$.

Suppose that besides r_i also another neighbor r_j of r_{i+1} is transmitting in a time slot $t \equiv i \pmod 3$. Then

$$(r_1, r_2, \ldots, r_j, r_{i+1}, r_{i+2} \ldots, r_L)$$

if $j < i$, or

$$(r_1, r_2, \ldots, r_{i+1}, r_j, r_{j+1} \ldots, r_L)$$

if $j > i$, would be a path from s_1 to d_1 which is shorter than $\mathcal{L}$, since $|i - j| \geq 3$. This contradicts the fact that $\mathcal{L}$ is a shortest path, and thus it can be concluded that all neighbors of r_{i+1} other than r_i are silent. Hence, this scheme shows no conflict with (1), and thus it is valid. Clearly, starting at time $t = L - 1$, one source symbol is delivered to the destination every three time slots, and thus, a rate of 1/3 is achieved by the proposed scheme. Furthermore, since R^{TR} is defined as the maximum achievable common rate over all possible schemes in TR mode, it follows that $R^{\mathrm{TR}} \geq 1/3$. ■

Theorem 1: For any network with K sessions,

$$I \leq 3K.$$

Proof: By applying a simple time sharing argument among the K sessions, it follows from Lemma 2 that $R^{\mathrm{TR}} \geq 1/(3K)$. Further, since every destination receives at most one symbol per time slot, it holds that $R^{\mathrm{CF}} \leq 1$. The result follows by applying these two bounds on (7). ■

IV. Example Networks

In the previous section, we have shown that for any network, the improvement factor can be at most $3K$. In this section, we will show that for any $K \geq 3$, there exists a network for which the improvement factor is at least $K/2$. More precisely, we propose a class of networks, denoted as $\mathrm{RN}(K)$, for which CF brings such an improvement. The network $\mathrm{RN}(K)$ is constructed as follows.

1) We construct a bipartite graph with the vertex set $\mathcal{P} \cup \mathcal{Q}$, where $\mathcal{P} \cap \mathcal{Q} = \emptyset$ and $|\mathcal{P}| = |\mathcal{Q}| = K \geq 3$. Denote $\mathcal{P} = \{p_1, p_2, \ldots, p_K\}$, $\mathcal{Q} = \{q_1, q_2, \ldots, q_K\}$. Any $p_i \in \mathcal{P}$ connects to all the vertices in $\mathcal{Q}$ except q_i. Hence, all vertices of this graph have degree $K - 1$.
2) We add a vertex z, called relay, to this graph, which is connected to all the other vertices. Thus it has degree $2K$. As a result, we have vertex set $\mathcal{P} \cup \{z\} \cup \mathcal{Q}$ and edge set $\{(p_i, q_j) | i \neq j, p_i \in \mathcal{P}, q_j \in \mathcal{Q}\} \cup \{(p_i, z) | p_i \in \mathcal{P}\} \cup \{(q_i, z) | q_i \in \mathcal{Q}\}$.
3) Sessions $S_1, S_2, \ldots, S_K$ are placed on this graph, in such a way that the source and destination of session S_i are $s_i = p_i$ and $d_i = q_i$, respectively.
4) As a result, we have the network $\mathrm{RN}(K)$.

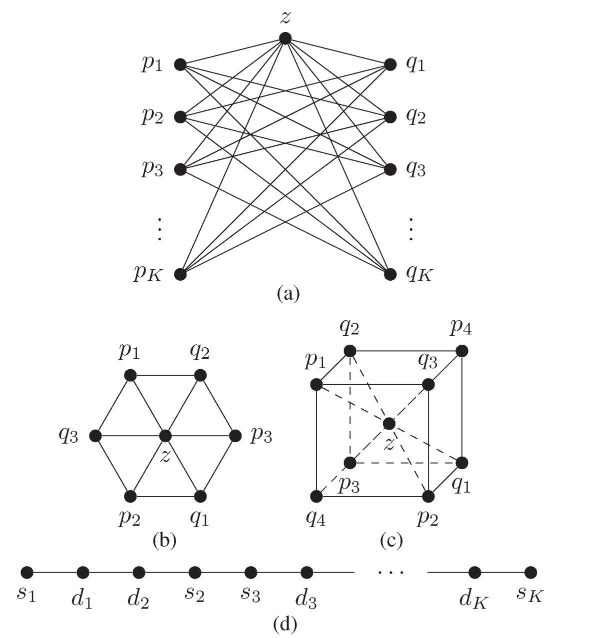

Fig. 1. (a) The network $\mathrm{RN}(K)$, the geometric representations of (b) $\mathrm{RN}(3)$ and (c) $\mathrm{RN}(4)$, and (d) the network $\mathrm{LN}(K)$.

$\mathrm{RN}(K)$ is illustrated in Fig. 1(a). Note that $\mathrm{RN}(3)$ and $\mathrm{RN}(4)$ can be geometrically represented as a hexagon and a cube, respectively, with the relay z in their centers (Fig. 1(b) and (c)).

In order to show that $\mathrm{RN}(K)$ has an improvement factor of at least $K/2$, we first present a lemma providing an upper bound on the common rate for $\mathrm{RN}(K)$ in the TR mode, and then another lemma giving a lower bound on the maximum achievable common rate for $\mathrm{RN}(K)$ in the CF mode.

