Long Cycles in Graphs with Large Degree Sums and Neighborhood Unions

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ABSTRACT

We present and prove several results concerning the length of longest cycles in 2-connected or 1-tough graphs with large degree sums. These results improve many known results on long cycles in these graphs. We also consider the sharpness of the results and discuss some possible strengthenings. © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION AND RESULTS

We use Bondy and Murty [5] for terminology and notation not defined here and consider simple graphs only. If G is a graph and $S \subseteq V(G)$, then $N(S)$ denotes the *neighborhood of S*, that is, the set of all vertices in G adjacent to at least one vertex in S. If $S = \{u\}$, we write $N(u)$ instead of $N(\{u\})$ and set $d(u) = |N(u)|$. We write $\omega(G)$ for the number of components of G and $\alpha(G)$ for its vertex independence number.

For an integer $t \geq 1$ we define the parameters $\sigma_t(G)$ and $NC_t(G)$ by

$$\sigma_t(G) = \min\left\{ \sum_{v \in S} d(v) \,|\, S \subseteq V(G) \text{ is an independent set with } |S| = t \right\},$$

$$NC_t(G) = \min\{|N(S)| \,|\, S \subseteq V(G) \text{ is an independent set with } |S| = t\}.$$

In these definitions we follow the convention that the minimum over an empty set is $+\infty$. The definitions imply that $\sigma_1(G) = \mathrm{NC}_1(G) = \delta(G)$ (the minimum vertex degree of G), and

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$NC_t \geq (1/t)\sigma_t(G) \geq \delta(G)$ in general. We define $\overline{\delta}(G) = \lceil \frac{1}{3}\sigma_3(G) \rceil$, so that

$$NC_3(G) \geq \overline{\delta}(G) \geq \tfrac{1}{3}\sigma_3(G) \geq \delta(G). \tag{1}$$

A graph G is called *1-tough* if $\omega(G - S) \leq |S|$ for every nonempty subset S of $V(G)$. Let $\mathcal{G}_1(n)$ denote the class of 1-tough graphs on $n \geq 3$ vertices such that $\sigma_3(G) \geq n$, and let $\mathcal{G}_2(n)$ denote the class of 2-connected graphs on n vertices such that $\sigma_3(G) \geq n + 2$.

The lengths of a longest path and a longest cycle in G (the numbers of edges in a longest path and a longest cycle) are denoted by $p(G)$ and $c(G)$, respectively.

A classical theorem from Dirac [7] states that a graph G on $n \geq 3$ vertices with $\delta(G) \geq \frac{1}{2}n$ contains a Hamilton cycle. This result was generalized as follows.

Theorem 1 (Nash-Williams [13]). If G is a 2-connected graph on n vertices such that $\delta(G) \geq \max\{\frac{1}{3}(n + 2), \alpha(G)\}$, then G contains a Hamilton cycle.

By (1), Theorem 1 is implied by Theorem 2.

Theorem 2 (Bauer, Morgana, Schmeichel and Veldman [2]). If $G \in \mathcal{G}_2(n) \cup \mathcal{G}_1(n)$, then $c(G) \geq \min\{n, n + \frac{1}{3}\sigma_3(G) - \alpha(G)\}$.

Theorem 2 implies several known results, as shown in the surveys Bauer, Broersma and Veldman [1] and Bauer, Schmeichel and Veldman [3]. Here we further generalize Theorem 2.

Theorem 3.

(a) If $G \in \mathcal{G}_2(n)$ and $t \in \mathbb{N}$, then

$$c(G) \geq \min\{n, \tfrac{1}{2}(n + 3\overline{\delta}(G) + 1 - t), n + NC_t(G) - \alpha(G)\}.$$

(b) If $G \in \mathcal{G}_1(n)$ and $t \in \mathbb{N}$, then

$$c(G) \geq \min\{n, \tfrac{1}{2}(n + 3\overline{\delta}(G) + 4 - t), n + NC_t(G) - \alpha(G)\}.$$

Theorem 3 implies Theorem 2 by (1). The proof of Theorem 3 is postponed to Section 5.

Theorem 3 would not be true if the lower bounds on $\sigma_3(G)$ were relaxed in the definitions of $\mathcal{G}_2(n)$ and $\mathcal{G}_1(n)$; this is already the case for Theorem 2 as is shown by examples in [2]. Here we present examples showing that both the lower bounds on $c(G)$ and the subscripts of $NC(G)$ in the conclusion of Theorem 3 (a) cannot be increased in general.

Let $k, \ell, m \in \mathbb{N}$ such that $2k \geq \ell + 2m + 2$ and $\ell + m \geq k + 1$ (hence $k \geq m + 3$). Define the graph $G_{k,\ell,m}$ as the join of K_k and $\ell K_1 + mK_2$, where "+" denotes the union of graphs (see Fig. 1). $G_{k,\ell,m}$ is a 2-connected graph with $n = |V(G_{k,\ell,m})| = k + \ell + 2m$, $\sigma_3(G_{k,\ell,m}) = 3k \geq n + 2$, $\alpha(G_{k,\ell,m}) = \ell + m$ and $c(G_{k,\ell,m}) = 2k + m$. Furthermore, we have $\overline{\delta}(G_{k,\ell,m}) = k$ and $NC_{\ell+x}(G_{k,\ell,m}) = k + x$ if $0 \leq x \leq m$. This means that for $t = \ell$ we have

$$\tfrac{1}{2}(n + 3\overline{\delta}(G_{k,\ell,m}) + 1 - t) = 2k + m + \tfrac{1}{2} > c(G_{k,\ell,m}),$$
$$n + NC_t(G_{k,\ell,m}) - \alpha(G_{k,\ell,m}) = 2k + m = c(G_{k,\ell,m}),$$

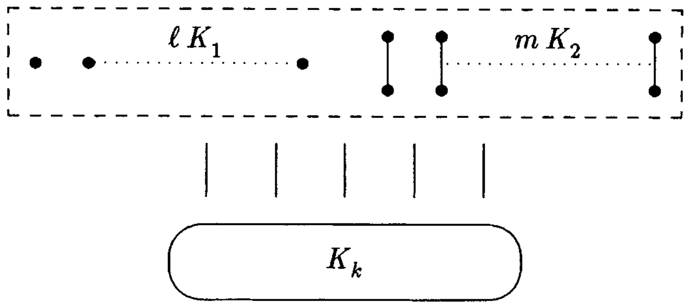

FIGURE 1. The graph $G_{k,\ell,m}$. The vertical lines indicate a join.

and for $t = \ell + 1$ we obtain

$$\tfrac{1}{2}(n + 3\overline{\delta}(G_{k,\ell,m}) + 1 - t) = 2k + m = c(G_{k,\ell,m}),$$

$$n + \mathrm{NC}_t(G_{k,\ell,m}) - \alpha(G_{k,\ell,m}) = 2k + m + 1 > c(G_{k,\ell,m}).$$

Hence Theorem 3(a) is best possible if $\sigma_3(G) \equiv 0 \pmod 3$. The graphs obtained from $G_{k,\ell,m}$ by deleting one or two edges incident with an isolated vertex of $G_{k,\ell,m} - V(K_k)$ show that Theorem 3(a) is also best possible if $\sigma_3(G) \not\equiv 0 \pmod 3$.

Note that the graphs $G_{k,\ell,m}$ satisfy $|V(G_{k,\ell,m})| - c(G_{k,\ell,m}) = \ell + m - k$. The graphs $H_{k,\ell,1}$ to be introduced in Section 2 can be used to show that Theorem 3(b) is sharp for graphs G in $\mathcal{G}_1(n)$ with $|V(G)| - c(G) = 1$. We do not know of the existence of graphs $G \in \mathcal{G}_1(n)$ with $|V(G)| - c(G) \geq 2$ for which Theorem 3(b) is best possible. This is why we believe an improvement of Theorem 3(b) is possible. We discuss this further in Section 2.

Theorem 3 generalizes several known results. A first corollary is the following.

Corollary 4 (Broersma, Van den Heuvel, and Veldman [6]).

(a) If $G \in \mathcal{G}_2(n)$, then $c(G) \geq \min\{n, n + \mathrm{NC}_{3\overline{\delta}(G)-n+2}(G) - \alpha(G)\}$.
(b) If $G \in \mathcal{G}_1(n)$, then $c(G) \geq \min\{n, n + \mathrm{NC}_{3\overline{\delta}(G)-n+5}(G) - \alpha(G)\}$.

