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The local disturbance decoupling problem with stability for nonlinear systems

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Abstract: In this paper the Disturbance Decoupling Problem with Stability (DDPS) for nonlinear systems is considered. The DDPS is the problem of finding a feedback such that after applying this feedback the disturbances do not influence the output anymore and $x = 0$ is an exponentially stable equilibrium point of the feedback system. For systems that can be decoupled by static state feedback it is possible to define (under fairly mild assumptions) a distribution Δ_s^* which is the nonlinear analogue of the linear $\mathscr{V}_s^*$, the largest stabilizable controlled invariant subspace in the kernel of the output mapping, and to prove that the DDPS is locally solvable if and only if the disturbance vector fields are contained in Δ_s^*.

Keywords: Nonlinear control system; disturbance decoupling with stability; controlled invariant distribution.

1. Introduction

Since the appearance of Wonham's well-known book on the geometric approach to linear control problems [13] much attention has been paid to the further development of this theory for linear systems. Also, a considerable amount of research has been done on the generalization of this theory to nonlinear systems using a differential-geometric approach. This has led to local solutions for several well-known design problems like the Disturbance Decoupling Problem and the Noninteracting Control Problem. A good overview of results in this area up to 1985 can be found in Isidori [5]. In recent years a couple of articles on stabilizability of nonlinear systems have appeared, e.g. Charlet [3], Aeyels [1], Tsinias [11], Jurdjevic & Quinn [7].

However, nor the book [5], neither these articles pay attention to design problems in connection with stability of the feedback system, in contrast to Isidori & Grizzle [6], Byrnes & Isidori [2] and Van der Wegen & Nijmeijer [12].

In [6], Isidori & Grizzle derive some negative results concerning the solution of the Noninteracting Control Problem with Stability. They show that this problem is not solvable by applying static feedback if the fixed internal dynamics inherent to the noninteracting requirement is unstable. In [2], Byrnes & Isidori solve the Disturbance Decoupling Problem (DDP) with BIBO-stability for systems which are (exponentially) minimum phase.

As a matter of fact, the minimum-phase requirement is rather strong. For linear systems it comes down to requiring that $\mathscr{V}^*$, the maximal controlled invariant subspace in the kernel of the output mapping, and $\mathscr{V}_s^*$, the largest stabilizable controlled invariant subspace in $\mathscr{V}^*$, coincide. Under this assumption, the conditions for solvability of the DDP and the Disturbance Decoupling Problem with Stability (DDPS) for linear systems are the same, whereas in general conditions for solvability of the latter problem are stronger.

The generalization of $\mathscr{V}^*$ for nonlinear systems is Δ^*, the largest locally controlled invariant distribution in the kernel of the output mapping. Necessary and sufficient conditions for the (local) solvability of the DDP are stated in terms of Δ^* (see [5]). These conditions are similar to the ones in the linear case. In this paper we will give a nonlinear analogue of $\mathscr{V}_s^*$, called Δ_s^*, and solve the DDPS for systems that can be decoupled by static state feedback. It will appear that the conditions for solvability of the DDPS look very much alike the linear and nonlinear case.

In Section 2 we will give the problem formulation, the definition of Δ_s^* and the solution to the DDPS. Some examples are worked out in Section 3, and in Section 4 our results are compared to those of Byrnes and Isidori. We end up with an appendix where some standard results on hyperbolic vector fields are summarized.

2. The local disturbance decoupling problem with stability

Consider the analytic SISO control system

$$\dot{x} = f(x) + g(x)u + e(x)d, \quad y = h(x), \quad x \in \mathbb{R}^n, \quad d \in \mathbb{R}, \tag{2.1}$$

with $f(0) = 0$ and $h(0) = 0$.

We want to solve the Disturbance Decoupling Problem with Stability (DDPS) for (2.1), i.e. find a controlled invariant distribution Δ and a feedback

$$u = \alpha(x) + \beta(x)v, \quad \alpha(0) = 0, \ \beta(x) \text{ invertible} \tag{2.2}$$

such that

$$e \in \Delta \subset \ker \mathrm{d}h \tag{2.3}$$

and $x = 0$ is an exponentially stable equilibrium point of the system

$$\dot{x} = (f + g\alpha)(x). \tag{2.4}$$

In general, it is hard to find a global solution. Therefore, we will be satisfied if we can find Δ and a feedback defined locally around $x = 0$ such that Δ is locally controlled invariant and $x = 0$ is locally exponentially stable.

We make the following assumptions:

(A1) The DDP is locally solvable for (2.1).

(This is equivalent with the condition that $e \in \Delta^*$.)

(A2) The system (2.1) is accessible in a neighborhood of $x = 0$, i.e.

$$\mathrm{sp}\langle f, g \,|\, \mathrm{sp}\{f, g\}\rangle(x) = T_x\mathbb{R}^n \tag{2.5}$$

for all x in a neighborhood of $x = 0$ (see Sussmann & Jurdjevic [10]).

