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Subpancyclicity of line graphs and degree sums along paths☆

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Abstract

A graph is called *subpancyclic* if it contains a cycle of length ℓ for each ℓ between 3 and the circumference of the graph. We show that if G is a connected graph on $n \geqslant 146$ vertices such that $d(u)+d(v)+d(x)+d(y) > (n+10/2)$ for all four vertices u, v, x, y of any path $P = uvxy$ in G, then the line graph $L(G)$ is subpancyclic, unless G is isomorphic to an exceptional graph. Moreover, we show that this result is best possible, even under the assumption that $L(G)$ is hamiltonian. This improves earlier sufficient conditions by a multiplicative factor rather than an additive constant.

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1. Introduction

During the last 50 years, a key issue in the vast literature on cyclic properties of graphs has been the development of sufficient conditions to guarantee the existence of certain cycles in graphs. Most of these conditions involve the neighborhood of single vertices, pairs of (nonadjacent) vertices, or larger sets of (mutually nonadjacent) vertices. Typical examples (see e.g. [2]) are the well-known minimum degree condition $\delta(G) \geqslant n/2$ (where n is the number of vertices of the graph G), due to Dirac, for the existence of a hamiltonian cycle in G, and its subsequent generalizations to all pairs of nonadjacent vertices having degree sum at least n (Ore), and other sets of vertices meeting some degree condition. These conditions have in common that the average degree in all conditions is bounded from below by (roughly) the same function of n, namely $n/2$. Even if one adds certain conditions to the graph, e.g. that the graph is highly connected or 1-tough, these bounds cannot be improved considerably, but at most by a small constant. A similar phenomenon is often encountered in other chains of subsequently improved results guaranteeing certain cyclic properties of graphs. We omit the details, but stress the common feature of the results in this area of graph theory. In contrast to this, the main result of this paper involves a degree condition on sets of four vertices along a path on four

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vertices, which improves earlier results on smaller sets along shorter paths considerably. The best previously known result guarantees the same conclusion with a lower bound on the degree sum of sets of three vertices along a path on three vertices which is roughly the same as our lower bound for four vertices. Hence the lower bound on the average degree in the condition is decreased by a factor rather than a small constant. A small drawback is that we have to exclude some exceptional graphs from the conclusion, but this is unavoidable since the previously known result is best possible (in the sense that the lower bound cannot be decreased without excluding some exceptional graphs). We refer to the next sections for the details, and start with some useful definitions, notation and related results.

2. Preliminaries

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. Let H be a subgraph of G. If S is a subgraph of H, then the *degree* of S in H, denoted by $d_H(S)$ (or just $d(S)$ if $H=G$), is defined to be the degree sum of the vertices of S, i.e., $d_H(S)=\sum_{u\in V(S)} d_H(u)$. With $c(G)$ we denote the circumference of G, i.e., the length of a longest cycle of G, and with $\lambda(G)$ the set of all different cycle lengths of cycles in G. G is called *pancyclic* if $\lambda(G)=[3,|V(G)|]=\{3,4,\ldots,|V(G)|\}$. G is said to be *subpancyclic* if $\lambda(G)=[3,c(G)]=\{3,4,\ldots,c(G)\}$. By the length of a path we refer to the number of edges on this path.

Define

$$\rho_i(G)=\min\{d(P): P \text{ is a path of length } i-1 \text{ in } G\}.$$

Obviously $\delta(G)=\rho_1(G)$. As introduced in [1], let $f_i(n)$ be the smallest integer such that for any graph G of order n with $\rho_i(G)>f_i(n)$, the line graph $L(G)$ of G is pancyclic whenever $L(G)$ is hamiltonian. Van Blanken et al. [1] proved that $f_1(n)$ has the order of magnitude $O(n^{1/3})$. If we do not impose the condition that $L(G)$ is hamiltonian, the following degree condition results guaranteeing subpancyclicity have been obtained. We note here that pancyclicity cannot be guaranteed by these conditions.

Theorem 1. *Let G be a connected graph of order n. If G satisfies one of the following conditions*:

(i) [10] $\rho_2(G)>(\sqrt{8n+1}+1)/2$ *and* $n\geqslant 600$;
(ii) [11] $\rho_3(G)>(n+6)/2$ *and* $n\geqslant 76$;
(iii) [11] $\rho_4(G)>(2n+16)/3$ *and* $n\geqslant 76$,

then $L(G)$ is subpancyclic and the bounds on $\rho_i(G)$ are all sharp.

Here sharpness means that the bounds cannot be lowered without excluding some exceptional graphs from the conclusion.

Concentrating on sufficient degree conditions for the line graph itself in order to guarantee subpancyclicity, Trommel, et al. showed a consequence of Theorem 1(i) in this respect for large line graphs.

Corollary 2 (*Trommel et al. [8]*). *Let G be a line graph with order $n>100577$. If*

$$\delta(G)=\rho_1(G)>(\sqrt{8n+1}-3)/2,$$

then G is subpancyclic.

As a first exercise, we show that we can obtain the following analogous consequences of Theorem 1(ii) and (iii) by using similar arguments as in the proof of Corollary 2.

Corollary 3. *Let G be a line graph with order $n>2775$. If G satisfies one of the following conditions*:

(i) $\rho_2(G)>(n-2)/2$;
(ii) $\rho_3(G)>(2n-2)/3$;

then G is subpancyclic.