Proof (a) Set $t = 3\overline{\delta}(G) - n + 2$, hence $\tfrac{1}{2}(n + 3\overline{\delta}(G) + 1 - t) = n - \tfrac{1}{2}$. Theorem 3(a) gives $c(G) \geq \lceil \min\{n, n - \tfrac{1}{2}, n + \mathrm{NC}_t(G) - \alpha(G)\} \rceil = \min\{n, n + \mathrm{NC}_t(G) - \alpha(G)\}$.

(b) The proof of (b) is completely analogous to the proof of (a). ∎

If G is a graph on n vertices and $t \in \mathbb{N}$ such that $t \leq \alpha(G)$, then clearly $\mathrm{NC}_t(G) \leq n - \alpha(G)$, hence $2\mathrm{NC}_t(G) \leq n + \mathrm{NC}_t(G) - \alpha(G)$. So Theorem 3 has the following consequences.

Corollary 5.

(a) If $G \in \mathcal{G}_2(n)$ and $t \in \mathbb{N}$, then $c(G) \geq \min\{n, \tfrac{1}{2}(n + 3\overline{\delta}(G) + 1 - t), 2\mathrm{NC}_t(G)\}$.
(b) If $G \in \mathcal{G}_1(n)$ and $t \in \mathbb{N}$, then $c(G) \geq \min\{n, \tfrac{1}{2}(n + 3\overline{\delta}(G) + 4 - t), 2\mathrm{NC}_t(G)\}$.

Corollary 6 (Broersma, Van den Heuvel and Veldman [6]).

(a) If $G \in \mathcal{G}_2(n)$, then $c(G) \geq \min\{n, 2\mathrm{NC}_{3\overline{\delta}(G)-n+2}(G)\}$.
(b) If $G \in \mathcal{G}_1(n)$, then $c(G) \geq \min\{n, 2\mathrm{NC}_{3\overline{\delta}(G)-n+5}(G)\}$.

Like Theorems 2 and 3, Corollaries 5 and 6 would not be true if the lower bounds on $\sigma_3(G)$ in the definitions of $\mathcal{G}_2(n)$ or $\mathcal{G}_1(n)$ were relaxed. Also, Corollaries 5(a) and 6(a) are sharp in the sense that longer cycles are not implied by the hypothesis, as shown by suitable complete bipartite graphs. However, in contrast with Theorem 3, the subscript of NC(G) in the corollaries can be improved, as is shown in the next theorem, which will be proved in Section 5.

Theorem 7. If $G \in \mathcal{G}_2(n) \cup \mathcal{G}_1(n)$ and $t \in \mathbb{N}$, then

$$c(G) \geq \min\{n, \tfrac{1}{3}(2n + 4\overline{\delta}(G) + 1 - 2t), 2\mathrm{NC}_t(G)\}.$$

We first show that Theorem 7 is a full generalization of Corollary 5(a). Both results give $c(G) \geq 2\mathrm{NC}_3(G) \geq 2\lceil \frac{1}{3}\sigma_3(G)\rceil = 2\overline{\delta}(G)$. Hence it suffices to show that

$$\lceil \tfrac{1}{3}(2n + 4\overline{\delta}(G) + 1 - 2t)\rceil \geq \lceil \tfrac{1}{2}(n + 3\overline{\delta}(G) + 1 - t)\rceil, \tag{2}$$

whenever t satisfies $\frac{1}{2}(n + 3\overline{\delta}(G) + 1 - t) > 2\overline{\delta}(G)$. The latter inequality is equivalent to $t < n - \overline{\delta}(G) + 1$ and hence to $t \leq n - \overline{\delta}(G)$, which does indeed imply (2).

Theorem 7 does not completely generalize Corollary 5(b). If $\sigma_3(G) \leq \frac{3}{2}(n - 13)$, hence $\overline{\delta}(G) \leq \lceil \frac{1}{2}(n - 13)\rceil \leq \frac{1}{2}(n - 13) + \frac{1}{2} = \frac{1}{2}n - 6$, then $n - \overline{\delta}(G) - 7 \geq \frac{1}{2}n - 1$. Since $\mathrm{NC}_{\lceil n/2\rceil - 1}(G) \geq \lceil \frac{1}{2}n\rceil$ for a 1-tough graph G on n vertices, this means that if $\sigma_3(G) \leq \frac{3}{2}(n - 13)$, then the best lower bound for $c(G)$ in both Corollary 5(b) and Theorem 7 will occur for $t \leq n - \overline{\delta}(G) - 8$, which implies $\lceil \frac{1}{3}(2n + 4\overline{\delta}(G) + 1 - 2t)\rceil \geq \lceil \frac{1}{2}(n + 3\overline{\delta}(G) + 4 - t)\rceil$. So Theorem 7 generalizes Corollary 5(b) if $\sigma_3(G) \leq \frac{3}{2}(n - 13)$.

Note that if $G \in \mathcal{G}_1(n)$ satisfies the hypothesis of Theorem 7 with $\sigma_3(G) \geq \frac{3}{2}n - 1$, then G is hamiltonian. This follows from $2\mathrm{NC}_3(G) \geq 2 \cdot \frac{1}{3}\sigma_3(G) \geq n - \frac{2}{3}$ and, since $n \geq 3$ and $\overline{\delta}(G) \geq \lceil \frac{1}{3}(\frac{3}{2}n - 1)\rceil = \lceil \frac{1}{2}n - \frac{1}{3}\rceil = \lceil \frac{1}{2}n\rceil$,

$$\tfrac{1}{3}(2n + 4\overline{\delta}(G) + 1 - 2 \cdot 3) \geq \tfrac{1}{3}(4n - 5) > n - 1.$$

Next we show that both the lower bound on $c(G)$ and the subscript of NC(G) in the conclusion of Theorem 7 cannot be increased in general if $G \in \mathcal{G}_2(n)$. This is shown by the graphs $G_{k,\ell,m}$ defined after Theorem 3. For $t = \ell + \lfloor \frac{1}{2}m\rfloor$ we have

$$\begin{aligned}\tfrac{1}{3}(2n + 4\overline{\delta}(G_{k,\ell,m}) + 1 - 2t) &\geq 2k + m + \tfrac{1}{3} > c(G_{k,\ell,m}),\\ 2\mathrm{NC}_t(G_{k,\ell,m}) &\leq 2k + m = c(G_{k,\ell,m}),\end{aligned}$$

and for $t = \ell + \lfloor \frac{1}{2}m\rfloor + 1$ we obtain

$$\begin{aligned}\tfrac{1}{3}(2n + 4\overline{\delta}(G) + 1 - 2t) &\leq 2k + m = c(G_{k,\ell,m}),\\ 2\mathrm{NC}_t(G_{k,\ell,m}) &\geq 2k + m + 1 > c(G_{k,\ell,m}).\end{aligned}$$

Hence Theorem 7 is best possible if $G \in \mathcal{G}_2(n)$ and $\sigma_3(G) \equiv 0 \pmod 3$. Again, by deleting one or two suitable edges of $G_{k,\ell,m}$, we can prove it is best possible for $\sigma_3(G) \not\equiv 0 \pmod 3$.

We do not believe that Theorem 7 is best possible for $G \in \mathcal{G}_1(n)$. A possible improvement will be discussed in Section 2.

Corollary 8. If $G \in \mathcal{G}_2(n) \cup \mathcal{G}_1(n)$ and $t \in \mathbb{N}$ with $t \le \frac{1}{2}(4\overline{\delta}(G) - n + 3)$, then $c(G) \ge \min\{n, 2\mathrm{NC}_t(G)\}$.

Proof We have $\frac{1}{3}(2n + 4\overline{\delta}(G) + 1 - 2t) \ge \frac{1}{3}(3n - 2) = n - \frac{2}{3}$. So by Theorem 7, $c(G) \ge \min\{n, n - \frac{2}{3}, 2\mathrm{NC}_t(G)\}$. We can conclude $c(G) \ge \min\{n, 2\mathrm{NC}_t(G)\}$. ∎

The result in Corollary 8 for $G \in \mathcal{G}_2(n)$ was conjectured in [6]. There a weaker version with $t \le \frac{1}{8}(6\sigma_3(G) - 5n + 17)$ was proved.

Theorem 7 also generalizes the following result.

Corollary 9 (Häggkvist [9]). Let G be a nonhamiltonian 2-connected graph on n vertices such that $\delta(G) \ge \frac{1}{3}(n + 2)$ and let $t = 2\delta(G) - \lceil \frac{1}{2}(n + 1) \rceil + 2$. Then $\mathrm{NC}_t(G) \le \frac{1}{2}(n - 1)$.

