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Note

Tight bounds for break minimization in tournament scheduling ☆

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Abstract

We consider round-robin sports tournaments with n teams and $n-1$ rounds. We construct an infinite family of opponent schedules for which every home-away assignment induces at least $\frac{1}{4}n(n-2)$ breaks. This construction establishes a matching lower bound for a corresponding upper bound from the literature.

Keywords: Latin squares; Sports scheduling; Sports tournament; Break minimization

1. Introduction

A central problem in sports scheduling is the planning of round-robin tournaments where an (even) number n of teams plays $n-1$ rounds of matches in which they meet all other teams exactly once; every round consists of $n/2$ matches. Tournament planning is often done in two phases. The first planning phase fixes the $n/2$ matches in every round, and thus generates a so-called *opponent schedule*; Table 1 shows the example of an opponent schedule for $n=16$ teams. The second planning phase decides the location for every match in the opponent schedule: Which team will play at home, and which team will play away? If a team must play two consecutive matches away or two consecutive matches at home, the team incurs a so-called *break*. In general,

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Table 1
An opponent schedule for 16 teams, for which every home-away assignment induces at least 56 breaks

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2	7	4	9	6	11	8	13	10	15	12	3	14	5	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	15	2	5	10	13	16	11	6	7	14	1	12	9	8
4	3	2	1	10	9	16	15	6	5	14	13	12	11	8	7
5	6	11	10	3	2	7	12	15	4	13	8	9	16	1	14
6	5	10	9	2	1	12	11	4	3	8	7	16	15	14	13
7	8	1	16	13	12	5	2	9	14	3	6	15	10	11	4
8	7	16	15	12	11	2	1	14	13	6	5	10	9	4	3
9	10	13	6	1	4	15	14	7	2	11	16	5	8	3	12
10	9	6	5	4	3	14	13	2	1	16	15	8	7	12	11
11	12	5	14	15	8	1	6	3	16	9	2	13	4	7	10
12	11	14	13	8	7	6	5	16	15	2	1	4	3	10	9
13	14	9	12	7	16	3	10	1	8	5	4	11	2	15	6
14	13	12	11	16	15	10	9	8	7	4	3	2	1	6	5
15	16	3	8	11	14	9	4	5	12	1	10	7	6	13	2
16	15	8	7	14	13	4	3	12	11	10	9	6	5	2	1

breaks are considered undesirable events, and one planning objective is to keep their total number small. For more information, we refer the reader to the paper [1] by de Werra (who started the mathematical treatment of the area) and the survey paper [3] by Rasmussen and Trick (who nicely summarize the current state of the area).

Post and Woeginger [2] studied the break minimization problem that arises in the second planning phase: For a given opponent schedule, find a home-away assignment with the smallest possible number of breaks.

Theorem 1. *(Post and Woeginger [2].) Every opponent schedule for n teams has a home-away assignment with at most $\frac{1}{4}n(n-2)$ breaks.*

Here is a quick probabilistic proof for this result in a graph-theoretic setting. For every team T and every round r, create a corresponding vertex (T, r). If team T and team T' meet each other in round r, create an edge between (T, r) and (T', r). For every *even* round r and team T, create an edge between $(T, r-1)$ and (T, r). In the resulting bipartite graph, every connected component is an even cycle or a path, and hence allows exactly two different proper vertex colorings with the colors HOME and AWAY. For each connected component, randomly choose one of these two colorings, with probability $p = 1/2$ and independently of the other choices. Consider an even round r: No team T will incur a break between rounds $r-1$ and r, and the probability that T incurs a break between rounds r and $r+1$ equals $1/2$. Hence, the expected overall number of breaks equals $\frac{1}{4}n(n-2)$. There must exist a point in the underlying probability space that does not exceed this bound.

Post and Woeginger [2] give a polynomial time algorithm that computes a home-away assignment with at most $\frac{1}{4}n(n-2)$ breaks; their algorithm essentially derandomizes the above probabilistic argument. In this note we will demonstrate that the simple bound in Theorem 1 is in fact best possible for infinitely many values of n (more precisely: whenever n is a power of two).

Table 2
A partial schedule for a quadruple in partition $\mathcal{P}_1$

Team\round	1	2	3
$B(x)$	$A(x)$	$B(x+c)$	$A(x+c)$
$B(x+c)$	$A(x+c)$	$B(x)$	$A(x)$
$A(x)$	$B(x)$	$*$	$B(x+c)$
$A(x+c)$	$B(x+c)$	$*$	$B(x)$

2. The construction

Let $k \geqslant 2$ be a power of two, and consider the finite field $\mathbb{GF}(k)$ of characteristic 2; denote the additive identity by 0 and the multiplicative identity by 1, and let c be a generator of the multiplicative group. For every element x in $\mathbb{GF}(k)$, create two corresponding sports teams $A(x)$ and $B(x)$. For these $n = 2k$ teams construct the following opponent schedule with rounds $r = 1, 2, \ldots, 2k - 1$.