(A3) The function $A(x)$ defined by

$$A(x) = L_g L_f^{\rho(x)} h(x) \tag{2.6}$$

is nonzero in a neighborhood of $x = 0$. Here the characteristic number $\rho(x)$, the smallest integer such that

$$L_g L_f^k h(x) = 0, \quad k < \rho(x), \tag{2.7}$$

and

$$L_g L_f^{\rho(x)} h(x) \neq 0 \tag{2.8}$$

is assumed to be constant in a neighborhood of $x = 0$. Note that (A3) implies that $\rho := \rho(x)$ is finite.

To stress the resemblance with the linear case as well as the differences we will in short sketch the solution of the DDPS in the linear case.

Consider the SISO linear system

$$\dot{x} = Ax + Bu + Ed, \quad y = Cx, \quad x \in \mathbb{R}^n, \quad d \in \mathbb{R}^r. \tag{2.9}$$

Suppose that (A1), (A2), and (A3) hold for this system. Note that (A1) is equal to the condition that $\operatorname{Im} E \subset \mathscr{V}^*$, while (A2) is equivalent to saying that (2.9) is controllable. Under these assumptions we can find a feedback

$$u = Fx + Gv, \quad G \text{ invertible}, \tag{2.10}$$

such that $(A + BF)\mathscr{V}^* \subset \mathscr{V}^*$. It is well known that the eigenvalues of $(A + BF)|_{\mathscr{V}^*}$, the so-called transmission zeros of the system, are independent of the choice of F. Hence, there exists a unique subspace of $\mathscr{V}^*$ spanned by the (generalized) eigenvectors corresponding to the exponentially stable eigenvalues of the matrix $A + BF$. This subspace is denoted by $\mathscr{V}_s^*$. Note that $\mathscr{V}_s^*$ is controlled invariant, because $\mathscr{V}_s^*$ is $(A + BF)$-invariant. Assume now that

$$\operatorname{Im} E \subset \mathscr{V}_s^* \tag{2.11}$$

then we can solve the DDPS for (2.9) by choosing a feedback (2.10) such that $(A + BF)\mathscr{V}_s^* \subset \mathscr{V}_s^*$ and $A + BF$ is an asymptotically stable matrix, because by (A2), the induced system on $\mathbb{R}^n/\mathscr{V}_s^*$ is controllable.

We look at the solution of this problem in a different way now. The linear subspaces $\mathscr{V}^*$ and $\mathscr{V}_s^*$ can be considered as integral manifolds through $x = 0$ of the flat distributions $\Delta_{\mathscr{V}^*} \cong \mathscr{V}^*$ and $\Delta_{\mathscr{V}_s^*} \cong \mathscr{V}_s^*$, respectively. The manifold $\mathscr{V}_s^*$ is invariant under the vector field $(A + BF)x$, for $x = 0$ is an equilibrium point of this vector field. Since $\Delta_{\mathscr{V}_s^*}$ is spanned by constant vector fields, this distribution is necessarily invariant under any vector field of the form $(A + BF)^i BG$, $i = 0, \ldots, n-1$. This implies that $X^t(\mathscr{V}_s^*)$ where $X = (A + BF)^i BG$ for some i is again an integral manifold of the distribution $\Delta_{\mathscr{V}_s^*}$. Of course, $X^t(\mathscr{V}_s^*) = x + \mathscr{V}_s^*$ for some $x \in \mathbb{R}^n$ (depending on t). As a matter of fact, it is possible to construct the foliation $\{x + \mathscr{V}_s^* \mid x \in \mathbb{R}^n\}$ from the integral manifold $\mathscr{V}_s^*$ through $x = 0$. By controllability,

$$\operatorname{sp}\{(A + BF)^i BG \mid i = 0, \ldots, n-1\} = T_x\mathbb{R}^n$$

for any $x \in \mathbb{R}^n$, and so it is possible to find independent vector fields $X_1, \ldots, X_{n-k}$ in $\{(A + BF)^i BG \mid i = 0, \ldots, n-1\}$ that are transversal to the manifold $\mathscr{V}_s^*$ of dimension k. (See the appendix for the definitions of independence and transversality.) The set

$$\{X_{n-k}^{t_{n-k}} \circ \cdots \circ X_1^{t_1}(\mathscr{V}_s^*) \mid t_1, \ldots, t_{n-k} \in \mathbb{R}\} \tag{2.12}$$

defines a foliation on $\mathbb{R}^n$. Note that the order of the X_i in (2.12) does not matter, since $[X_i, X_j] = 0$ for j, $i = 1, \ldots, n-k$. This foliation (2.12) is the same as the foliation $\{x + \mathscr{V}_s^* \mid x \in \mathbb{R}^n\}$, for the X_i are constant vector fields. This implies that it is possible to find the distribution $\Delta_{\mathscr{V}_s^*}$ from the foliation (2.12) that was defined using only the integral manifold $\mathscr{V}_s^*$ through $x = 0$ and a particular set of vector fields transversal to this integral manifold.

It is in this way that we are going to construct the generalization of $\mathscr{V}_s^*$ for nonlinear systems.