Proof. We have $\sigma_3(G) \ge 3\delta(G) \ge n + 2$. Furthermore, $t \le 2\delta(G) - \frac{1}{2}(n + 1) + 2 \le \frac{1}{2}(4\overline{\delta}(G) - n + 3)$, since $\delta(G) \le \overline{\delta}(G)$. By Corollary 8 this means $c(G) \ge \min\{n, 2\mathrm{NC}_t(G)\}$. Since G is nonhamiltonian, $c(G) \le n - 1$, hence $\mathrm{NC}_t(G) \le \frac{1}{2}(n - 1)$. ∎

The remainder of this paper is organized as follows. In Section 2 we discuss some possible strengthenings of our results for 1-tough graphs. In Sections 3 and 4 we prove and cite some preliminary results. More particularly, in Section 3 we prove a variant of the Hopping Lemma from Woodall [14]. The main results, Theorems 3 and 7, are proved in Section 5.

2. SOME CONJECTURES ON 1-TOUGH GRAPHS

In this section we discuss some possible strengthenings of the results for 1-tough graphs.

Conjecture 10. If $G \in \mathcal{G}_1(n)$ and $t \in \mathbb{N}$, then

$$c(G) \ge \min\{n, \tfrac{1}{4}(3n + 3\overline{\delta}(G) + 2 - t), n + \tfrac{1}{2}(\mathrm{NC}_t(G) - \alpha(G) - 1)\}.$$

Conjecture 10, if true, is a full generalization of Theorem 3(b). This follows from the fact that for $t \ge 3\overline{\delta}(G) - n + 6$ we have $\frac{1}{4}(3n + 3\overline{\delta}(G) + 2 - t) \ge \frac{1}{2}(n + 3\overline{\delta}(G) + 4 - t)$, for $t \le 3\overline{\delta}(G) - n + 5$ we have $\frac{1}{2}(n + 3\overline{\delta}(G) + 4 - t) \ge n - \frac{1}{2}$ and $\frac{1}{4}(3n + 3\overline{\delta}(G) + 2 - t) \ge n - \frac{3}{4}$, for $\alpha(G) \ge \mathrm{NC}_t(G) + 1$, we have $n + \frac{1}{2}(\mathrm{NC}_t(G) - \alpha(G) - 1) \ge n + \mathrm{NC}_t(G) - \alpha(G)$, and for $\alpha(G) \le \mathrm{NC}_t(G)$ we have $n + \mathrm{NC}_t(G) - \alpha(G) \ge n$ and $n + \frac{1}{2}(\mathrm{NC}_t(G) - \alpha(G) - 1) \ge n - \frac{1}{2}$. So we obtain that the lower bound on $c(G)$ in Conjecture 10 is always greater than or equal to the lower bound in Theorem 3(b).

If true, Conjecture 10 is best possible, as shown by the following examples. For $m \in \mathbb{N}$, let F_m be the graph obtained from $K_{2m+1} + \overline{K_{2m+1}}$ by adding the edges of a matching between K_{2m+1} and $\overline{K_{2m+1}}$. For $k, \ell, m \in \mathbb{N}$ with $k \ge \ell + 1 \ge 4m + 1$, define the 1-tough graph $H_{k,\ell,m}$ as the join of K_k and $\ell K_1 + (k - \ell - 1)K_2 + F_m$ (see Fig. 2). We have $n = |V(H_{k,\ell,m})| = 3k - \ell + 4m$, $\sigma_3(H_{k,\ell,m}) = 3k \ge n$, $\alpha(H_{k,\ell,m}) = k + 2m$ and $c(H_{k,\ell,m}) = 3k - \ell + 3m$. Furthermore, $\overline{\delta}(H_{k,\ell,m}) = k$ and $\mathrm{NC}_{\ell+x}(H_{k,\ell,m}) = k + x$ if $0 \le x \le k - \ell + 2m$. This means that for $t = \ell + 1$ we have

$$\tfrac{1}{4}(3n + 3\overline{\delta}(H_{k,\ell,m}) + 2 - t) = 3k - \ell + 3m + \tfrac{1}{4} > c(H_{k,\ell,m}),$$
$$n + \tfrac{1}{2}(\mathrm{NC}_t(H_{k,\ell,m}) - \alpha(H_{k,\ell,m}) - 1) = 3k - \ell + 3m = c(H_{k,\ell,m}),$$

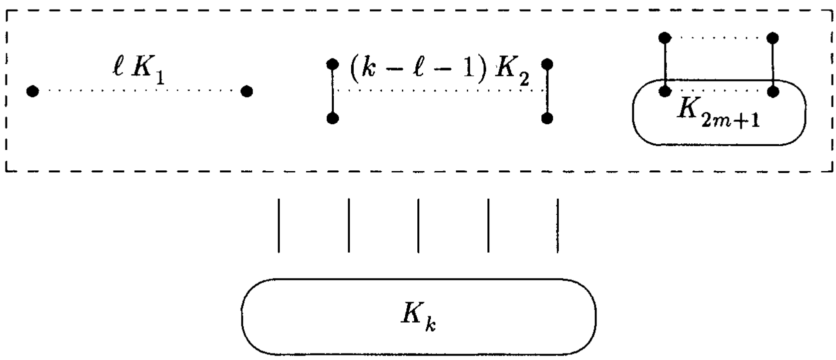

FIGURE 2. The graph $H_{k,\ell,m}$. The vertical lines indicate a join.

and for $t = \ell + 2$ we obtain

$$\tfrac{1}{4}(3n + 3\overline{\delta}(H_{k,\ell,m}) + 2 - t) = 3k - \ell + 3m = c(H_{k,\ell,m}),$$
$$n + \tfrac{1}{2}(\mathrm{NC}_t(H_{k,\ell,m}) - \alpha(H_{k,\ell,m}) - 1) = 3k - \ell + 3m + \tfrac{1}{2} > c(H_{k,\ell,m}).$$

If Conjecture 10 is true, then this implies the truth of the following conjectures, in the same way as Corollaries 4(b) and 5(b) are consequences of Theorem 3(b).

Conjecture 11. If $G \in \mathcal{G}_1(n)$, then $c(G) \geq \min\{n, n + \frac{1}{2}(\mathrm{NC}_{3\overline{\delta}(G)-n+5}(G) - \alpha(G) - 1)\}$.

Conjecture 12. If $G \in \mathcal{G}_1(n)$ and $t \in \mathbb{N}$, then $c(G) \geq \min\{n, \frac{1}{4}(3n + 3\overline{\delta}(G) + 2 - t), \frac{1}{2}(n - 1) + \mathrm{NC}_t(G)\}$.

The consequence of Conjecture 10 that is the analog of Corollary 6(b), reads that a graph $G \in \mathcal{G}_1(n)$ satisfies $c(G) \geq \min\{n, \frac{1}{2}(n - 1) + \mathrm{NC}_{3\overline{\delta}(G)-n+5}(G)\}$. But this immediately follows from Corollary 4(b), since for a 1-tough graph G on $n \geq 3$ vertices we have $\alpha(G) \leq \frac{1}{2}n$.

We do not believe that Conjecture 12 is best possible. In fact, we conjecture the following.

Conjecture 13. If $G \in \mathcal{G}_1(n)$ and $t \in \mathbb{N}$, then $c(G) \geq \min\{n, \frac{1}{6}(5n + 4\overline{\delta}(G) + 2 - 2t), \frac{1}{2}(n - 1) + \mathrm{NC}_t(G)\}$.

If true, Conjecture 13 is best possible as can be seen by considering the graphs $H_{k,\ell,m}$ defined above, and the possibilities $t = \ell + m + \lfloor\frac{1}{2}(k - \ell + 1)\rfloor$ and $t = \ell + m + \lfloor\frac{1}{2}(k - \ell + 1)\rfloor + 1$. The details are left to the reader. Conjecture 13 also means an improvement of Theorem 7.

In the same way as Corollary 8 follows from Theorem 7, the truth of the following conjecture follows from the truth of Conjecture 13.

Conjecture 14. If $G \in \mathcal{G}_1(n)$ and $t \in \mathbb{N}$ with $t \leq \frac{1}{2}(4\overline{\delta}(G) - n + 7)$, then $c(G) \geq \min\{n, \frac{1}{2}(n - 1) + \mathrm{NC}_t(G)\}$.