Proof. Let $G = L(H)$ be a line graph with order $n > 2775$ and satisfying one of the conditions (i) and (ii) of Corollary 3. Since

$$n > \binom{75}{2},$$

$|V(H)| \geqslant 76$. If H is a tree, then G is subpancyclic. If H is not a tree, then $n = |E(H)| \geqslant |V(H)| \geqslant 76$. Note that $\rho_i(H) = \rho_{i-1}(G) + 2(i-1)$ for $i = 3, 4$. We obtain that either $\rho_3(H) = \rho_2(G) + 4 \geqslant (n-2)/2 + 4 \geqslant (|V(H)| + 6)/2$ or $\rho_4(H) = \rho_3(G) + 5 \geqslant (2n-2)/3 + 6 \geqslant (2|V(H)| + 16)/3$. Hence H satisfies the conditions of Theorem 1 and so $G = L(H)$ is subpancyclic. □

Theorem 1 shows that the graphs in [4,7,9] are pancyclic.

Rather than looking at line graphs, one could consider the larger class of so-called claw-free graph and ask for similar degree conditions guaranteeing subpancyclicity. One such result related to Theorem 1 has appeared in [8].

Theorem 4 (*Trommel et al. [8]*). *Let G be a claw-free graph on at least* 5 *vertices. If* $\delta(G) > \sqrt{3n+1} - 2$, *then G is subpancyclic and the bound is sharp.*

More recently, Gould and Pfender improved Theorem 4 by considering a degree sum condition on pairs of nonadjacent vertices instead of a minimum degree condition, as follows.

Theorem 5 (*Gould et al. [5]*). *Let G be a claw-free graph on at least* 5 *vertices. If* $d(u) + d(v) > 2\sqrt{3n+1} - 4$ *for any pair of vertices* u, v *with* $uv \notin E(G)$, *then G is subpancyclic and the bound is sharp.*

3. Main result of this paper

We turn back to degree conditions on the graph G for guaranteeing the subpancyclicity of its line graph $L(G)$. We will show that we can considerably decrease the lower bound in the best known degree condition for this by excluding one class of exceptional graphs. In fact, Theorem 6 below shows that when we exclude an exceptional graph, the lower bound on the degree sums of the vertices along 4-paths which ensures that its line graph is subpancyclic can be improved roughly from $2n/3$ to $n/2$. Moreover, this bound is then almost the same as the bound on degree sums of the vertices along 3-paths (comparing the statements in Theorem 1).

Theorem 6. *Let G be a connected graph of order* $n \geqslant 146$. *If*

$$\rho_4(G) > (n+10)/2,$$

then $L(G)$ *is subpancyclic unless G is isomorphic to the exceptional graph F defined below. Moreover, the bound on* $\rho_4(G)$ *is sharp, even under the assumption that* $L(G)$ *is hamiltonian.*

The exceptional graph F is defined as follows: let $n \equiv 1 \pmod 3$, and let $C_1, C_2, \ldots, C_{(n-1)/3}$ be $(n-1)/3$ edge-disjoint cycles of length 4. Now F is obtained from those cycles such that $C_1, C_2, \ldots, C_{(n-1)/3}$ have exactly one common vertex in F and $E(F) = E(C_1) \cup E(C_2) \cup \cdots \cup E(C_{(n-1)/3})$. Obviously $|V(F)| = n$ and $\rho_4(F) = (2n+16)/3 = f_4(n)$, by Theorem 1.

Using similar arguments as in the proof of Corollary 3, we obtain the following consequence of Theorem 6.

Corollary 7. *Let G be a line graph with order* $n > 10440$. *If* $\rho_3(G) > (n-2)/2$, *then G is subpancyclic unless G is isomorphic to the line graph of F.*

4. Proof of Theorem 6

Before we present our proof of the main result, we introduce some additional terminology and notation, and state a number of preliminary results.

By a *circuit* of a graph G we will mean an eulerian subgraph of G, i.e., a connected subgraph in which every vertex has even degree. Note that by this definition (the trivial subgraph induced by) a single vertex is also a circuit. If C is a circuit of G, then $\overline{E}(C)$ denotes the set of edges of G incident with at least one vertex of C and we let $UP_4(C)=\{P : P \text{ is a path of length 3 in } C\}$. We write $\varepsilon(C)$ for $|E(C)|$ and $\overline{\varepsilon}(C)$ for $|\overline{E}(C)|$. The distance $d_H(G_1, G_2)$ between two subgraphs G_1 and G_2 of H is defined to be $\min\{d_H(v_1, v_2) : v_1 \in V(G_1) \text{ and } v_2 \in V(G_2)\}$. The *diameter* of a connected subgraph H, denoted by $dia(H)$, is defined to be $\max\{d_H(u, v) : u, v \in V(H)\}$. By C_k we denote a cycle of length k. For any subgraph H of G, let $N(H)=\bigcup_{u\in V(H)} N(u)$.

Harary and Nash-Williams [6] characterized those graphs with line graphs that are hamiltonian. One can easily prove a more general result (see, e.g., [3]).

Theorem 8 (*Broersma [3]*). *The line graph $L(G)$ of a graph G contains a cycle of length $k \geqslant 3$ if and only if G contains a circuit C such that $\varepsilon(C) \leqslant k \leqslant \overline{\varepsilon}(C)$.*

Before we present our proof of the main result, we start with a useful technical lemma that will avoid ending up in too many subcases.