We return to the nonlinear system (2.1) now. Note that (A3) implies that $S^*(\Delta^*)$, the largest local controllability distribution in Δ^* (and so, in $\ker dh$) is equal to the zero distribution (see Nijmeijer [8]). Again by assumption (A3) it is possible to choose new coordinates in the following way. Let

$$\xi_1 = [h(x), L_f h(x), \ldots, L_f^\rho h(x)]^{\mathrm{T}}. \tag{2.13}$$

If $\rho + 1 < n$, then choose an extra set of coordinates ξ_2. It is well known (see Isidori [5], Chapter 4) that by choosing

$$\beta(x) = (A(x))^{-1}, \qquad \alpha(x) = -(A(x))^{-1} L_f^{\rho+1} h, \qquad u = \alpha(x) + \beta(x)\tilde{v}, \tag{2.14}$$

the system (2.1), (2.2) with $d \equiv 0$ becomes in the new coordinates

$$\begin{aligned} \dot{\xi}_1 &= A_1\xi_1 + B_1\tilde{v}, \\ \dot{\xi}_2 &= \bar{f}_2(\xi_1, \xi_2) + \bar{g}_2(\xi_1, \xi_2)\tilde{v}, \\ y &= \xi_{11}, \end{aligned} \tag{2.15}$$

with

$$A_1 = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{2.16}$$

Obviously, (A_1, B_1) is a controllable pair. Hence it is possible to apply another feedback

$$\tilde{v} = N\xi_1 + v \tag{2.17}$$

such that $M := A_1 + B_1 N$ is an *unstable* matrix, i.e. $\sigma(M) \subset \mathbb{C}^+ := \{s \in \mathbb{C} \mid \text{Re } s > 0\}$.

Now the system (2.15), (2.17) has the form

$$\begin{aligned} \dot{\xi}_1 &= M\xi_1 + B_1 v, \\ \dot{\xi}_2 &= \bar{f}_2(\xi_1, \xi_2) + \bar{g}_2(\xi_1, \xi_2)N\xi_1 + \bar{g}_2(\xi_1, \xi_2)v, \\ y &= \xi_{11}. \end{aligned} \tag{2.18}$$

Next, choose new coordinates $(x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}}$ as follows:

$$x_1 = \xi_2, \qquad x_2 = \xi_1. \tag{2.19}$$

Then the system (2.18) including the disturbances has the following form:

$$\begin{aligned} \dot{x}_1 &= \hat{f}(x_1, x_2) + \hat{g}_1(x_1, x_2)Nx_2 + \hat{g}_1(x_1, x_2)v + e_1(x_1, x_2)d, \\ \dot{x}_2 &= Mx_2 + \hat{g}_2 v, \\ y &= \hat{h}(x_2). \end{aligned} \tag{2.20}$$

Note that it follows from (2.18) that $\hat{g}_2$ is constant. Moreover, by (A1), e belongs to $\Delta^* = \text{sp}\{\partial/\partial x_1\} = \text{sp}\{\partial/\partial \xi_2\}$. It follows directly from (2.20) that

$$[\tilde{f}, \Delta^*] \subset \Delta^*, \qquad [\tilde{g}, \Delta^*] \subset \Delta^* \tag{2.21}$$

where

$$\tilde{f}(x) = \begin{pmatrix} \hat{f}(x_1, x_2) + \hat{g}_1(x_1, x_2)Nx_2 \\ Mx_2 \end{pmatrix}, \qquad \tilde{g}(x) = \begin{pmatrix} \hat{g}_1(x_1, x_2) \\ \hat{g}_2 \end{pmatrix}. \tag{2.22}$$

Equation (2.20) will be our starting point for the definition of a possible generalization of $\mathcal{V}_s^*$. At this point, we make another assumption:

(A4) The zero dynamics of the system has an hyperbolic equilibrium at $x = 0$.

This implies in particular that the set of eigenvalues of the matrix

$$F = \frac{\partial \tilde{f}}{\partial x}(0) = \begin{pmatrix} \dfrac{\partial \hat{f}}{\partial x_1} & \dfrac{\partial \hat{f}}{\partial x_2} + \hat{g}_1(x_1, x_2)N \\ 0 & M \end{pmatrix}_{x_1 = 0,\ x_2 = 0} \tag{2.23}$$

has no intersection with the imaginary axis. Note that this is a restriction on $f(x)$ and $g(x)$ only, since N can be chosen arbitrarily. In that case there exist uniquely determined analytic stable and unstable manifolds through $x=0$ for the system $\dot{x}=\tilde{f}(x)$ (see the appendix). Since $\Delta^* \cap \operatorname{sp}\{g\}=0$ we have that the feedback is uniquely determined on the integral manifold M_0 of Δ^* through $x=0$ (see [6]). This implies that the vector field $\tilde{f}$ is uniquely determined on M_0. Therefore, the stable manifold S_0 through $x=0$ is completely contained in M_0, because S_0 as well as M_0 is invariant under $\tilde{f}$. As noted earlier we will, if possible, construct a distribution Δ_s^* now (which will a priori depend on the choice of the vector N) that plays the role of $\mathscr{V}_s^*$ in this nonlinear setting.