Let $G \in \mathcal{G}_1(n)$ and set $t = \overline{\delta}(G) + 1$. Then we have $\frac{1}{4}(3n + 3\overline{\delta}(G) + 2 - t) = \frac{1}{4}(3n + 1) + \frac{1}{2}\overline{\delta}(G)$. Also, $\mathrm{NC}_t(G) \geq t + 1$ and $\alpha(G) \leq \frac{1}{2}n$, hence $n + \frac{1}{2}(\mathrm{NC}_t(G) - \alpha(G) -$

$1) \geq \frac{1}{4}(3n + 2) + \frac{1}{2}\overline{\delta}(G)$. This shows that the truth of Conjecture 10 implies the truth of the following conjecture.

Conjecture 15. (Bauer, Morgana, Schmeichel, and Veldman [2]). If $G \in \mathcal{G}_1(n)$, then $c(G) \geq \min\{n, \frac{1}{4}(3n + 1) + \frac{1}{2}\overline{\delta}(G)\} \geq \frac{1}{12}(11n + 3)$.

Setting $t = \overline{\delta}(G) + 1$ and $\alpha(G) \leq \frac{1}{2}n$ in Theorem 3(b) we obtain the following result.

Conjecture 16. If $G \in \mathcal{G}_1(n)$, then $c(G) \geq \min\{n, \frac{1}{2}(n+3) + \overline{\delta}(G)\} \geq \min\{n, \frac{1}{6}(5n+9)\}$.

Recently, the following partial improvement of Corollary 16 was proved, as a first step toward a proof of Conjecture 15.

Theorem 17 (Li[11]). If G is a 1-tough graph on n vertices such that $\frac{1}{3}n \leq \delta(G) \leq \frac{1}{2}(n - 7)$, then

$$c(G) \geq \min\{n, \tfrac{1}{3}(2n + 1 + 2\delta(G))\} \geq \tfrac{1}{9}(8n + 3).$$

3. A HOPPING LEMMA USING ALL VERTICES OUTSIDE THE CYCLE

The Hopping Lemma was obtained in Woodall [14] as a tool for problems concerning paths and cycles in graphs. In this section we prove a version of it for graphs G that satisfy $c(G) \geq p(G)$. This is stronger than the condition imposed on G in the original Hopping Lemma, and this results in a stronger conclusion.

We need some additional terminology and notation. A cycle C in a graph G is called a *dominating cycle* if $G - V(C)$ contains no edges. By $\vec{C}$ we denote the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle with the reverse orientation. If $u, v \in V(C)$, then $u\vec{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. If $u \in V(C)$, then u^+ denotes the successor of u on $\vec{C}$, and u^- denotes its predecessor. If $U \subseteq V(C)$, then $U^+ = \{u^+ | u \in U\}$ and $U^- = \{u^- | u \in U\}$. When more than one cycle is under consideration, we sometimes write $u^{+C}, u^{-C}, \ldots$ instead of just $u^+, u^-, \ldots$ in order to avoid ambiguity.

Similar notation as described above is used for paths. We will use the notation $P = v_1\vec{P}v_p$ to say that P is a path with origin v_1, terminus v_p, and orientation from v_1 to v_p.

In the remainder of this section let G be a nonhamiltonian connected graph with $c(G) \geq p(G)$. Clearly this implies that every longest cycle in G is a dominating cycle. Let C be a longest cycle in G and set $R = V(G) - V(C)$. Fix an orientation $\vec{C}$ on C. Set $Y_0 = \emptyset$ and for $i \geq 1$,

$$X_i = N(Y_{i-1} \cup R) \cap V(C), \qquad Y_i = X_i^+ \cap X_i^-.$$

Set $X = \bigcup_{i=1}^{\infty} X_i$ and $Y = \bigcup_{i=1}^{\infty} Y_i$.

Lemma 18. The sets X and Y satisfy

(a) $X \cap X^+ = \emptyset$ and $X \cap Y = \emptyset$;
(b) $N(Y) \subseteq X$;
(c) Y is an independent set.

For the proof of Lemma 18 we use the notion of a hopping path. If $x \in X$, then the *height* *h(x)* of x is defined by $h(x) = \min\{i | x \in X_i\}$. A path $P = x_1\vec{P}x_2$ is called a *hopping path* if it satisfies the following requirements:

(HP1) $x_1, x_2 \in X$;
(HP2) $V(P) = V(C)$;
(HP3) if $i < \max\{h(x_1), h(x_2)\}$ and $y \in Y_i - \{x_1, x_2\}$, then y^{-P}, $y^{+P} \in X_i$.

The *height h(P)* of a hopping path $P = x_1\vec{P}x_2$ is defined by $h(P) = \max\{h(x_1), h(x_2)\}$.

Lemma 19. If there exists a hopping path, then there exists a hopping path of height 1.

Proof Let $P = x_1\vec{P}x_2$ be a hopping path, chosen such that

(i) *h(P)* is minimum, and
(ii) $h(x_1) + h(x_2)$ is minimum, subject to (i).

Without loss of generality, we may assume $h(x_1) \geq h(x_2)$. If $h(P) = 1$, we are done, so assume $h(P) = i + 1 \geq 2$. We distinguish two cases, in each of which we reach a contradiction.

Case 1. $h(x_2) < h(x_1) = i + 1$.
Then $x_1 \in X_{i+1} - X_i$ and hence there exists a vertex $y \in Y_i - Y_{i-1}$ with $x_1y \in E(G)$. If $y = x_2$, then $y \in X_i \cap Y_i$, so, by the definition of Y_i, also $y^{+C} \in X_i$. Then the path $y^{+C}\vec{C}y$ is a hopping path containing all vertices of $V(C)$ and satisfies $h(y^{+C}\vec{C}y) \leq i \leq h(P)$, contradicting the choice of P. So we have $y \neq x_1, x_2$. By (HP2) and (HP3) this means $y \in V(P)$ and y^{-P}, $y^{+P} \in X_i$. Set $Q = y^{-P}\overleftarrow{P}x_1y\vec{P}x_2$, then Q satisfies (HP1). Since $V(Q) = V(P)$, Q also satisfies (HP2). We have $x_1 \notin Y_{i-1}$, otherwise $y \in X_i \cap Y_i$ and we reach a contradiction as in the case $x_2 \in Y_i$, and $y \notin Y_{i-1}$. Finally, $\{v^{-P}, v^{+P}\} = \{v^{-Q}, v^{+Q}\}$ for each internal vertex v of Q with $v \neq x_1, y$. It follows that Q also satisfies (HP3), whence Q is a hopping path with $h(Q) \leq i < h(P)$, contradicting the chose of P.

Case 2. $h(x_1) = h(x_2) = i + 1$.
In this case there exist vertices $y_1, y_2 \in Y_i - Y_{i-1}$ with $x_1y_1, x_2y_2 \in E(G)$. If $y_1 = x_2$, then $y_1 \in Y_i \cap X_{i+1}$, so, by the definition of $Y_i, y_1^{+C} \in X_i$. Then the path $y_1^{+C}\vec{C}y_1$ is a hopping path that satisfies $h(y_1^{+C}\vec{C}y_1) \leq h(P)$ and $h(y_1^{+C}) + h(y_1) < h(x_1) + h(x_2)$, contradicting the choice of P. So we have $y_1 \neq x_2$ and, similarly, $y_2 \neq x_1$. Now if $y_1 \in x_1\vec{P}y_2$ set $Q = y_1^{-P}\overleftarrow{P}x_1y_1\vec{P}y_2x_2\overleftarrow{P}y_2^{+P}$, and if $y_2 \in x_1\vec{P}y_1$ set $Q = y_2^{-P}\overleftarrow{P}x_1y_1\overleftarrow{P}y_2x_2\overleftarrow{P}y_1^{+P}$. As in Case 1, Q is a hopping path with $h(Q) < h(P)$, contradicting the choice of P. ∎

Lemma 20. There exists no hopping path.

Proof. Suppose $P = x_1\vec{P}x_2$ is a hopping path. By Lemma 19 we may assume $h(P) = 1$. Since $x_1, x_2 \in X_1$, we have $r_1x_1 \in E(G)$ and $r_2x_2 \in E(G)$ for some $r_1, r_2 \in R$. If $r_1 = r_2$, then $r_1x_1\vec{P}x_2r_1$ is a cycle of length $|V(C)| + 1$, contradicting the choice of C; and if $r_1 \neq r_2$, then $r_1x_1\vec{P}x_2r_2$ is a path of length $|V(C)| + 1$, contradicting $|V(C)| = c(G) \geq p(G)$. ∎

Now we can prove the statements in Lemma 18.