- In round $r = 1$, teams $A(x)$ and $B(x)$ play each other.
- In round $r = 2s + 1$ with $1 \leqslant s \leqslant k - 1$, teams $A(x)$ and $B(x + c^s)$ play each other.
- In round $r = 2s$ with $1 \leqslant s \leqslant k - 1$, teams $A(x)$ and $A(x + (c + 1)c^{s-1})$ play each other, and teams $B(x)$ and $B(x + c^s)$ play each other.

It is easily verified that this construction indeed yields a feasible opponent schedule: Since the field has characteristic 2, in every fixed round the opponent of any team is the team itself. Since $c^1, c^2, \ldots, c^{k-1}$ is an enumeration of the non-zero elements of the field, every team plays every other team exactly once.

The opponent schedule depicted in Table 1 illustrates this construction for $k = 8$ and $n = 16$; the elements in $\mathbb{GF}(8)$ are $x_0 = 0$, and $x_i = c^i$ for $i = 1, \ldots, 7$, with $c^7 = 1$. In Table 1 the team with number $2i + 1$ corresponds to $A(x_i)$, and the team with number $2i$ corresponds to $B(x_{i-1})$.

Now let us analyze the home-away assignments for this opponent schedule. For every integer s with $1 \leqslant s \leqslant k - 1$, define a partition $\mathcal{P}_s$ of the $n = 2k$ teams into $k/2$ quadruples. Every quadruple in partition $\mathcal{P}_s$ consists of two A-teams and two B-teams. Loosely speaking, the partition is centered around round $2s$.

- For $s = 1$, the two B-teams in every quadruple meet each other in round 2. Hence, for an appropriate choice of x, this quadruple contains the teams $B(x)$ and $B(x + c)$. Furthermore, the quadruple contains $A(x)$ and $A(x + c)$, the opponents of the B-teams in round 1.
- If $2 \leqslant s \leqslant k - 1$, then the two A-teams in a quadruple play each other in round $2s$. Hence, for an appropriate choice of x, this quadruple contains teams $A(x)$ and $A(x + (c + 1)c^{s-1})$. Furthermore, the quadruple contains teams $B(x + c^{s-1})$ and $B(x + c^s)$, the opponents of the two A-teams in the preceding round $2s - 1$.

Tables 2 and 3 depict part of the opponent schedule for a quadruple in rounds $2s - 1$, $2s$, and $2s + 1$. The asterisks represent matches against opponents outside the quadruple; these matches are irrelevant for our further argumentation. A more compact, isomorphic version of these opponent schedules is depicted in Table 4.

Table 3
A partial schedule for a quadruple in partition $\mathcal{P}_s$ with $s \geqslant 2$

Team\round	$2s-1$	$2s$	$2s+1$
$A(x)$	$B(x+c^{s-1})$	$A(x+(c+1)c^{s-1})$	$B(x+c^s)$
$A(x+(c+1)c^{s-1})$	$B(x+c^s)$	$A(x)$	$B(x+c^{s-1})$
$B(x+c^{s-1})$	$A(x)$	$*$	$A(x+(c+1)c^{s-1})$
$B(x+c^s)$	$A(x+(c+1)c^{s-1})$	$*$	$A(x)$

Table 4
An opponent schedule isomorphic to the schedules in Tables 2 and 3

Team\round	$2s-1$	$2s$	$2s+1$
1	3	2	4
2	4	1	3
3	1	$*$	2
4	2	$*$	1

Lemma 2. *Any home-away assignment for the schedule in Table* 4 (*or equivalently, for the schedules in Tables* 2 *and* 3) *induces at least two breaks between the rounds* $2s-1$, $2s$, *and* $2s+1$.

Proof. The statement concerns only five matches, and can easily be verified by enumerating all corresponding home-away assignments. (A crucial observation on the irrelevant asterisk entries: If team 3 plays one of its two matches against teams 1 and 2 at home and the other one away, then it must incur a break between rounds $2s-1$ and $2s$ or between rounds $2s$ and $2s+1$. An analogous observation holds for team 4.) □

Hence, for every s with $1 \leqslant s \leqslant k-1$ there are $k/2$ corresponding quadruples in $\mathcal{P}_s$, that each induce at least two breaks between rounds $2s-1$, $2s$, and $2s+1$. Altogether, this yields at least $k(k-1) = \frac{1}{4}n(n-2)$ breaks.

Theorem 3. *For* $n = 2^m$ *teams with* $m \geqslant 2$, *there exists an opponent schedule for which every home-away assignment induces at least* $\frac{1}{4}n(n-2)$ *breaks.*

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