Definition 2.1. A stable distribution Δ_s is a distribution that is locally controlled invariant, contained in Δ^*, and for which the linearization of the dynamics restricted to the leaf of Δ_s through $x=0$ is asymptotically stable.

Obviously, the class of stable distributions is nonempty, since the zero distribution is contained in this class. First, we try to construct a stable distribution Δ_s for which S_0 is the leaf through $x=0$. Suppose that the vectorfield X belongs to D, where D denotes the set of vector fields

$$D=\left\{\operatorname{ad}_{\tau_l}\operatorname{ad}_{\tau_{l-1}}\cdots\operatorname{ad}_{\tau_1}\tau_0 \mid l\in\mathbb{N},\ \tau_0,\ldots,\tau_l\in\{\tilde{f},\tilde{g}\}\right\}. \tag{2.24}$$

Then the manifold $X^t(S_0)$ should be an integral manifold of this distribution Δ_s (for, if Δ_s is invariant under $\tilde{f}$ and $\tilde{g}$, then it is invariant under all Lie brackets of these vector fields). Assume that S_0 has dimension k. By the accessibility condition (A2) it is possible to find, locally around $x=0$, independent vector fields $X_1,\ldots,X_{n-k}$ in D that are transversal to S_0. As a matter of fact, none of these X_i is equal to $\tilde{f}$, since $\tilde{f}(0)=0$. Once the order of the X_i's is fixed, the set

$$\left\{X_{n-k}^{t_{n-k}}\circ X_{n-k-1}^{t_{n-k-1}}\circ\cdots\circ X_1^{t_1}(S_0)\mid -\varepsilon\le t_i\le\varepsilon,\ 1\le i\le n-k\right\} \tag{2.25}$$

defines a foliation in a neighborhood of $x=0$. To explain this, we construct the foliation (2.25) for a one-dimensional S_0 in $\mathbb{R}^3$; the general case follows along the same lines. In this special case there exist locally around $x=0$ two independent vector fields X_1 and X_2 in the set D (see (2.24)) that are transversal to S_0. Obviously, the set $\{X_1^{t_1}(S_0)\mid -\varepsilon\le t_1\le\varepsilon\}$ defines (locally around $x=0$) a foliation on a two-dimensional manifold L in $\mathbb{R}^3$. Consider a point p outside L, but sufficiently close to $x=0$. Then there exists a t_2 such that $q:=X_2^{-t_2}(p)$ lies on one of the leaves of $\{X_1^{t_1}(S_0)\mid -\varepsilon\le t_1\le\varepsilon\}$, say S_1. Hence $p\in X_2^{t_2}(S_1)=X_2^{t_2}\circ X_1^{t_1}(S_0)$. Since p is arbitrary, the foliation (2.25) is defined on a neighborhood of $x=0$ in $\mathbb{R}^3$. Note that it depends on the order of the X_i's. The foliation (2.25) will be called stable, because it is generated by the stable manifold S_0 (that is associated with the stable eigenvalues of the matrix F in (2.23)). This terminology is in accordance with that given in Palis & de Melo [9]. (Note that a stable distribution is *not* just the distribution associated with a stable foliation (cf. Definition 2.1).) If the foliation (2.25) is invariant under $\tilde{f}$ and $\tilde{g}$, it defines a distribution Δ_s that is invariant under these vector fields. In that case the foliation (2.25) does not depend on the order of the X_i's.

Lemma 2.1. *Suppose that the foliation* (2.25) *is invariant under $\tilde{f}$ and $\tilde{g}$. Then it uniquely defines a stable distribution Δ_s. Moreover, Δ_s is independent of the choice of N in* (2.17).

Proof. The uniqueness of Δ_s (for a fixed N) follows immediately from the definition of the stable manifold S_0 and the construction of the foliation (2.25). The inclusion $\Delta_s\subset\Delta^*$ follows from the fact that both Δ_s and Δ^* are invariant under $\tilde{f}$ and $\tilde{g}$ and $S_0\subset M_0$. Consider eq. (2.20) now with $d\equiv 0$ and assume that $x_1=(x_{11}^{\mathrm{T}}, x_{12}^{\mathrm{T}})^{\mathrm{T}}$ is chosen such that

$$\Delta_s=\operatorname{sp}\left\{\frac{\partial}{\partial x_{11}}\right\}. \tag{2.26}$$

Then we have, by the invariance of Δ_s under $\tilde{f}$ and $\tilde{g}$,

$$\begin{aligned}\dot{x}_{11} &= \hat{f}_{11}(x_{11}, x_{12}, x_2) + \hat{g}_{11}(x_{11}, x_{12}, x_2)Nx_2 + \hat{g}_{11}(x_{11}, x_{12}, x_2)v,\\ \dot{x}_{12} &= \hat{f}_{12}(x_{12}, x_2) + \hat{g}_{12}(x_{12}, x_2)Nx_2 + \hat{g}_{12}(x_{12}, x_2)v,\\ \dot{x}_2 &= Mx_2 + \hat{g}_2 v,\\ y &= \hat{h}(x_2).\end{aligned} \tag{2.27}$$