Lemma 3: For network $\mathrm{RN}(K)$,

$$R^{\mathrm{TR}} \leq 1/K.$$

Proof: We will show that for any scheme on $\mathrm{RN}(K)$ in TR mode, the common rate is at most $1/K$. First, for any time slot t, define

$$B_t = \sum_{i=1}^{K} \left(A_t(p_i, z) + A_t(z, q_i) + \sum_{j=1, j \neq i}^{K} A_t(p_i, q_j) \right) \quad (8)$$

with the $A_t(u, v)$ as introduced in Section II-B. Note that the expression in (8) counts the number of successful transmissions in time slot t leaving from a vertex in $\mathcal{P}$ and/or arriving in a vertex in $\mathcal{Q}$.

Since there is no direct link between source and destination of any session S_i in $\mathrm{RN}(K)$, at least two successful transmissions are required per retrievable source symbol: one transmission from the source vertex p_i to a vertex in $\{z\} \cup \mathcal{Q}\backslash\{q_i\}$, and another from a vertex in $\{z\} \cup \mathcal{P}\backslash\{p_i\}$ to the destination q_i. Hence, when running any scheme in TR mode during T time slots, the total number of successfully obtained source symbols at the destinations, denoted as N_T, satisfies

$$2N_T \leq \sum_{t=1}^{T} B_t. \tag{9}$$

It follows from Lemma 1 that any term in the summation in (8) being equal to one implies that all other terms equal zero, with the possible exception of $A_t(p_j, q_i)$ in case $A_t(p_i, q_j) = 1$. Hence, we have

$$B_t \leq 2 \tag{10}$$

for all t, and thus it follows from (9) that the total number of obtained source symbols at the destinations is at most T. Since rate has been defined as the long-term ratio of the number of retrieved source symbols and the number of time slots used, we thus conclude from (9) and (10) that the common rate satisfies

$$R \leq \frac{1}{K}\sum_{i=1}^{K} R_i = \frac{1}{K}\lim_{T\to\infty}\frac{N_T}{T}$$
$$\leq \frac{1}{K}\lim_{T\to\infty}\frac{\frac{1}{2}\sum_{t=1}^{T} B_t}{T} \leq \frac{1}{K}\lim_{T\to\infty}\frac{\frac{1}{2}\cdot 2T}{T} = \frac{1}{K}.$$

Since this holds for any scheme and R^{TR} is defined as the maximum achievable common rate over all possible schemes in TR mode, we have $R^{\mathrm{TR}} \leq 1/K$. ■

Lemma 4: For network $\mathrm{RN}(K)$,

$$R^{\mathrm{CF}} \geq 1/2.$$

Proof: We prove this result by presenting a scheduling scheme in CF mode, which allows all the destinations of all sessions to retrieve one source symbol in every two time slots:

1) If $t \equiv 1 \pmod 2$, then all sources p_i simultaneously transmit a new source message, while all other vertices are in the receive state. From (6), we have $Y_t(q_j) = \sum_{i=1,i\neq j}^{K} X_t(p_i)$ for all j, $Y_t(z) = \sum_{i=1}^{K} X_t(p_i)$.
2) If $t \equiv 0 \pmod 2$, then relay z broadcasts its reception of the previous time slot to all destinations. Each destination can retrieve its desired source symbol by subtracting its reception in the previous time slot from the reception in this time slot: $Y_t(q_j) - Y_{t-1}(q_j) = \sum_{i=1}^{K} X_{t-1}(p_i) - \sum_{i=1,i\neq j}^{K} X_{t-1}(p_i) = X_{t-1}(p_j)$ for all j.

Since this scheme achieves a common rate of 1/2 and R^{CF} is defined as the maximum achievable common rate over all possible schemes in CF mode, it follows that $R^{\mathrm{CF}} \geq 1/2$. ■

By applying the bounds from Lemmas 3 and 4 on (7), we obtain the main result of this section, as stated in the next theorem.

Theorem 2: For network $\mathrm{RN}(K)$,

$$I \geq K/2.$$

As a contrasting example, we briefly consider the line network $\mathrm{LN}(K)$ illustrated in Fig. 1(d). It consists of vertex set $\{v_1, v_2, \ldots, v_{2K}\}$ and edge set $\{(v_i, v_{i+1}) | i \in \{1, 2, \ldots, 2K-1\}\}$, on which K sessions $S_1, S_2, \ldots, S_K$ are specified as follows: session S_i has source $s_i = v_{2i-1}$ and destination $d_i = v_{2i}$ if i is odd, and source $s_i = v_{2i}$ and destination $d_i = v_{2i-1}$ if i is even. The simple TR scheme in which all sources send a new symbol to their destinations in every time slot achieves a common rate of 1, which cannot be beaten by CF since every destination can receive at most 1 symbol per time slot. Hence, $R^{\mathrm{CF}} = R^{\mathrm{TR}} = 1$ and thus from (7) we have $I = 1$ for any $\mathrm{LN}(K)$. Note that $\mathrm{LN}(K)$ and $\mathrm{RN}(K)$ both have K sessions, but that their improvement factors are very different. We conclude that the benefit of CF depends very much on the network topology and session placement.

V. Conclusion

In this letter, we have given an upper bound on the throughput gain of CF over traditional routing for multiple unicast. More precisely, we have proved that although CF provides a more efficient utilization of the broadcast and superposition nature of wireless networks over traditional routing, the improvement factor will not exceed $3K$ for K unicast sessions, regardless of the network structure and the session placement. On the other hand, we proposed a network with K sessions in which the gain of using CF can be as high as $K/2$. This shows that the throughput gain of CF in some networks can be in the order of K, i.e. it is at most a constant factor away from the maximum throughput gain of CF. An interesting research challenge is to find out to what extent the fundamental CF improvement limit as presented in this letter can be approached in practical networks.

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