Lemma 9. *If G is a graph of order n which satisfies the conditions of Theorem 6 but whose line graph $L(G)$ is not subpancyclic, then G contain neither a circuit C_0 with $\varepsilon(C_0) \leqslant k \leqslant \overline{\varepsilon}(C_0)$ nor a cycle of length $k+1$, where $k=\max\{i : i \in [3, c(L(G))]\backslash\lambda(L(G))\}$.*

Proof. Assume G is a graph satisfying the conditions of Lemma 9. It follows from Theorem 8 that G does not contain a circuit C_0 with $\varepsilon(C_0) \leqslant k \leqslant \overline{\varepsilon}(C_0)$. We claim that G does not contain a cycle of length $k+1$. Suppose to the contrary that G has such a cycle C. Note that $[3, \Delta(G)] \subseteq \lambda(L(G))$. Hence $k \geqslant \Delta(G)+1$. Since $\rho_4(G) > (n+10)/2 \geqslant 78$,

$$\varepsilon(C)=k+1 \geqslant \Delta(G)+2 \geqslant \rho_4(G)/4+2 > (n+26)/8 \geqslant 21. \tag{4.1}$$

We are going to derive a contradiction, and start with the following claim.

Claim. *G does not contain a cycle C'* with $\varepsilon(C)/2 < \varepsilon(C') \leqslant k$.

Proof of Claim. Suppose otherwise that C' is a cycle of G such that $\varepsilon(C)/2 < \varepsilon(C') \leqslant k$. Note that in $\sum_{P\in UP_4(C)} d(P)$, every edge in $\overline{E}(C')$ is counted at most 8 times. Hence, by (4.1) and $\rho_4(G) \geqslant (n+10)/2 \geqslant 78$,

$$\begin{aligned}\overline{\varepsilon}(C') &\geqslant \sum_{P\in UP_4(C')} (d(P)-8)/8+\varepsilon(C')\\ &\geqslant (\rho_4-8)\varepsilon(C')/8+\varepsilon(C')=\rho_4\varepsilon(C')/8\\ &\geqslant \rho_4\varepsilon(C)/16\\ &\geqslant k+1.\end{aligned}$$

On the other hand, $\varepsilon(C') \leqslant k$. Thus $L(G)$ contains a C_k, a contradiction. This completes the proof of Claim. □

By Claim, C has no chord. Since $\rho_4 \geqslant 78$, C cannot be a hamiltonian cycle of G. Let u be a vertex in $V(G)\backslash V(C)$. Since G does not contain a circuit C_0 with $\varepsilon(C_0) \leqslant k \leqslant \overline{\varepsilon}(C_0)$, u is adjacent to at most three vertices of C. By (4.1), we obtain the following:

$$\overline{\varepsilon}(C) \leqslant 3|V(G)\backslash V(C)|+\varepsilon(C)=3(n-\varepsilon(C))+\varepsilon(C) < (11n-26)/4. \tag{4.2}$$

On the other hand, since C has no chord,

$$\begin{aligned}\bar{\varepsilon}(C) &\geqslant \sum_{P\in UP_4(C)} (d(P)-8)/4+\varepsilon(C)\\ &\geqslant (\rho_4-8)\varepsilon(C)/4+\varepsilon(C)\\ &=(\rho_4-4)\varepsilon(C)/4\\ &\geqslant (n^2+28n+52)/64,\end{aligned}$$

which contradicts (4.2) and $n\geqslant 146$. This completes the proof of Lemma 9. □

We now present the proof of Theorem 6.

Proof of Theorem 6. We will prove the theorem by contradiction.

Assuming G is a graph of order n which satisfies the conditions of Theorem 6 but whose line graph $L(G)$ is not subpancyclic, we define

$$k=\max\{i : i \in [3, c(L(G))]\backslash\lambda(L(G))\},$$

so this k is the same as in Lemma 9.

By the definition of k, $L(G)$ contains a cycle C_{k+1} of length $k+1$. Hence, by Theorem 8, we obtain that G contains a circuit C with $\varepsilon(C)\leqslant k+1\leqslant\bar{\varepsilon}(C)$. By Lemma 9, $\varepsilon(C)=k+1$. Since C is a circuit, there exist edge-disjoint cycles $D_1, D_2, \ldots, D_r$ such that $C=\bigcup_{i=1}^r D_i$ and we assume that these cycles are chosen in such a way that r is maximized. By Lemma 9, it suffices to consider the case that $r\geqslant 2$. Hence,

$$|V(D_i)\cap V(D_j)|\leqslant 2 \quad \text{for } \{i, j\}\subseteq\{1, 2, \ldots, r\}. \tag{4.3}$$

It is easy to verify that (4.1) also holds here.

Let H be the graph with $V(H)=\{D_1, D_2, \ldots, D_r\}$ and $D_iD_j \in E(H)$ if and only if $V(D_i)\cap V(D_j)\neq\emptyset$. Since C is a circuit, H is connected. Without loss of generality, we assume that D_1 and D_r are two vertices of H such that

$$d_H(D_1, D_r)=dia(H). \tag{4.4}$$

Hence, any element of $\{D_1, D_r\}$ is not a cut vertex of H, so $C^1=\bigcup_{i=2}^r D_i$ and $C^r=\bigcup_{i=1}^{r-1} D_i$ are two circuits of G. Let

$$E_1(D_i)=E(D_i)\cap\overline{E}(C^i) \quad \text{and} \quad E_2(D_i)=E(D_i)\backslash E_1(D_i)$$

and

$$V_1(D_i)=V(D_i)\cap V(C^i) \quad \text{and} \quad V_2(D_i)=\{u, v : uv \in E_2(D_i)\},$$

where $i \in \{1, r\}$.