It follows immediately from (2.27) that Δ_s is invariant under $\tilde{f}$ and $\tilde{g}$ for every feedback that leaves $\Delta^* = \mathrm{sp}\{\partial/\partial x_{11}, \partial/\partial x_{12}\}$ invariant, whether this feedback is linear or nonlinear. This is implied by the fact that every such feedback

$$v = \phi(x_2) + w, \quad \phi(x_2) = 0, \tag{2.28}$$

with ϕ arbitrary only depends on x_2. By choosing $v = -Nx_2 + \tilde{N}x_2 + w$ with $\tilde{N}$ an arbitrary vector for which $A_1 + B_1\tilde{N}$ is unstable, it is easy to see that Δ_s does not depend on the choice of N in (2.17). □

In the linear case, $\mathscr{V}_s^*$ is the maximal controlled invariant subspace in ker C such that the dynamics of the system restricted to this subspace is asymptotically stable. Motivated by this linear paradigm, we investigate if there exists a maximal element in the class of stable distributions, i.e. we search for a locally controlled invariant distribution Δ_s^* contained in Δ^* for which the linearization of the dynamics restricted to the leaf of Δ_s^* through $x = 0$ is asymptotically stable and that contains all stable distributions.

In case the foliation (2.25) is invariant under $\tilde{f}$ and $\tilde{g}$, the stable distribution Δ_s that is defined by this foliation is maximal, since S_0 has maximal dimension (for S_0 is the stable manifold through $x = 0$ of the vector field $\tilde{f}$ in (2.22)). Hence, the maximal stable distribution Δ_s^* equals Δ_s.

The next question is of course if Δ_s^* exists when (2.25) is not invariant under $\tilde{f}$ and $\tilde{g}$. Clearly the construction of a foliation like the one given in (2.25) may be repeated for any stable manifold S contained in S_0 (see the appendix). If the foliation generated by S is invariant under $\tilde{f}$ and $\tilde{g}$, then it can be proved along the lines of Lemma 2.1 that there exists a stable distribution $\Delta(S)$ associated with it. Since the zero distribution is stable, we can define $\bar{\Delta}_s$ as the involutive sum of all stable distributions. Clearly, the integral manifold $\tilde{S}$ of $\bar{\Delta}_s$ through x = 0 is stable, maximal and contains all integral manifolds S through $x = 0$ of the stable distributions $\Delta(S)$ that are contained in $\bar{\Delta}_s$. Along the lines of Lemma 2.1 we can prove now that $\bar{\Delta}_s$ is a uniquely defined stable distribution independent of the choice of N in (2.17). In fact $\bar{\Delta}_s$ equals Δ_s^*, since the integral manifold $\tilde{S}$ of $\bar{\Delta}_s$ through $x = 0$ is the largest manifold that generates a stable distribution.

We have proved the following lemma.

Lemma 2.2. *There exists a uniquely defined maximal stable distribution* Δ_s^*. *Moreover, if* Δ^* *is invariant under* $\tilde{f}$ *and* $\tilde{g}$, *then so is* Δ_s^*. □

Corollary. *For systems with m inputs and m outputs* Δ_s^* *can be defined along the same lines as above if we replace assumption* (A3) *by the following*:

(A3′) The characteristic numbers $\rho_1(x), \ldots, \rho_m(x)$ are constant and the decoupling matrix $A(x)$ is invertible in a neighborhood of $x = 0$.

Remarks. (i) Unfortunately, the existence result on Δ_s^* does not give a method to construct this distribution in practice.

(ii) Although the definition of Δ_s^* is fairly analogous to the construction of $\mathscr{V}_s^*$ there is an important difference, mainly due to the fact that we have the extra (strong!) requirement that Δ_s^* should be invariant under $\tilde{g}$. This is the reason why the dimension of Δ_s^* can be strictly less than the number of stable eigenvalues of the matrix F in (2.23).

By now, the solution to the DDPS for (2.1) is straightforward. For convenience, we choose new coordinates $x=(z_1^{\mathrm{T}}, z_2^{\mathrm{T}})^{\mathrm{T}}$ such that

$$\Delta_s^* = \operatorname{sp}\left\{\frac{\partial}{\partial z_1}\right\}. \tag{2.29}$$

This yields, instead of (2.19),

$$\begin{aligned}\dot{z}_1 &= \tilde{\tilde{f}}_1(z_1, z_2) + \tilde{\tilde{g}}_1(z_1, z_2)v + \tilde{\tilde{e}}_1(z_1, z_2)d,\\ \dot{z}_2 &= \tilde{\tilde{f}}_2(z_2) + \tilde{\tilde{g}}_2(z_2)v + \tilde{\tilde{e}}_2(z_1, z_2)d,\\ y &= \tilde{\tilde{h}}(z_2).\end{aligned} \tag{2.30}$$

Assume:

(A5) $((\partial\tilde{\tilde{f}}_2/\partial z_2)(0),\ \tilde{\tilde{g}}_2(0))$ is a controllable pair.