Proof of Lemma 18. (a) If $x \in X \cap X^+$ or $x \in X \cap Y$, then $x^- \in X$, hence $x\vec{C}x^-$ is a hopping path, contradicting Lemma 20.

(b) By the definition of X we have $N(R) \subseteq X$ and $N(Y) \cap V(C) \subseteq X$. Since $N(R) \cap Y = \emptyset$ by (a), it follows that $N(Y) \cap R = \emptyset$, hence $N(Y) \subseteq V(C)$. This means $N(Y) \subseteq X$.

(c) This is an immediate consequence of (a) and (b). ∎

Set $U = X^+ - Y$ and $W = X^- - Y$. Let $u_1, \ldots, u_t$ be the vertices of U, occurring on $\vec{C}$ in the order of their indices. For $i = 1, \ldots, t$, let w_i be the unique vertex of W in the component of $C - X$ containing u_i.

Corollary 21.

(a) U and W are independent sets.
(b) If $v \in w_j^{++}\vec{C}u_i^{--}$ and $u_i v \in E(G)$, then $w_j v^- \notin E(G)$ and $w_j v^+ \notin E(G)(i, j = 1, \ldots, t, i \neq j + 1 \pmod t))$.
(c) If $v \in u_i^{++}\vec{C}u_j^-$ and $u_i v \in E(G)$, then $u_j v^- \notin E(G)$ $(i, j = 1, \ldots, t, i \neq j)$.
(d) If $v \in w_i^{++}\vec{C}w_j^-$ and $w_i v \in E(G)$, then $w_j v^- \notin E(G)(i, j = 1, \ldots, t, i \neq j)$.

Proof. (a) We only prove that U is an independent set, the proof for W is similar. Suppose $u_i u_j \in E(G)$ for some $i \neq j$. Then, since u_i, u_j, $\notin Y$, the path $u_i^- \overleftarrow{C} u_j u_i \vec{C} u_j^-$ is a hopping path, contradicting Lemma 20.

(b) If $v \in w_j^{++}\vec{C}u_i^{--}$ and $u_i v$, $w_j v^- \in E(G)$, then we have $v, v^- \notin Y$. So the path $u_i^- \overleftarrow{C} v u_i \vec{C} w_j v^- \overleftarrow{C} w_j^+$ is a hopping path, a contradiction. The rest of (b) is proved similarly.

(c) Suppose $v \in u_i^{++}\vec{C}u_j^-$ and $u_i v$, $u_j v^- \in E(G)(i \neq j)$. Since $u_i, u_j, v, v^- \notin Y$, we obtain the hopping path $u_i^- \overleftarrow{C} u_j v^- \overleftarrow{C} u_i v \vec{C} u_j^-$, a contradiction.

(d) The proof of (d) is similar to the proof of (c). ∎

The results in this section were proved in Min Aung [12] for graphs satisfying a different condition. In the remainder of the paper we only use Corollary 21(a). We choose to give Corollary 21(b)–(d) as well, in order to make it possible to compare our results with known extensions of the original Hopping Lemma (see, e.g., Jackson [10]).

4. FURTHER PRELIMINARY RESULTS

The following results, together with Lemma 18 and Corollary 21(a), play an essential role throughout the proofs of the main results of this paper. If a cycle C in a graph G is not a Hamilton cycle, then we define $\mu(C) = \max\{d(v) | v \in V(G) - V(C)\}$.

Theorem 22. Let $G \in \mathcal{G}_2(n) \cup \mathcal{G}_1(n)$. Then we have

(i) $c(G) \geq p(G)$;
(ii) every longest cycle in G is a dominating cycle;
(iii) if G is nonhamiltonian, then G contains a longest cycle C with $\mu(C) \geq \frac{1}{3}\sigma_3(G)$.

Proof. Part (i) is in Enomoto, Van den Heuvel, Kaneko and Saito [8]. Part (ii) follows immediately from (i); also, (ii) is in Bondy [4] for $G \in \mathcal{G}_2(n)$ and in Bauer, Morgana, Schmeichel, and Veldman [2] for $G \in \mathcal{G}_1(n)$. Part (iii) is implicit in the proofs of [2, Theorems 9 and 10]. Parts (ii) and (iii) also appear in Bauer, Broersma, and Veldman [1]. ∎

If C is a cycle in a graph, with orientation $\vec{C}$, and $B \subseteq V(C)$ with $B \cap B^+ = \emptyset$, then the components of $C - B$ will be called the *B-segments* of C. A B-segment is a *$B - p$-segment* if it contains p vertices.

Lemma 23. Let C be a cycle in a graph, with orientation $\vec{C}$, and $B \subseteq V(C)$ with $B \cap B^+ = \emptyset$. Set $Z_B = V(C) - (B \cup B^+ \cup B^-)$ and let b_1 denote the number of B − 1-segments on C. Then $b_1 = 3|B| + |Z_B| - |V(C)| \geq 3|B| - |V(C)|$.

Proof. There are exactly $|B|$ B-segments. The number of B-segments with two or more vertices is equal to $|B| - b_1$. So we have $|V(C)| = |B| + b_1 + 2(|B| - b_1) + |Z_B| = 3|B| - b_1 + |Z_B|$, hence $b_1 = 3|B| + |Z_B| - |V(C)|$. ∎

For the remainder of this section, let G be a nonhamiltonian graph in $\mathcal{G}_2(n)$ or $\mathcal{G}_1(n)$. Let C be a longest cycle of G for which $\mu(C)$ is maximal, and fix an orientation $\vec{C}$ on C. By Theorem 22 (i), $c(G) \geq p(G)$, and so we can use the variant of the Hopping Lemma established in Section 3. Define R, X, and Y as in Section 3. Let $y_0 \in V(G) - V(C)$ with $d(y_0) = \mu(C)$, and let

$$A = N(y_0) \subseteq N(R) \subseteq X \subseteq V(C) \tag{3}$$

by Theorem 22 (ii) and the definition of X. By (3) and Theorem 22 (iii) we have

$$|X| \geq |A| = d(y_0) \geq \lceil \tfrac{1}{3}\sigma_3(G) \rceil = \overline{\delta}(G). \tag{4}$$

Let a_1 denote the number of A − 1-segments on C.

Lemma 24. We have $a_1 \geq 3\overline{\delta}(G) - n + |R|$, with equality if and only if $d(y_0) = \overline{\delta}(G)$ and, apart from the A − 1-segments, C contains A − 2-segments only.

Proof. Since C is a longest cycle, $A \cap A^+ = \emptyset$. By Lemma 23 and (4), $a_1 \geq 3|A| - |V(C)| \geq 3\overline{\delta}(G) - |V(C)|$, with equality if and only if $|A| = \overline{\delta}(G)$ and $|V(C) - (A \cup A^+ \cup A^-)| = 0$. The result follows. ∎

Lemma 25. There exists an A − 1-segment y_1 such that $d(y_1) \geq \overline{\delta}(G)$.

Proof. Label the A − 1-segments as $y_1, y_2, \ldots$ with $d(y_1) \geq d(y_2) \geq \cdots$. If $a_1 \geq 3$, then $\{y_1, y_2, y_3\}$ is an independent set. So we have $d(y_1) + d(y_2) + d(y_3) \geq \sigma_3(G)$, which means $d(y_1) \geq \overline{\delta}(G)$. If $a_1 = 2$, then by (4) and Lemma 23, $\overline{\delta}(G) \leq |A| \leq \frac{1}{3}(a_1 + n - |R|) \leq \frac{1}{3}(n + 1)$. Suppose $d(y_1) < \overline{\delta}(G)$. Since $d(y_1)$ is an integer, $d(y_2) \leq d(y_1) \leq \overline{\delta}(G) - 1$. Thus we have

$$\begin{aligned} n \leq \sigma_3(G) \leq d(y_0) + d(y_1) + d(y_2) &\leq \tfrac{1}{3}(n + 1) + 2(\overline{\delta}(G) - 1) \\ &\leq n - 1, \end{aligned}$$

a contradiction.