For any path P of C, let $d_2(P)=d(P)-d_C(P)$. Since $\bar{\varepsilon}(C^i)\geqslant\varepsilon(C)-|E_2(D_i)|=k+1-|E_2(D_i)|$,

$$|V_2(D_i)|-1\geqslant|E_2(D_i)|\geqslant 2, \tag{4.5}$$

where $i \in \{1, r\}$. Otherwise $\varepsilon(C^i)\leqslant k\leqslant\bar{\varepsilon}(C^i)$ which contradicts Lemma 9.

Since $\bar{\varepsilon}(C^t)\geqslant\varepsilon(C)-|E_2(D_t)|+|\overline{E}(D_s)\backslash E(C)|$,

$$|\overline{E}(D_s)\backslash E(C)|\leqslant|E_2(D_t)|-2, \tag{4.6}$$

where $\{s, t\}=\{1, r\}$. Otherwise $\varepsilon(C^t)\leqslant k\leqslant\bar{\varepsilon}(C^t)$ which contradicts Lemma 9.

Before we turn to a case distinction, we need one more claim.

Claim 1. *Let P be a path of length 3 in D_s. Then*

$$d_C(P)>(n+14)/2-|E_2(D_t)| \tag{4.7}$$

and

$$|E_2(D_t)| \leqslant 2|V_2(D_t)|/3 \quad \text{and} \quad d_C(P) > (n+14)/2 - 2|V_2(D_t)|/3, \tag{4.8}$$

where $\{s, t\} = \{1, r\}$.

Proof of Claim 1. Let P be a path of length 3 in D_s. Then

$$|\overline{E}(D_s) \backslash E(C)| \geqslant d(P) - d_C(P).$$

Hence be (4.6) and $\rho_3(G) > (n+10)/2$,

$$d_C(P) > (n+10)/2 - (|E_2(D_t)| - 2),$$

i.e., (4.7) is true.

By (4.5) and (4.7),

$$d_C(P) > (n+16)/2 - |V_2(D_t)|. \tag{4.9}$$

In order to obtain (4.8), it suffices to prove the following assertion.

Each component of $C[E_2(D_1) \cup E_2(D_r)]$ is a path of length at most two. (4.10)

Suppose to the contrary that there exists a $t \in \{1, r\}$ and a path $P_0 = u_0 v_0 x_0 y_0$ of D_t such that $\{u_0, v_0, x_0, y_0\} \subseteq V_2(D_t)$. By (4.9), $d_C(P_0) > (n+16)/2 - |V_2(D_s)|$ where $\{s, t\} = \{1, r\}$. Since $d_C(P_0) = 8$,

$$|V_2(D_s)| > n/2 \geqslant 78. \tag{4.11}$$

Hence there exists a path $P'_0 = u'_0 v'_0 x'_0 y'_0$ in D_s such that $u'_0 v'_0 \in E_2(D_s)$ and $\{x'_0, y'_0\} \cap V_1(D_t) = \emptyset$. By (4.9),

$$d_C(P') > (n+16)/2 - |V_2(D_t)|. \tag{4.12}$$

For any $x \in N_C(x'_0) \cap N_C(y'_0)$, $C - x$ has at least a nontrivial component, denoted by Q_x, which does not contain any vertex of D_s (otherwise $\varepsilon(C') \leqslant k \leqslant \overline{\varepsilon}(C')$, where $C' = C - \{xx'_0, xy'_0, x'_0 y'_0\}$, a contradiction).

It is easy to see that

$$|V(Q_x)| \geqslant 3. \tag{4.13}$$

Otherwise $\varepsilon(C') \leqslant k \leqslant \overline{\varepsilon}(C')$, where $C' = C - Q_x$, a contradiction.

Let B denote the cut-vertex set of $N_C(x'_0) \cap N_C(y'_0)$ such that for any $x \in B$, $C - x$ has a nontrivial component, denoted by Q_x, which does not contain any vertex of $V(D_1) \cup V(D_r)$. Set

$$\beta = |N_C(x'_0) \cap N_C(y'_0)|.$$

Obviously,

$$|\{Q_x : x \in B\}| = |B| \geqslant \beta - 1. \tag{4.14}$$

Using (4.11), (4.13) and (4.14), we obtain

$$\begin{aligned} d_C(x'_0) + d_C(y'_0) &= |N_C(x'_0) \cup N_C(y'_0)| + |N_C(x'_0) \cap N_C(y'_0)| \\ &\leqslant \begin{cases} n - (|V_2(D_1)| + |V_2(D_r)| - 4) + 1 & \text{if } \beta \leqslant 1, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 2 + 3(\beta - 1)) + \beta & \text{if } \beta \geqslant 2, \end{cases} \\ &\leqslant n - (|V_2(D_1)| + |V_2(D_r)|) + 5 \\ &\leqslant n - n/2 - |V_2(D_t)| + 5 \\ &= (n+10)/2 - |V_2(D_t)|, \end{aligned}$$

which contradicts (4.12) and $d_C(u'_0) = d_C(v'_0) = 2$. This implies that assertion (4.10) is true. Hence (4.8) also holds. This completes the proof of Claim 1. □

For convenience, we now introduce the following notation.

$S = \{x, y, x', y'\}$,
$N_i = \{u \in V(C) : |N_C(u) \cap S| = i\}$,
$M_1 = ((N_C(x) \cap N_C(y)) \cup (N_C(x') \cap N_C(y'))) \cap N_2$,
$M_2 = N_2 \backslash M_1$,
$n_i = |N_i|$ and $m_i = |M_i|$.