(A6) $e \in \Delta_s^*$.

Assumption (A6) implies that $\tilde{\tilde{e}}_2$ is identically equal to zero and (A5) that there exists a linear feedback

$$v = Gz_2 + w \tag{2.31}$$

such that the system (2.30), (2.31),

$$\begin{aligned}\dot{z}_1 &= \tilde{\tilde{f}}_1(z_1, z_2) + \tilde{\tilde{g}}_1(z_1, z_2)Gz_2 + \tilde{\tilde{g}}_1(z_1, z_2)w + \tilde{\tilde{e}}_1(z_1, z_2)d,\\ \dot{z}_2 &= \tilde{\tilde{f}}_2(z_2) + \tilde{\tilde{g}}_2(z_2)Gz_2 + \tilde{\tilde{g}}_2(z_2)w,\\ y &= \tilde{\tilde{h}}(z_2),\end{aligned} \tag{2.32}$$

is locally exponentially stable around $z=0$ (see [7]). Notice that Δ_s^* is invariant under both the drift vector field

$$\check{f}(z) = \begin{pmatrix}\tilde{\tilde{f}}_1(z_1, z_2) + \tilde{\tilde{g}}_1(z_1, z_2)Gz_2\\ \tilde{\tilde{f}}_2(z_2) + \tilde{\tilde{g}}_2(z_2)Gz_2\end{pmatrix} \quad\text{and}\quad \check{g}(z) = \begin{pmatrix}\tilde{\tilde{g}}_1(z_1, z_2)\\ \tilde{\tilde{g}}_2(z_2)\end{pmatrix}. \tag{2.33}$$

We conclude that $z=0$ is an exponentially stable equilibrium point of the system $\dot{z}=\check{f}(z)$ and that $e\in\Delta_s^*$, $[\check{f}, \Delta_s^*]\subset\Delta_s^*$, $[\check{g}, \Delta_s^*]\subset\Delta_s^*$, and $\Delta_s^*\subset\ker \mathrm{d}h$; hence the DDPS for (2.1) is solved.

For convenience, we summarize the result in the following theorem.

Theorem 2.1. *Consider the system* (2.1). *Assume that* (A1) *up to* (A5) *hold. Then the* DDPS *for* (2.1) *is locally solvable if and only if* $e\in\Delta_s^*$.

Proof. The only assertion that has to be proved is the 'only if' part. Assume that the DDPS is solvable. It follows from [5, p. 131] that there exists a distribution Δ that is invariant under $\check{f}$ and $\check{g}$ such that

$$e\in\Delta\subset\ker \mathrm{d}h. \tag{2.34}$$

Since the feedback system has $x=0$ as a (locally) exponentially stable equilibrium point, the integral manifold of Δ through $x=0$ is certainly stable. Hence, by definition of Δ_s^*, $\Delta\subset\Delta_s^*$, and consequently $e\in\Delta_s^*$. □

Obviously, Theorem 2.1 is also valid for systems with as many inputs as outputs for which assumption (A3′) holds and for systems with an arbitrary number of disturbances.

Remark. Consider the following controllable linear system with nonlinear disturbances:

$$\dot{x} = Ax + Bu + e(x)d,$$
$$y = Cx. \tag{2.35}$$

Suppose that (A1), (A2) and (A3) hold. Choose $u = Fx + v$ in such a way that $(A + BF)\mathscr{V}^* \subset \mathscr{V}^*$ and the matrix $A + BF$ has no eigenvalues on the imaginary axis (i.e. (A4) is valid). Then Δ_s^* is just the flat distribution $\Delta_{\mathscr{V}_s^*} \cong \mathscr{V}_s^*$. This implies that the DDPS for this system is solvable if and only if $e \in \Delta_s^*$ (i.e. (A6)). (Note that assumption (A5) is induced by the controllability of the system.) This result is in agreement with Theorem 2.2 in [12].

3. Examples

In this section we give two examples to illustrate the theory given in the previous section. In the first one the dimension of Δ_s^* is equal to the number of stable eigenvalues of the matrix F in (2.23), whereas in the second one this dimension is strictly less.

Example 3.1. Consider the analytic control system

$$\dot{x} = f(x) + g(x)u + e(x)d, \quad y = h(x), \tag{3.1}$$

with

$$f(x) = \begin{pmatrix} -x_1 + x_3 \\ x_2 + x_3 \\ (x_2+1)x_3 \end{pmatrix}, \quad g(x) = \begin{pmatrix} x_1(1+x_1) \\ 0 \\ 1 + x_1 \end{pmatrix}, \quad e(x) = \begin{pmatrix} x_2^2\,e^{x_3} \\ 0 \\ 0 \end{pmatrix}, \quad h(x) = x_3. \tag{3.2}$$