So we can assume $a_1 \leq 1$. By Lemma 24, (1) and the definitions of $\mathcal{G}_2(n)$ and $\mathcal{G}_1(n)$, $a_1 \geq \sigma_3(G) - n + |R| \geq 1$, and equality implies $G \in \mathcal{G}_1(n)$, $3\overline{\delta}(G) = \sigma_3(G) = n$, $|R| = 1$ and C contains A − 1-segments and A − 2-segments only. Lemma 23 and (4) now give $d(y_0) = |A| = \frac{1}{3}(1 + n - 1) = \frac{1}{3}\sigma_3(G)$. Since $y_1^+ \vec{C} y_1^- y_0 y_1^+$ is a longest cycle not containing y_1, we have $d(y_1) \leq d(y_0)$ by the choice of C. Let $x_1, \ldots, x_{|A|}$ be the vertices of A in order round $\vec{C}$. Since C is a longest cycle, clearly $\{y_0\} \cup A^+$ and $\{y_0\} \cup A^-$ are independent sets. Since G is 1-tough, $\omega(G - A) \leq |A|$. Hence there exist two A − 2-segments $x_i^+ x_{i+1}^-$ and $x_j^+ x_{j+1}^-$ such that $x_i^+ x_{j+1}^- \in E(G)$ (indices mod $|A|$), otherwise $\omega(G - A) = |A| + |R| > |A|$.

Since $x_i^+ x_{j+1}^- \overleftarrow{C} x_{i+1} y_0 x_{j+1} \overrightarrow{C} x_i^+$ is a longest cycle not containing x_{i+1}^-, we have $d(x_{i+1}^-) \leq d(y_0)$, by the choice of C. Obviously, $\{y_0, y_1, x_{i+1}^-\}$ is an independent set. So we have

$$\sigma_3(G) \leq d(y_0) + d(y_1) + d(x_{i+1}^-) \leq 3d(y_0) = \sigma_3(G).$$

Thus $d(y_1) = \frac{1}{3}\sigma_3(G) = \overline{\delta}(G)$. ∎

5. PROOFS OF THEOREMS 3 AND 7

Assume that G is a nonhamiltonian graph in $\mathcal{G}_2(n)$ or $\mathcal{G}_1(n)$, and $t \in \mathbb{N}$. Let C be a longest cycle of G with orientation $\overrightarrow{C}$ such that $\mu(C)$ is maximal, and let $y_0 \in V(G) - V(C)$ with $d(y_0) = \mu(C)$. Copy the notation and terminology used in former sections. In particular, define R, X_1, X, Y, U, W, A, and a_1 as in Sections 3 and 4.

By Lemma 25 there exists an $A - 1$-segment $y_1 \in Y$ such that $d(y_1) \geq \overline{\delta}(G)$. Since the $A - 1$-segments are elements of Y, we have, by Lemma 24,

$$|Y| \geq 3\overline{\delta}(G) - n + |R|. \tag{5}$$

By Lemma 18 (a), (b), (c) and the fact that $N(R) \subseteq X$ by (3), it follows that

$$R \cup Y \text{ is an independent set with } N(R \cup Y) \subseteq X \subseteq V(C). \tag{6}$$

Since $N(R) \subseteq X$ and $X^+ = Y \cup U$, from Corollary 21 (a) and Lemmas 18 (a), (b), (c) we have

$$R \cup X^+ \text{ is an independent set.} \tag{7}$$

Let $Z = V(C) - (X \cup Y \cup U \cup W) = V(C) - (X \cup X^+ \cup X^-)$. By Lemma 23 we have

$$0 \leq |Z| = |V(C)| + |Y| - 3|X| = n + |Y| - 3|X| - |R|. \tag{8}$$

Proof of Theorem 3(a). If $c(G) \geq \frac{1}{2}(n + 3\overline{\delta}(G) + 1 - t)$, then we are done. So we may assume $n - |R| = c(G) \leq \frac{1}{2}(n + 3\overline{\delta}(G) - t)$, which implies

$$t \leq n + 3\overline{\delta}(G) - 2(n - |R|) = 3\overline{\delta}(G) - n + 2|R|.$$

Set $S = Y \cup R$. By (5) this means

$$|S| = |Y| + |R| \geq 3\overline{\delta}(G) - n + 2|R| \geq t. \tag{9}$$

By (6) and (7), both S and $R \cup N(S)^+$ are independent sets. So we obtain

$$\begin{aligned}\alpha(G) &\geq |R \cup N(S)^+| = |R| + |N(S)^+| = |V(G)| - |V(C)| + |N(S)| \\ &\geq n - |V(C)| + \mathrm{NC}_{|S|}(G).\end{aligned}$$

Together with (9) this gives $c(G) = |V(C)| \geq n + \mathrm{NC}_t(G) - \alpha(G)$. ∎

Proof of Theorem 3(b). Again, we may assume that $c(G) \leq \frac{1}{2}(n + 3\overline{\delta}(G) + 3 - t)$, hence

$$t \leq n + 3\overline{\delta}(G) + 3 - 2c(G) = 3\overline{\delta}(G) - n + 2|R| + 3. \tag{10}$$

Let $Y = \{y_1, \ldots, y_{|Y|}\}$, with $d(y_2) \geq d(y_3) \geq \cdots$. Let $u_1, \ldots, u_{|U|}$ be the vertices of U in order round $\vec{C}$ such that u_1 is the first vertex of U following y_1 on $\vec{C}$. For $i = 1, \ldots, |U|$, let w_i be the unique vertex of W in the X-segment containing u_i.

We state some observations which will be used repeatedly. We will say that a property $\mathcal{P}$ of G holds by (*lca*) (*l*ongest *c*ycle *a*rgument) if the contrary to $\mathcal{P}$ implies the existence of a cycle C' longer than C. (lca) often represents an argument which is standard in hamiltonian graph theory. The cycle C' will often be given between brackets after the statement of $\mathcal{P}$.

(OB1) Let $y \in Y$ and assume $N(y_0) = N(y) = X$ and $u_i w_j \in E(G)$ for some $i \neq j$. Then $y \in w_j \vec{C} u_i$, $w_i y^+ \notin E(G)$ and $w_i y^- \notin E(G)$.

Proof. By (lca) we have $y \in w_j \vec{C} u_i$ $(y^- y_0 w_j^+ \vec{C} u_i^- y \vec{C} w_j u_i \vec{C} y^-)$. Also, $w_i y^+ \notin E(G)$ $(y u_i^- \overleftarrow{C} y^+ w_i \overleftarrow{C} u_i w_j \overleftarrow{C} w_i^+ y_0 w_j^+ \vec{C} y)$ and $w_i y^- \notin E(G)$ $(y \vec{C} u_i^- y_0 w_j^+ \vec{C} y^- w_i \overleftarrow{C} u_i w_j \overleftarrow{C} w_i^+ y)$. ▮

(OB2) Assume $N(y_0) = N(y_1) = X$ and $u_i w_i \in E(G)$, where $i \neq j$ and i is chosen minimal. Then $N(w_i) \cap (U \cup W) \subseteq \{u_i\}$.

Proof. By Corollary 21 (a) we have $N(w_i) \cap W = \emptyset$. The minimality of i implies $u_l w_i \notin E(G)$ for $l = 1, \ldots, i - 1$. By (OB1), if $u_l \in N(w_i) \cap U$, then $y_1 \in w_i \vec{C} u_l$, so also $u_l w_i \notin E(G)$ for $l = i + 1, \ldots, |U|$. ▮

(OB3) If $|X| = \overline{\delta}(G)$, then $d(y_0) = \cdots = d(y_{|Y|-\varepsilon}) = \overline{\delta}(G)$, where $\varepsilon = 3\overline{\delta}(G) - \sigma_3(G)$.

Proof. First note that by (5) we have $|Y| - \varepsilon \geq |R| \geq 1$. By (4), $d(y_0) = \overline{\delta}(G)$. For $y \in Y$ it yields $N(y) \subseteq X$, so $d(y) \leq |X| = \overline{\delta}(G)$. By the choice of y_1 we have $d(y_1) \geq \overline{\delta}(G)$, hence $d(y_1) = \overline{\delta}(G)$ and we are done if $|Y| = 1$. Now assume $|Y| \geq 2$ and $d(y_{|Y|-\varepsilon}) \leq \overline{\delta}(G) - 1$. By definition, $d(y_2) \geq d(y_3) \geq \cdots \geq d(y_{|Y|})$, which gives the contradiction

$$\begin{aligned}\sigma_3(G) &\leq d(y_{|Y|-2}) + d(y_{|Y|-1}) + d(y_{|Y|}) \leq 3\overline{\delta}(G) - \varepsilon - 1 \\ &= \sigma_3(G) - 1.\end{aligned}$$

This shows $d(y_{|Y|-\varepsilon}) \geq \overline{\delta}(G)$, hence $d(y_0) = d(y_1) = \cdots = d(y_{|Y|-\varepsilon}) = \overline{\delta}(G)$. ▮

A subset $S \subseteq V(G)$ will be called *suitable* if $|S| \geq t$, $N(S) \subseteq V(C)$ and both S and $R \cup N(S)^+$ are independent sets. If a suitable set S exists, then we are done, since

$$\begin{aligned}\alpha(G) &\geq |R \cup N(S)^+| = |R| + |N(S)^+| = |V(G)| - |V(C)| + |N(S)| \\ &\geq n - |V(C)| + \mathrm{NC}_{|S|}(G),\end{aligned}$$

hence $c(G) = |V(C)| \geq n + \mathrm{NC}_t(G) - \alpha(G)$.