Obviously,

$$n_2 = m_1 + m_2. \tag{4.15}$$

We will complete the proof by deriving contradictions in the following three cases.

Case 1. $dia(H) \geqslant 2$.

This implies that $V(D_1) \cap V(D_r) = \emptyset$.

We can take two paths $P = uvxy$ and $P' = u'v'x'y'$ of length 3 in D_1 and D_r respectively with $\{uv, u'v'\} \subseteq E_2(D_1) \cup E_2(D_r)$ and $\{x, x'\} \subseteq V_1(D_1) \cup V_1(D_r)$ such that $V(P) \cap V(P') = \emptyset$.

We now prove three claims in order to get contradictions.

Claim 2. $|N_3 \cup N_4| \leqslant 1$, *i.e.*, $n_3 + n_4 \leqslant 1$.

Proof of Claim 2. Suppose otherwise that $|N_3 \cup N_4| \geqslant 2$, say, $w, w' \in N_3 \cup N_4$. Obviously,

$$w, w' \in (N_C(x) \cap N_C(y)) \cup (N_C(x') \cap N_C(y')).$$

Without loss of generality, we assume that $wx, wy \in E(C)$. Hence $C' = C - \{wx, wy, xy\}$ is a circuit with $\varepsilon(C') = \varepsilon(C) - 3 \leqslant k \leqslant \overline{\varepsilon}(C')$, a contradiction. This completes the proof of Claim 2. □

Claim 3. *Each element of M_1 is a cut vertex of C.*

Proof of Claim 3. Suppose otherwise that there exists a vertex $w \in M_1$, say, $w \in N_C(x) \cap N_C(y) \cap N_2$, which is not a cut vertex of C. Hence $C' = C - \{wx, wy, xy\}$ is a circuit with $\varepsilon(C) - 3 = \varepsilon(C') \leqslant k \leqslant \overline{\varepsilon}(C')$, a contradiction. This completes the proof of Claim 3. □

Let W_1 denote the cut vertex set of C in M_1 such that for any $z \in W_1$, $C - z$ has a nontrivial component which does not contain any element of S. Every vertex of M_1 is in W_1 except when it is a cut vertex such that deleting it creates two components which contain $\{x, y\}$ and $\{x', y'\}$, respectively; in the exceptional case, $dia(H) \geqslant 3$. Hence at most two vertices of M_1 are not in W_1 (in case of two they are in $N_C(x) \cap N_C(y) \cap N_2$ and $N_C(x') \cap N_C(y') \cap N_2$, respectively, note that in this case, $dia(H) \geqslant 4$), i.e.,

$$|W_1| \geqslant m_1 - 2. \tag{4.16}$$

Similarly, we obtain

$$\text{if } N_3 \cup N_4 \cup M_2 \neq \emptyset \quad \text{then } |W_1| = m_1. \tag{4.17}$$

Claim 4. *If $m_2 \geqslant 3$, then for any pair of vertices $\{w, w'\}$ of M_2, a cycle of $C[\{w, w'\} \cup ((N_C(w) \cup N_C(w')) \cap S)]$ which does not contain the possible edge ww', is a nontrivial cut set of C, i.e., deleting it creates at least two nontrivial components of C.*

Proof of Claim 4. Otherwise, $C[\{w, w'\} \cup ((N_C(w) \cup N_C(w')) \cap S)]$ has a cycle C' which does not contain ww', such that $\varepsilon(C'') \leqslant k \leqslant \overline{\varepsilon}(C'')$ where $C'' = C - C'$, a contradiction. This completes the proof of Claim 4. □

Let W_2 denote the cut vertex set of C in M_2 such that for any vertex $y \in W_2$, $C - y$ has a nontrivial component which does not contain any element of S. By Claim 4, every vertex in M_2 is in W_2 except at most two vertices, w_0, w_{00}

(say); in this exceptional case, deleting the edges of $(\cup_{u\in S}\{w_0u, w_{00}u\}) \cap E(C)$ we will get exactly one component of C that does not contain any element of S (In fact, the component is nontrivial if $m_2 \geqslant 3$). Hence we obtain

$$|W_2| \geqslant m_2 - 2. \tag{4.18}$$

In a similar way, we obtain

$$\text{if } n_3 = 1 \quad \text{then } |W_2| \geqslant m_2 - 1, \tag{4.19}$$

and

$$\text{if } n_4 = 1 \quad \text{then } |W_2| = m_2. \tag{4.20}$$

For $y \in W_1 \cup W_2$, take one nontrivial component of $C - y$ which does not contain any element of S, denoted by Q_y. Then it is easy to see that

$$|V(Q_y)| \geqslant 3. \tag{4.21}$$

Otherwise $\varepsilon(C') \leqslant k \leqslant \bar{\varepsilon}(C')$, where $C'_y = C - Q_y$. We also obtain

$$|\{Q_y : y \in W_1 \cup W_2\}| = |W_1 \cup W_2|. \tag{4.22}$$

If $N_3 \cup N_4 \neq \emptyset$ then, using Claims 1–4 and (4.17) up to (4.22), we obtain

$$\begin{aligned} d_C(S) &= \left|\bigcup_{i=1}^{4} N_i\right| + n_2 + 2n_3 + 3n_4 \\ &\leqslant \begin{cases} n - (|V_2(D_1)| + |V_2(D_r)| - 4 + 3n_2) + n_2 + 3 & \text{if } n_4 = 1, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 5 + 3(n_2 - 1)) + n_2 + 2 & \text{if } n_2 \geqslant 2, \ \ n_3 = 1, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 5) + 3 & \text{if } n_2 \leqslant 1, \ \ n_3 = 1, \end{cases} \\ &\leqslant n - (|V_2(D_1)| + |V_2(D_r)|) + 8. \end{aligned}$$