Now $L_g h = x_1 + 1 \neq 0$ in a neighborhood of $x = 0$. Hence $\rho = 0$ and $\Delta^* = \mathrm{sp}\{\partial/\partial x_1, \partial/\partial x_2\}$. Choose

$$u = \frac{1}{1+x_1}[-(x_2+1)x_3 + x_3 + v]. \tag{3.3}$$

Notice that (3.3) is well defined in a neighborhood of $x = 0$. Now equations (3.1)–(3.3) yield

$$\dot{x} = \tilde{f}(x) + \tilde{g}(x)v + e(x)d, \quad y = h(x), \tag{3.4}$$

with

$$\tilde{f}(x) = \begin{pmatrix} -x_1 - x_1x_2x_3 + x_3 \\ x_2 + x_3 \\ x_3 \end{pmatrix}, \qquad \tilde{g}(x) = \begin{pmatrix} x_1 \\ 0 \\ 1 \end{pmatrix}. \tag{3.5}$$

Obviously, Δ^* is invariant under $\tilde{f}$ and $\tilde{g}$. In this case

$$F = \frac{\partial \tilde{f}}{\partial x}(0) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.6}$$

so the stable eigenspace of F is $\mathrm{sp}\{e_1\}$. Moreover, since

$$\left[\tilde{f}, \frac{\partial}{\partial x_1}\right] = \begin{pmatrix} 1 + x_2x_3 \\ 0 \\ 0 \end{pmatrix} \in \mathrm{sp}\left\{\frac{\partial}{\partial x_1}\right\}, \qquad \left[\tilde{g}, \frac{\partial}{\partial x_1}\right] = -\frac{\partial}{\partial x_1}, \tag{3.7}$$

it is obvious that Δ_s^* equals $\mathrm{sp}\{\partial/\partial x_1\}$ in this case. And so, for this simple example it is not necessary to calculate the foliation (2.25) starting from the stable manifold $\{x \mid x_2 = x_3 = 0\}$. Note that $e \in \Delta_s^*$ thus

(A6) is satisfied. Easy calculations show that assumptions (A2) and (A5) are fulfilled. Choose another feedback

$$v = -4x_2 - 4x_3 + w \tag{3.8}$$

then the system (3.4), (3.5), (3.8) has the form

$$\dot{x} = \tilde{\tilde{f}}(x) + \tilde{g}(x)w + e(x)d, \quad y = h(x), \tag{3.9}$$

with

$$\tilde{\tilde{f}}(x) = \begin{pmatrix} -x_1 + x_3 - x_1x_2x_3 - 4x_1x_2 - 4x_1x_3 \\ x_2 + x_3 \\ -4x_2 - 3x_3 \end{pmatrix} \tag{3.10}$$

It is easy to verify that $\tilde{\Delta}_s^*$ is invariant under $\tilde{\tilde{f}}$ and that $x = 0$ is an exponentially stable equilibrium point of the equation $\dot{x} = \tilde{\tilde{f}}(x)$. This implies that the local DDPS for (3.1), (3.2) is solvable by applying the feedback (3.3), (3.8).

Example 3.2. Consider the system (3.1) with

$$f(x) = \begin{pmatrix} -2x_1 \\ -x_2 + x_4 \\ x_3 - x_2x_3 \\ 3x_4 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad h(x) = x_4, \quad e(x) \text{ arbitrary.} \tag{3.11}$$

In this case $L_g h = 1$, hence $\rho = 0$ and $\Delta^* = \text{sp}\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3\}$. Oviously, Δ^* is invariant under f and g and

$$\frac{\partial f}{\partial x}(0) = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \tag{3.12}$$

and the stable manifold through $x = 0$ is $S_0 = \{x \mid x_3 = x_4 = 0\}$. Define

$$k := [f,g] = \begin{bmatrix} 2 \\ -2 \\ -x_3 - 1 + x_2 \\ -3 \end{bmatrix}, \quad l := [g,k] = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \end{bmatrix},$$
$$p := [f,k] = \begin{bmatrix} 4 \\ 1 \\ -x_2 - 3x_3 + x_4 + x_2x_3 - (1 - x_2)(-x_3 - 1 + x_2) \\ 9 \end{bmatrix}. \tag{3.13}$$

In $x = 0$ we have

$$\text{rank}[g, k, l, p](0) = 4, \tag{3.14}$$

and so, the system is accessible.

We will now construct a stable foliation starting from S_0. Let

$$S_{s,t} := l^s \circ g^t(S_0). \tag{3.15}$$

Since $S_0 = \{(x_{10}, x_{20}, 0, 0)^{\mathrm{T}} \mid x_{10} \in \mathbb{R},\ x_{20} \in \mathbb{R}\}$, $S_{s,t}$ is given by

$$S_{s,t} := \left\{ (t + x_{10},\ -t + x_{20},\ t - 2s,\ t)^{\mathrm{T}} \mid x_{10} \in \mathbb{R},\ x_{20} \in \mathbb{R},\ t,\ s \in \mathbb{R} \right\}. \tag{3.16}$$

Obviously, the set $\{S_{s,t} \mid s,\ t \in \mathbb{R}\}$ gives a foliation in $\mathbb{R}^4$. The distribution Δ associated with it is $\Delta = \mathrm{sp}\{\partial/\partial x_1,\ \partial/\partial x_2\}$. Unfortunately, this Δ is not invariant under f as follows from

$$\left[f, \frac{\partial}{\partial x_2} \right] = \begin{bmatrix} 0 \\ 1 \\ x_3 \\ 0 \end{bmatrix} \notin \Delta \quad (\text{unless } x_3 = 0). \tag{3.17}$$

This implies that Δ_s^* has dimension one at most. Since $\mathrm{sp}\{\partial/\partial x_1\}$ is invariant under f and g and has stable manifold $S_0^1 = \{x \mid x_2 = x_3 = x_4 = 0\}$ through $x = 0$ it follows that $\Delta_s^* = \mathrm{sp}\{\partial/\partial x_1\}$.