We will distinguish a number of cases, in each of which we either exhibit a suitable set, or reach a contradiction.

Case 1. $|Y| = 3\overline{\delta}(G) - n + |R|$.
By (4) and (8) we conclude $|Z| = 0$ and $|X| = \overline{\delta}(G)$. So apart from the $X - 1$-segments, C contains $X - 2$-segments only. Furthermore, we have $N(y_0) = N(y_1) = X$. Since G is

1-tough, $\omega(G - X) \leq |X|$. Hence there exist $i, j, i \neq j$, such that $u_i w_j \in E(G)$. Assume i is chosen minimal. By (OB1) and (OB2), $N(w_i) \subseteq (X - \{y_1^+, y_1^-\}) \cup \{u_i\}$, so $d(w_i) \leq \overline{\delta}(G) - 1$. Now consider the greatest h such that $u_g w_h \in E(G)$ for some $g \neq h$. By (OB1), $y_1 \in w_h \vec{C} u_g$, so $i \leq g < h$. By (OB1) and (OB2) (now applied to $\overleftarrow{C}$), $N(u_h) \subseteq (X - \{y_1^+, y_1^-\}) \cup \{w_h\}$, so $d(u_h) \leq \overline{\delta}(G) - 1$. For $y \in Y$ we have $N(y) \subseteq X$, hence $d(y) \leq \overline{\delta}(G)$. This gives

$$n \leq \sigma_3(G) \leq d(w_i) + d(u_h) + d(y) \leq 3\overline{\delta}(G) - 2.$$

So we obtain $|Y| \geq |R| + 2 \geq 3$ and $d(y) = \overline{\delta}(G)$ for all $y \in Y$. But then, in fact, we have $N(w_i) \subseteq (X - \bigcup_{y \in Y}\{y^+, y^-\}) \cup \{u_i\}$. Since $|\bigcup_{y \in Y}\{y^+, y^-\}| \geq |Y| + 1$, this means $d(w_i) \leq \overline{\delta}(G) - |Y|$. Similarly, $d(u_h) \leq \overline{\delta}(G) - |Y|$. We reach the contradiction

$$\sigma_3(G) \leq d(w_i) + d(u_h) + d(y_1) \leq 3\overline{\delta}(G) - 2|Y| \leq 3\overline{\delta}(G) - 6.$$

Case 2. $|Y| = 3\overline{\delta}(G) - n + |R| + 1$.
By (4) and (8) we obtain $|X| = \overline{\delta}(G)$ and $|Z| = 1$. So apart from the X − 1- and X − 2-segments, C contains exactly one X − 3-segment, say $u_l u_l^+ w_l$. Furthermore, we have $|Y| - (3\overline{\delta}(G) - \sigma_3(G)) \geq |R| + 1 \geq 2$, so by (OB3), $N(y_0) = N(y_1) = N(y_2) = X$. If $u_l w_l \in E(G)$, then, by (lca), $u_l^+ u_i \notin E(G)$ for $i \neq l$ $(u_i \vec{C} u_l^- y_0 u_i^- \overleftarrow{C} w_l u_l u_l^+ u_i)$ and $u_l^+ w_i \notin E(G)$ for $i \neq l$ $(w_i \overleftarrow{C} w_l^+ y_0 w_i^+ \vec{C} u_l w_l u_l^+ w_i)$, hence $N(u_l^+) \subseteq X \cup \{u_l, w_l\}$. Since $\omega(G - X) \leq |X|$, there exist $i, j, i \neq j$, such that $u_i w_j \in E(G)$. This is also true if $u_l w_l \notin E(G)$, otherwise $\omega(G - (X \cup \{u_l^+\})) > |X \cup \{u_l^+\}|$. Choose i minimal. By (OB1) and (OB2) we have $N(w_i) \subseteq (X - (\{y_1^+, y_1^-\} \cup \{y_2^+, y_2^-\})) \cup \{u_i, u_l^+\}$, hence $d(w_i) \leq \overline{\delta}(G) - 1$. Now consider the greatest h such that $u_g w_h \in E(G)$ for some $g \neq h$, then we have $d(u_h) \leq \overline{\delta}(G) - 1$. Reasoning as in Case 1 we obtain a contradiction.

Case 3. $|Y| = 3\overline{\delta}(G) - n + |R| + 2$.
By (4) and (8), $|X| = \overline{\delta}(G)$ and $|Z| = 2$. So apart from the X − 1- and X − 2-segments, C contains either one X − 4-segment, or two X − 3-segments. Furthermore, by (OB3), $N(y_0) = N(y_1) = N(y_2) = N(y_3) = X$.

Case 3.1. C contains a X − 4-segment, say $u_l u_l^+ w_l^- w_l$, and $u_i w_j \in E(G)$ for some $i, j, i \neq j$.

Choose i minimal and h maximal such that $u_i w_j$, $u_g w_h \in E(G)$ for some $j \neq i$, $g \neq h$. By (OB1) and (OB2) we have $i \leq g < h$, $N(w_i) \subseteq (X - (\{y_1^+, y_1^-\} \cup \{y_2^+, y_2^-\} \cup \{y_3^+, y_3^-\})) \cup \{u_i, u_l^+, w_l^-\}$ and $N(u_h) \subseteq (X - (\{y_1^+, y_1^-\} \cup \{y_2^+, y_2^-\} \cup \{y_3^+, y_3^-\})) \cup \{w_h, u_l^+, w_l^-\}$, so $d(w_i)$, $d(u_h) \leq \overline{\delta}(G) - 1$. Following Case 1, we reach a contradiction.

Case 3.2. C contains a X − 4-segment, say $u_l u_l^+ w_l^- w_l$, and $u_i w_j \notin E(G)$ for all $i, j, i \neq j$.

Since G is 1-tough, some X − 2-segment contains a vertex adjacent to u_l^+ or w_l^-. Without loss of generality, we may assume $w_l^- u_i \in E(G)$ or $w_l^- w_i \in E(G)$ for some $i \neq l$. First consider the case $w_l^- u_i \in E(G)$. By (lca), $w_i u_l^+ \notin E(G)$ $(u_l^+ w_i u_i w_l^- \vec{C} u_i^- y_0 w_i^+ \vec{C} u_l^+)$.

Next we suppose $w_l u_l^+ \in E(G)$. Then by (lca) we have $i < l$ $(y_1 \vec{C} u_l^+ w_l w_l^- u_i \vec{C} y_1^- y_0 w_l^+ \vec{C} u_i^- y_1)$ and $w_i w_l^- \notin E(G)$ $(w_i w_l^- w_l u_l^+ \overleftarrow{C} w_i^+ y_0 w_l^+ \vec{C} w_i)$. Also, by (lca), $w_i y_1^+ \notin E(G)$ $(y_1 w_i^+ \vec{C} u_l^+ w_l w_l^- u_i w_i y_1^+ \vec{C} u_i^- y_0 w_l^+ \vec{C} y_1)$, $w_i y_1 \notin E(G)$ $(y_1^- w_i u_i w_l^- w_l u_l^+ \overleftarrow{C} w_i^+ y_1 \vec{C} u_i^- y_0 w_l^+ \vec{C} y_1^-)$. Similarly, we obtain $w_i y_2^+$, $w_i y_2^-$, $w_i y_3^+$, $w_i y_3^- \notin E(G)$. This gives

$N(w_i) \subseteq (X - (\{y_1^+, y_1^-\} \cup \{y_2^+, y_2^-\} \cup \{y_3^+, y_3^-\})) \cup \{u_i\}$, so $d(w_i) \leq \overline{\delta}(G) - 3$. We obtain the inequality

$$\sigma_3(G) \leq d(w_i) + d(y_0) + d(y_1) \leq 3\overline{\delta}(G) - 3\,.$$

This contradiction shows that $w_l u_l^+ \notin E(G)$. Also, by (lca), $w_l u_l \notin E(G)$ $(u_i \vec{C} u_l^- y_0 u_i^- \overleftarrow{C} w_l u_l u_l^+ w_l^- u_i)$. This means $N(w_l) \subseteq X \cup \{w_l^-\}$ and $X^+ \cup \{w_l\}$ is an independent set. So, by (6), (7), and (10), $R \cup Y \cup \{w_l\}$ is a suitable set.