If $N_3 \cup N_4 = \emptyset$ then, using Claims 3, 4 and (4.15), (4.16), (4.18), (4.21), (4.22), we obtain

$$\begin{aligned} d_C(S) &= \left|\bigcup_{i=1}^{4} N_i\right| + n_2 + 2n_3 + 3n_4 = \left|\bigcup_{i=1}^{4} N_i\right| + n_2 \\ &\leqslant \begin{cases} n - (|V_2(D_1)| + |V_2(D_r)| - 5 + 3(m_1 - 2) + 3(m_2 - 2)) + m_1 + m_2 & \text{if } m_1 \geqslant 2, \ \ m_2 \geqslant 3, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 5 + 3(m_1 - 2)) + m_1 + 2 & \text{if } m_1 \geqslant 2, \ \ m_2 \leqslant 2, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 5 + 3(m_2 - 2)) + 1 + m_2 & \text{if } m_1 \leqslant 1, \ \ m_2 \geqslant 3, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 5) + 3 & \text{if } m_1 = 1, \ \ m_2 \leqslant 2, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 6) + 2 & \text{if } m_1 = 0, \ \ m_2 \leqslant 2, \end{cases} \\ &\leqslant n - (|V_2(D_1)| + |V_2(D_r)|) + 8. \end{aligned}$$

Hence we conclude that

$$d_C(S) \leqslant n - |V_2(D_1)| - |V_2(D_r)| + 8. \tag{4.23}$$

On the other hand, by (4.8), $d_C(P) + d_C(P') > n + 14 - 2(|V_2(D_1)| + |V_2(D_r)|)/3$. Hence

$$d_C(S) > n + 6 - 2(|V_2(D_1)| + |V_2(D_r)|)/3. \tag{4.24}$$

Using (4.23) and (4.24), we obtain

$$|V_2(D_1)| + |V_2(D_r)| < 6,$$

which contradicts (4.5).

Case 2: $dia(H)=1$ and $|V(D_1)\cap V(D_r)|=1$.

Then H is a complete graph.

Let $V(D_1)\cap V(D_r)=\{y\}$. We will consider the following two subcases.

Subcase 2.1: $|V(D_i)|\geqslant 5$ for $i\in\{1,r\}$.

Hence, we can take two paths $P=u'v'x'y'$ and $P'=u''v''x''y''$ of length 3 in D_1 and D_r, respectively, such that $\{uv,u'v'\}\subseteq E_2(D_1)\cup E_2(D_r)$ and $V(P)\cap V(P')=\emptyset$. By (4.10), $|V(P)\cap V_1(D_1)|\geqslant 1$ and $|V(P')\cap V_1(D_r)|\geqslant 1$. Using similar arguments as in Case 1, we derive a contradiction. We omit the details.

Subcase 2.2: There exists a D_i $(i\in\{1,r\})$, say D_1, such that $|V(D_1)|=4$.

We will prove that in this case $G\cong F$. To see this, we need the following three claims.

Claim 5. $|V(D_r)|=4$.

Proof of Claim 5. Suppose otherwise that $|V(D_r)|\geqslant 5$. Hence by (4.10), we can take a path $P'=u'v'x'y'$ in D_r such that $u'v'\in E_2(D_r)$, $y\notin\{x',y'\}$ and $|\{x',y'\}\cap V_2(D_r)|\leqslant 1$. By (4.5), we can take a path $P=uvxy$ in D_1 such that $\{uv,vx\}=E_2(D_1)$.

In a way similar to Claims 2, 3, we obtain

$$n_3=n_4=0 \quad\text{and}\quad n_2\leqslant 1. \tag{4.25}$$

By (4.7),

$$d_C(P)+d_C(P')>n+14-|E_2(D_1)|-|E_2(D_r)|. \tag{4.26}$$

If $|E_2(D_r)|=2$ or 3, then by (4.26),

$$d_C(S)>n+6-|E_2(D_1)|-|E_2(D_r)|\geqslant n+1. \tag{4.27}$$

On the other hand, by (4.25),

$$d_C(S)=\left|\bigcup_{i=1}^{4}N_i\right|+n_2+2n_3+3n_4\leqslant n+1,$$

which contradicts (4.27).

If $|E_2(D_r)|\geqslant 4$, then using (4.8), we obtain $|V_2(D_r)|\geqslant 3|E_2(D_r)|/2\geqslant 6$. This implies that by (4.5),

$$|V_2(D_1)|+|V_2(D_r)|\geqslant 9. \tag{4.28}$$

On the other hand, using (4.25), we obtain

$$\begin{aligned}d_C(S)&=\left|\bigcup_{i=1}^{4}N_i\right|+n_2+2n_3+3n_4\\&\leqslant n-(|V_2(D_1)|+|V_2(D_r)|-8)+1\\&=n-|V_2(D_1)|-|V_2(D_r)|+9,\end{aligned}$$

implying that by (4.24),

$$|V_2(D_1)|+|V_2(D_r)|<3(9-6)=9,$$

which contradicts (4.28). This completes the proof of Claim 5. □

Claim 6. $|V(D_i)|=4$ *for* $i\in\{1,2,\ldots,r\}$.