Now consider the dynamics modulo Δ_s^*. This dynamics is given by ($d \equiv 0$)

$$\dot{\bar{x}} = \bar{f}(\bar{x}) + \bar{g}(\bar{x})u \tag{3.18}$$

where

$$\bar{f}(\bar{x}) = \begin{pmatrix} -x_2 + x_4 \\ x_3 - x_2 x_3 \\ 3x_4 \end{pmatrix}, \qquad \bar{g}(\bar{x}) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}. \tag{3.19}$$

It can easily be verified that the pair $((\partial\bar{f}/\partial\bar{x})(0),\ \bar{g}(0))$ is controllable. This implies that the DDPS for this system is locally solvable if $e \in \Delta_s^* = \mathrm{sp}\{\partial/\partial x_1\}$.

4. Conclusions

In this section we want to compare our results to those derived by Byrnes & Isidori [2]. They consider a smooth (i.e. C^∞) system of the form (2.1). Under certain conditions they solve the Disturbance Decoupling Problem with BIBO-stability.

If we focus our attention to local results for the time being (ignoring the global aspects of the solution in [2]) we see that Byrnes & Isidori have three important conditions that have to be fulfilled in order to solve this problem, namely they assume that (A1) and (A3) hold and:

(H4) The system (2.1) is exponentially minimum phase.

I.e., $x = 0$ is an exponentially stable equilibrium point of the zero dynamics. This is a strong condition, as can easily be seen if the system (2.1), (2.2) is linear. As noted earlier, under the assumption (A3), $\mathscr{R}^* = 0$ and so for a linear system the zero dynamics is equal to $(A + BF)\mid_{\mathscr{V}^*}$ where F is such that $(A + BF)\mathscr{V}^* \subset \mathscr{V}^*$. The minimum-phase condition implies that $\sigma((A + BF)\mid_{\mathscr{V}^*}) \subset \mathbb{C}^-$, hence $\mathscr{V}_s^*$ equals $\mathscr{V}^*$. (Also in case $\mathscr{R}^* \neq 0$, (H4) is equivalent to the equality of $\mathscr{V}_s^*$ and $\mathscr{V}^*$.) Since we do not require a condition like (H4) to hold in Section 2, in this sense our results are more general.

The paper [2] has an interesting feature that we do not find back in our treatment, namely the existence of a globally defined feedback. Even if we assume that the characteristic numbers $\rho_i(x)$ are constant for all x, then still our result is local, because the feedback (2.31) can only assure local stability in general. The search for the existence of globally defined stabilizing feedbacks is an interesting problem.

In order to solve the DDPS in practice the problem of finding an algorithm to calculate Δ_s^* explicitly needs attention. Some other problems that are related to the one we treated in Section 2 are e.g. the definition of Δ_s^* in case the largest controllability distribution in the kernel of the output mapping is not the zero distribution and the Noninteracting Control Problem with Stability.

Appendix

Consider a smooth vector field $f: \mathbb{R}^n \to \mathbb{R}^n$ with $f(0)=0$. If the Jacobian $Df(0)$ has no eigenvalues on the imaginary axis then f is called *hyperbolic*. For such a vector field f the following theorem holds.

Theorem (Hartman [4]). *In a neighborhood of $x=0$ there exist uniquely defined stable and unstable manifolds S_s and S_u that are invariant under f with the same dimensions n_s and n_u as the stable and unstable subspaces W_s and W_u of the system $\dot{z} = Df(0)z$, while in $x=0$, S_s and S_u are tangent to W_s and W_u, respectively. Moreover, if f is C^∞ (C^ω), then so are S_s and S_u.* □

In Section 2 we sometimes call a manifold $S \subset S_s$ of dimension strictly less than n_s a stable manifold if S is invariant under f and the tangent space of S in $x=0$ is an invariant subspace of the system $\dot{z} = Df(0)z$. It is clear from the context what is meant. A foliation is called stable (unstable) if the leaf through $x=0$ is a stable (unstable) manifold of some known hyperbolic vector field f. Furthermore, a vector field g on a manifold M is called *transversal* to a submanifold S of M if $g(x)$ is not tangent to S at x for all $x \in S$. Finally, the vector fields $\{X_i \mid i=1,\ldots,p\}$ on a manifold M are called *independent* if $\dim \mathrm{sp}\{X_i(x) \mid i=1,\ldots,p\} = p$ for all $x \in M$.

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