If $w_l^- w_i \in E(G)$, then similar conclusions are obtained by considering u_i instead of w_i.

Case 3.3. C contains two $X - 3$-segments, say $u_l u_l^+ w_l$ and $u_m u_m^+ w_m$ $(l < m)$.

Let $k \in \{l, m\}$. If $N(u_k^+) \cap (U \cup W) \neq \{u_k, w_k\}$, then we have, by (lca), $u_k w_k \notin E(G)$ $(u_i \vec{C} u_k^- y_0 u_i^- \overleftarrow{C} w_k u_k u_k^+ u_i$, if $u_k^+ u_i \in E(G)$ for some $i \neq k$; $w_i u_k^+ w_k u_k \overleftarrow{C} w_i^+ y_0 w_k^+ \vec{C} w_i$, if $u_k^+ w_i \in E(G)$ for some $i \neq k$). Since G is 1-tough, $\omega(G - T) \leq |T|$ for $T = X, X \cup \{u_l^+\}, X \cup \{u_m^+\}, X \cup \{u_l^+, u_m^+\}$. It follows that $u_i w_j \in E(G)$ for some i, j, $i \neq j$. Choose i minimal and h maximal such that $u_i w_j$, $u_g w_h \in E(G)$ for some $j \neq i$, $g \neq h$. By (OB1) and (OB2), $i \leq g < h$, $N(w_i) \subseteq (X - (\{y_1^+, y_1^-\} \cup \{y_2^+, y_2^-\} \cup \{y_3^+, y_3^-\})) \cup \{u_i, u_l^+, u_m^+\}$ and $N(u_h) \subseteq (X - (\{y_1^+, y_1^-\} \cup \{y_2^+, y_2^-\} \cup \{y_3^+, y_3^-\})) \cup \{w_h, u_l^+, u_m^+\}$. As in Case 3.1, we obtain a contradiction.

Case 4. $|Y| \geq 3\overline{\delta}(G) - n + |R| + 3$.
We have $|Y \cup R| \geq 3\overline{\delta}(G) - n + 2|R| + 3$, so by (6), (7), and (10), $R \cup Y$ is a suitable set. ∎

Proof of Theorem 7. Suppose first that $|Y \cup R| \geq t$. Set $S = Y \cup R$. By Lemma 18 (a) and (6) we obtain

$$|V(C)| \geq |X| + |X^+| = 2|X| \geq 2|N(S)| \geq 2\mathrm{NC}_{|S|}(G) \geq 2\mathrm{NC}_t(G)\,,$$

and hence we are done.

So from now on we may assume $|Y \cup R| \leq t - 1$. Hence (5) gives

$$t \geq |Y| + |R| + 1 \geq 3\overline{\delta}(G) - n + 2|R| + 1\,. \tag{11}$$

If $c(G) \geq \frac{1}{3}(2n + 4\overline{\delta}(G) + 1 - 2t)$, then we are also done. Hence we may assume $c(G) \leq \frac{1}{3}(2n + 4\overline{\delta}(G) - 2t)$. Together with $-\frac{1}{2}n \leq \frac{1}{2}t - \frac{3}{2}\overline{\delta}(G) - |R| - \frac{1}{2}$, by (11), this gives

$$t \leq n + 2\overline{\delta}(G) - \tfrac{3}{2}(n - |R|) \leq \tfrac{1}{2}t + \tfrac{1}{2}\overline{\delta}(G) + \tfrac{1}{2}|R| - \tfrac{1}{2}\,. \tag{12}$$

This is equivalent to $t + 1 \leq \overline{\delta}(G) + |R|$, and so (4) gives

$$|Y| + |W| + |R| = |X| + |R| \geq \overline{\delta}(G) + |R| \geq t + 1\,.$$

So we can choose $Q_1 = y_1 \vec{C} w_q$ with $w_q \in W$ and $|W \cap Q_1| + |Y| + |R| = t$. Note that by this definition

$$|U \cap Q_1| = |W \cap Q_1| = t - |Y| - |R|\,. \tag{13}$$

Let $T = (W \cap Q_1) \cup Y \cup R$. Then, by Lemma 18, Corollary 21 (a) and the definitions of X and W, T is an independent set of t vertices with $N(T) \subseteq V(C) - X^-$. If $|N(T)| \leq \frac{1}{2}|V(C)|$, then we are done. So we assume

$$|N(T)| \geq \tfrac{1}{2}(|V(C)| + 1) = \tfrac{1}{2}(n - |R| + 1). \tag{14}$$

We will show that this assumption eventually leads to a contradiction.

Set $Q_2 = V(C) - Q_1$ and note that

$$\begin{aligned} Q_2 - (X \cup Z) &= V(C) - (Q_1 \cup X \cup (Z \cap Q_2)) \\ &= V(C) - ((Q_1 - X) \cup X \cup (Z \cap Q_2)). \end{aligned}$$

Let $J = N(T) \cap (Q_2 - (X \cup Z))$. Since $N(T) \subseteq V(C)$, we have

$$|J| = |N(T)| - |N(T) \cap (Q_1 - X)| - |N(T) \cap X| - |N(T) \cap (Z \cap Q_2)|. \tag{15}$$

Clearly,

$$|N(T) \cap X| \leq |X|, \tag{16}$$

$$|N(T) \cap (Z \cap Q_2)| \leq |Z \cap Q_2|, \tag{17}$$

and, since $N(T) \subseteq V(C) - (X^- \cup Y)$,

$$|N(T) \cap (Q_1 - X)| \leq |U \cap Q_1| + |Z \cap Q_1|. \tag{18}$$

Using (8) we obtain

$$|Y| = |Z| + 3|X| - n + |R|. \tag{19}$$

Combing (13)–(19) gives

$$\begin{aligned} |J| &\geq |N(T)| - |U \cap Q_1| - |Z \cap Q_1| - |X| - |Z \cap Q_2| \\ &\geq \tfrac{1}{2}(n - |R| + 1) - t + |Y| + |R| - |X| - |Z| \\ &= \tfrac{1}{2}(n - |R| + 1) - t + |Z| + 3|X| - n + |R| + |R| - |X| - |Z| \\ &= 2|X| + \tfrac{3}{2}|R| - \tfrac{1}{2}n - t + \tfrac{1}{2}. \end{aligned} \tag{20}$$

Let $K = N(y_0) \cap N(y_1)$. Since $N(y_0), N(y_1) \subseteq X$ and $d(y_0), d(y_1) \geq \overline{\delta}(G)$, we have

$$|K| = |N(y_0)| + |N(y_1)| - |N(y_0) \cup N(y_1)| \geq 2\overline{\delta}(G) - |X|. \tag{21}$$

Combining (20) and (21) gives, using (12),

$$\begin{aligned} |J^- \cap K| + |J^- \cup K| &= |J^-| + |K| = |J| + |K| \\ &\geq 2|X| + \tfrac{3}{2}|R| - \tfrac{1}{2}n - t + \tfrac{1}{2} + 2\overline{\delta}(G) - |X| \\ &\geq |X| + \tfrac{3}{2}|R| - \tfrac{1}{2}n - 2\overline{\delta}(G) - \tfrac{3}{2}|R| + \tfrac{1}{2}n + \tfrac{1}{2} + 2\overline{\delta}(G) \\ &= |X| + \tfrac{1}{2}. \end{aligned} \tag{22}$$

Since $N(T) \subseteq V(C) - X^-$, it follows that $J \subseteq X^+$. Therefore we have $J^- \cup K \subseteq X$. Hence, by (22) there exists a vertex $x \in J^- \cap K$. Since $K \subseteq X$, we have $x \in X$. Let $w \in T$ such that $wx^+ \in E(G)$, then we have $w \in W \cap Q_1$, since $N(Y \cup R) \subseteq X$. Let P be the path $w^+\vec{C}xy_1\vec{C}wx^+\vec{C}y_1^-$. Then P is a hopping path, as defined in Section 3. Conditions (HP1) and (HP2) are obvious and the only vertex in Y for which (HP3) is not obvious, is y_1. But the neighbors of y_1 on P are x and y_1^+ which are both neighbors of y_0, hence are elements of X_1. By Lemma 20, the existence of the path P gives us the promised contradiction. ∎

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