Proof of Claim 6. Suppose otherwise that there exists a D_i $(i\in\{2,3,\ldots,r-1\})$, say D_2, such that $|V(D_2)|\neq 4$. If $|V(D_2)|=3$, then $C'=C-D_2$ is a circuit with $\varepsilon(C')\leqslant k\leqslant\bar{\varepsilon}(C')$, a contradiction. Hence $|V(D_2)|\geqslant 5$, and D_2 plays the same role as D_r in this subcase since $|V(D_1)\cap V(D_2)|=1$. In a way similar to the proof of Claim 5, we obtain a contradiction which completes the proof of Claim 6. □

Claim 7. $|V(D_i) \cap V(D_j)| = 1$ *for* $\{i, j\} \subseteq \{1, 2, \ldots, r\}$.

Proof of Claim 7. Suppose otherwise that there exists a pair of $\{i, j\}$ such that $|V(D_i) \cap V(D_j)| = 2$, then by Claim 6 there exists a triangle C_3 in $D_i \cup D_j$. Thus, $C' = C - C_3$ is a circuit with $\varepsilon(C') \leqslant k \leqslant \bar{\varepsilon}(C')$, a contradiction. □

By Claims 6 and 7, $C \cong F$. We claim that $d_C(u) = d(u)$ for any $u \in V(C)$. Suppose otherwise that there exists a vertex $x \in V(C)$ such that $d_C(x) < d(x)$, then $C_i = C - D_i$ ($x \notin V(D_i)$ unless $x = y$) is a circuit with $\varepsilon(C_i) \leqslant k \leqslant \bar{\varepsilon}(C_i)$, a contradiction. This settles our claim. Since G is connected, $V(G) = V(C)$ and $G \cong C \cong F$.

Case 3: $dia(H) = 1$ and $|V(D_1) \cap V(D_r)| = 2$.

Then again H is a complete graph.

Let $V(D_1) \cap V(D_r) = \{u, v\}$. Hence there exist four paths P_1, P_2, P_3, P_4 such that $D_1 = P_1 \cup P_2$, $D_r = P_3 \cup P_4$, and $V(P_i) \cap V(P_j) = \{u, v\}$ for $\{i, j\} \subseteq \{1, 2, 3, 4\}$.

In order to derive contradictions, we first prove the following two claims.

Claim 8. $|(V(P_s) \cup V(P_t)) \cap V(D_i)| = 2$ *for any pair of* $\{s, t\} \subseteq \{1, 2, 3, 4\}$ *and* $i \in \{2, 3, \ldots, r-1\}$.

Proof of Claim 8. Suppose otherwise that $|(V(P_s) \cup V(P_t)) \cap V(D_i)| \neq 2$ for some pair of $\{s, t\} \subseteq \{1, 2, 3, 4\}$ and some $i \in \{2, 3, \ldots, r-1\}$, say D_2, such that $|(V(P_s) \cup V(P_t)) \cap V(D_2)| \neq 2$. By the fact that r is maximized, it follows that $|(V(P_s) \cup V(P_t)) \cap V(D_2)| \leqslant 1$. Hence let $D'_1 = P_s \cup P_t$, $D'_r = (D_1 \cup D_r) - (P_s \cup P_t)$ and $D'_j = D_j$ for $j \in \{2, 3, \ldots, r-1\}$. Let H' be a graph with vertex set $V(H') = \{D'_1, D'_2, \ldots, D'_r\}$, $D'_i D'_j \in E(H')$ if and only if $V(D'_i) \cap V(D'_j) \neq \emptyset$. Obviously, H' is a complete graph. Note that D'_1 and D'_2 in H' play the same role as D_1 and D_r in H, respectively. Since $|V(D'_1) \cap V(D'_2)| \leqslant 1$, we derive contradictions in a similar way as in the proof of Case 1 or 2. □

Claim 9. $\{u, v\} \subseteq V(D_i)$ *for* $i \in \{2, 3, \ldots, r-1\}$.

Proof of Claim 9. Suppose otherwise that there exists a D_i, say D_2, such that $|\{u, v\} \cap V(D_2)| \leqslant 1$.

If $|\{u, v\} \cap V(D_2)| = 1$, say, $u \in V(D_2)$, then by (4.3) there exist two paths P_s and P_t such that $V(P_s) \cap V(D_2) = V(P_t) \cap V(D_2) = \{u\}$, where $s \in \{1, 2\}$ and $t \in \{3, 4\}$. This contradicts Claim 8.

If $\{u, v\} \cap V(D_2) = \emptyset$, then we claim that there exists a P_i, $i \in \{1, 2, 3, 4\}$, such that $|V(P_i) \cap V(D_2)| = 0$ or 2. Suppose otherwise that $|V(P_j) \cap V(D_2)| = 1$ for $j \in \{1, 2, 3, 4\}$, then there exist four edge-disjoint cycles C_1, C_2, C_3, C_4 in $D_1 \cup D_2 \cup D_r$ such that $D_1 \cup D_2 \cup D_r = C_1 \cup C_2 \cup C_3 \cup C_4$ which contradicts the maximality of r. Thus, $|V(P_i) \cap V(D_2)| = 0$ or 2. Without loss of generality we assume that $|V(P_1) \cap V(D_2)| = 0$ or 2. By (4.3) and $\{u, v\} \cap V(D_2) = \emptyset$, there exists an $s \in \{3, 4\}$ such that $|V(P_s) \cap V(D_2)| \leqslant 1$; if $|V(P_1) \cap V(D_2)| = 2$ then, by (4.3), $|(V(P_s) \cup V(P_2)) \cap V(D_2)| \leqslant 1$which contradicts Claim 8; if $|V(P_1) \cap V(D_2)| = 0$ then, $|(V(P_s) \cup V(P_1)) \cap V(D_2)| \leqslant 1$ which contradicts Claim 8. This completes the proof of Claim 9. □

It follows from Claim 9 that there exist $2r$ $(= d_C(u) = d_C(v))$ edge-disjoint paths $P_1, P_2, \ldots, P_{2r}$ such that $C = \bigcup_{i=1}^{2r} P_i$ and $V(P_i) \cap V(P_j) = \{u, v\}$ $(i \neq j)$.

Hence by (4.10), we obtain the following:

$$|V(P_i)| \leqslant 5 \quad \text{for } i \in \{1, 2, \ldots, 2r\}. \tag{4.29}$$

Now it is easy to verify that

$$|E_2(D_1)| + |E_2(D_r)| \leqslant 4 + \min\{4, |\{i \in \{1, 2, \ldots, 2r\} : |V(P_i)| = 5\}|\}. \tag{4.30}$$

By (4.5), there exist two paths P'_1 and P'_2 of length 3 in D_1 and D_r, respectively such that each path contains exactly one vertex in $V_1(D_1) \cup V_1(D_r)$. Let u, v denote two such vertices in P'_1 and P'_2, respectively, i.e., $u \in V(P'_1) \cap V_1(D_1)$ and $v \in V(P'_2) \cap V_1(D_r)$. Then by (4.7), $d_C(P'_1) + d_C(P'_2) \geqslant n + 14 - |E_2(D_1)| - |E_2(D_r)|$. Hence

$$d_C(u) + d_C(v) > n + 2 - |E_2(D_1)| - |E_2(D_r)|. \tag{4.31}$$

If there exists a pair $\{i, j\}$ such that $|V(P_i)| \leqslant 3$ and $|V(P_j)| \leqslant 4$, then $C' = C - (P_i \cup P_j)$ is a circuit with $\varepsilon(C') \leqslant k \leqslant \bar{\varepsilon}(C')$, which contradicts Lemma 9. Hence we obtain

$$\{(i, j) : |V(P_i)| \leqslant 3 \quad \text{and} \quad |V(P_j)| \leqslant 4\} = \emptyset. \tag{4.32}$$

Without loss of generality, we can assume that $|V(P_1)|, |V(P_2)|, \ldots, |V(P_{2r})|$ is an increasing sequence and $D_i = P_{2i-1} \cup P_{2i}$ for $i \in \{1, 2, \ldots, r\}$.

We need one more claim.

Claim 10. *If* $|V(P_1)| = 2$, *i.e.*, $uv \in E(C)$, *then* $r \geqslant 4$.

Proof of Claim 10. Suppose otherwise that $r = 2$ or 3. By (4.29) and (4.32), $|V(P_i)| = 5$ for $i \in \{2, 3, \ldots, 2r\}$. Since $n \geqslant 146$, there exists a vertex x of C with $d_C(x) \leqslant d(x) - 1$. Hence there exists a circuit C' such that $\varepsilon(C') \leqslant k \leqslant \bar{\varepsilon}(C')$, where $C' = C - (P_i \cup P_1)$ ($P_i \neq P_1$, and $x \notin V(P_i)$ unless $x \in \{u, v\}$), which contradicts Lemma 9. This completes the proof of Claim 10. □

Finally, we will obtain some inequalities which contradict (4.30).

If $3 \leqslant |V(P_1)| \leqslant 5$ then, using (4.32), we obtain

$$d_C(u) + d_C(v) \leqslant n - 2 - |\{i \in \{1, 2, \ldots, 2r\} : |V(P_i)| = 5\}|. \tag{4.33}$$

Combining (4.31) and (4.33) yields an inequality that contradicts (4.30).

If $|V(P_1)| = 2$, i.e., $uv \in E(C)$, then using (4.29) and (4.32), we obtain that $|V(P_i)| = 5$ for $i \geqslant 2$, and

$$d_C(u) + d_C(v) \leqslant n - |\{i \in \{1, 2, \ldots, 2r\} : |V(P_i)| = 5\}|. \tag{4.34}$$

Using (4.31), (4.32), (4.34) and Claim 10, we obtain

$$|E_2(D_1)| + |E_2(D_r)| > 2 + |\{i \in \{1, 2, \ldots, 2r\} : |V(P_i)| = 5\}| = 2 + (2r - 1) \geqslant 9,$$

which contradicts (4.30).

This completes the proof of the first conclusion of the theorem. We now prove that the results in Theorem 6 are best possible in the sense that the condition $\rho_4(G) > (n + 10)/2$ cannot be relaxed, even under the condition that $L(G)$ is hamiltonian. To see this, we construct a graph G_0 as follows:

Let $s = (n-2)/2$ ($n \equiv 2 \pmod 4$) and let $V(G_0) = \{u_1, v_1, u_2, v_2, \ldots, u_s, v_s, x, y\}$ and $E(G_0) = \bigcup_{i=1}^{s} \{xu_i, u_i v_i, v_i y\}$. Clearly G_0 is a graph with $\rho_4(G_0) = s + 6 = (n + 10)/2$. Theorem 8 implies that $L(G_0)$ is hamiltonian and $3s - 1 \in [3, \varepsilon(G_0)] \backslash \lambda(L(G_0))$, which implies that $L(G_0)$ is not (sub)pancyclic. This completes the proof of Theorem 6. □

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