# THE EXPLICIT STRUCTURE OF THE PROLONGATION ALGEBRA OF THE HIROTA–SATSUMA SYSTEM

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For a coupled system of KdV equations the prolongation Lie algebra is explicitly determined. It turns out to be of Kac–Moody type.

*1. Introduction.* Recently Hirota and Satsuma [1] introduced the system of equations:

$$u_t - a(u_{xxx} + 6uu_x) - 2b\phi\phi_x = 0,$$

$$\phi_t + \phi_{xxx} + 3u\phi_x = 0, \qquad (1)$$

a system with parameters $a$ and $b$ of coupled KdV equations for two functions $u$ and $\phi$ of the independent variables $x$ and $t$. It turns out that the value $a = \frac{1}{2}$ is most interesting. $b$ can be scaled to any nonzero number, e.g. $b = -3$. Dodd and Fordy [2] used the Wahlquist–Estabrook prolongation technique [3] to study system (1) and they found, that for $a = \frac{1}{2}$ this system possesses a linear scattering problem and a Lax pair and so they proved system (1) to be completely integrable in a certain sense. In a recent paper Wilson [4] pointed out that system (1) is an example of a general construction of Drinfel'd and Sokolov [5]. This construction involves infinite-dimensional Kac–Moody algebras. He showed that system (1) is connected with the affine Kac–Moody algebra $C_2^{(1)}$. In the construction of Drinfel'd and Sokolov the KdV equation itself is related to the algebra $A_1^{(1)}$. In the present note the structure of the general prolongation algebra of system (1) will be explicitly determined for the value $a = \frac{1}{2}$ and $b = -3$.

It will turn out to be a direct product $H_5 \times S$ of a 5-dimensional Heisenberg algebra $H_5$ and S. S is an infinite subalgebra of $C_2 \otimes C[t]$ generated by 10 elements. Here $C_2$ is the classical 10-dimensional simple Lie algebra. The KdV prolongation algebra is isomorphic to the direct product $H \times (A_1 \otimes C[t])$, see refs. [6,7]. For the nonlinear Schrödinger equation the general structure of the prolongation algebra is also known [8].

The results mentioned above strongly suggest that there must be a close relation, yet to be discovered, between the Drinfel'd–Sokolov construction and the Wahlquist–Estabrook prolongation method. The results in this note have been derived with intensive use of symbolic computations in REDUCE.

*2. Determination of the prolongation form.* In accordance with Dodd and Fordy we describe system (1) by the following differential ideal I, generated by the differential forms:

$$\alpha_1 = du \wedge dt - p\,dx \wedge dt, \quad \alpha_2 = dp \wedge dt - r\,dx \wedge dt,$$

$$\alpha_3 = d\phi \wedge dt - q\,dx \wedge dt, \quad \alpha_4 = dq \wedge dt - s\,dx \wedge dt,$$

$$\alpha_5 = du \wedge dx - a\,dt \wedge dr - 6(q\phi - aup)dx \wedge dt,$$

$$\alpha_6 = dx \wedge d\phi + dt \wedge ds - 3uq\,dx \wedge dt. \qquad (2)$$

Here $p = u_x; r = u_{xx}; q = \phi_x; s = \phi_{xx}$.

The ideal I is prolonged to the ideal I′ by adding to it the Lie algebra valued 1-form:

$$\omega = dy + F\,dx + G\,dt,$$

where $F$ and $G$ are functions of $u, \phi, p, q, r, s$ and $y$, still to be determined. Requiring the ideal I′ to be closed, yields the following conditions for $F$ and $G$:

$$F_s = F_r = F_q = F_p = 0, \quad F_\phi + G_s = F_u - 2G_r = 0,$$

$$6\phi q F_u + pG_u + qG_\phi + rG_p + sG_q$$

$$- 3upF_u + 3uqF_\phi - [F, G], \tag{3}$$

where $[F, G] = FG_y - GF_y$. This system of equations can be integrated with respect to $u, \phi, p, r$ and $s$. The integration constants, which are Lie algebra elements and functions of $y$, are not all linear independent. The dependencies are discovered by checking various Jacobi identities.

Finally the following expression for $F$ and $G$ are found:

$$F = (u^2 - 2\phi^2)X_4 + \phi X_3 + uX_2 + X_1,$$

$$G = 2uX_{10} - 4\phi X_9 + (u^2 - 2\phi^2)X_8 + 4X_7 + 2pX_6 - 4qX_5$$

$$+ 2(8\phi s - 4q^2 - p^2 + 2ru + 4u^3)X_4 - 4(u\phi + s)X_3$$

$$+ 2(r + 3u^2 - 6\phi^2)X_2. \tag{4}$$

This result is somewhat more general than obtained by Doff and Fordy [2]. In addition the $X_i$ have to satisfy the following commutator relations, where for simplicity we write $X_i X_j$ in place of $[X_i, X_j]$:

$$X_1X_4 = X_1X_6 = X_1X_7 = X_2X_3 = X_2X_4 = X_3X_4$$

$$= X_4X_5 = X_4X_6 = X_4X_8 = X_4X_9 = X_4X_{10} = 0,$$

$$X_1X_2 = X_6, \quad X_1X_3 = X_5, \quad X_1X_5 = X_9,$$

$$X_1X_6 = X_{10}, \quad X_1X_9 = X_3X_7, \quad X_1X_{10} = -2X_2X_7,$$

$$X_2X_5 = -2X_3, \quad X_2X_6 = X_8, \quad X_2X_{10} = X_3X_9$$

$$= (-4X_4X_7 - 6X_6)/3, \quad X_3X_5 = X_8,$$

$$X_3X_6 = -2X_3, \quad X_3X_{10} = -2X_5. \tag{5}$$

Using the Jacobi identity, the following relations must hold too:

$$X_2X_8 = X_3X_8 = X_5X_6 = 0,$$

$$X_1X_8 = X_3X_9 = X_2X_9 = -2X_5,$$

$$X_5X_8 = -4X_3, \quad X_5X_{10} = X_6X_9, \quad X_6X_8 = 2X_8.$$

Table A

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $X_1$ | 0 | $X_6$ | $X_5$ | 0 | $X_9$ | $X_{10}$ | 0 | $X_{11}$ | $X_{14}$ | $X_{12}$ |
| $X_2$ | | 0 | 0 | 0 | $-2X_3$ | $X_8$ | $-X_{12}/2$ | 0 | $-2X_5$ | $X_{11}$ |
| $X_3$ | | | 0 | 0 | $X_8$ | $-2X_3$ | $X_{14}$ | 0 | $X_{11}$ | $-2X_5$ |
| $X_4$ | | | | 0 | 0 | 0 | $(-3X_{11}-6X_6)/4$ | 0 | 0 | 0 |
| $X_5$ | | | | | 0 | 0 | $-X_{16}/4$ | $-4X_3$ | $(X_{13}-2X_{10})/2$ | $-2X_9$ |
| $X_6$ | | | | | | 0 | $X_{15}/2$ | $2X_8$ | $-2X_9$ | $X_{13}$ |
| $X_7$ | | | | | | | 0 | $-X_{12}$ | $X_{19}/4$ | $X_{18}/16$ |
| $X_8$ | | | | | | | | 0 | $4X_5$ | $-2X_{11}$ |
| $X_9$ | | | | | | | | | 0 | $2X_{14}$ |
| $X_{10}$ | | | | | | | | | | 0 |

| | $X_{11}$ | $X_{12}$ | $X_{13}$ | $X_{14}$ | $X_{15}$ |
|---|---|---|---|---|---|
| $X_1$ | $-2X_{10}$ | $-X_{15}$ | 0 | $-X_{16}/4$ | $-X_{18}/8$ |
| $X_2$ | $-2X_8$ | $-X_{13}-2X_{10}$ | $-2X_{11}$ | 0 | $2X_{12}$ |
| $X_3$ | $4X_3$ | 0 | $4X_5$ | $(-X_{13}-2X_{10})/2$ | $-4X_{14}$ |
| $X_4$ | 0 | 0 | 0 | 0 | 0 |
| $X_5$ | 0 | $-4X_{14}$ | $4X_9$ | $-X_{12}$ | $-X_{16}$ |
| $X_6$ | 0 | 0 | $4X_{10}$ | 0 | $-X_{17}/2$ |
| $X_7$ | $X_{15}$ | $X_{12}X_{17}/16$ | 0 | $X_9X_{18}/64$ | |
| $X_8$ | $4X_8$ | $2X_{13}+4X_{10}$ | $4X_{11}$ | 0 | $-4X_{12}$ |
| $X_9$ | $-4X_9$ | 0 | $4X_{14}$ | $(X_{17}+4X_{15})/8$ | $-X_{19}$ |
| $X_{10}$ | $2X_{13}$ | $-X_{17}/2$ | $4X_{12}$ | $X_{16}/2$ | 0 |
| $X_{11}$ | 0 | 0 | $-8X_{10}$ | 0 | $X_{17}$ |
| $X_{12}$ | | 0 | $-4X_{15}$ | $X_{19}$ | $-2X_{14}X_{16}+X_{12}X_{17}/4$ |
| $X_{13}$ | | | 0 | $X_{16}$ | $X_{18}/2$ |
| $X_{14}$ | | | | 0 | $X_9X_{18}/16$ |
| $X_{15}$ | | | | | 0 |

| | $X_{16}$ | $X_{17}$ | $X_{18}$ | $X_{19}$ |
|---|---|---|---|---|
| $X_1$ | $X_{19}$ | 0 | $X_{12}X_{17}$ | $X_9X_{18}/16$ |
| $X_2$ | 0 | $8X_{12}$ | $-4X_{17}+16X_{15}$ | $-2X_{16}$ |
| $X_3$ | 0 | $-16X_{14}$ | $-16X_{16}$ | $(-X_{17}+4X_{15})/2$ |
| $X_4$ | 0 | 0 | 0 | 0 |
| $X_5$ | $(X_{17}-4X_{15})/2$ | $4X_{16}$ | 0 | 0 |
| $X_6$ | $2X_{16}$ | $-8X_{15}$ | 0 | 0 |
| $X_7$ | $X_{16}X_{17}/16$ | 0 | $-2X_{16}X_{19}-X_{15}X_{18}/4$ | $-X_{17}X_{19}/16$ |
| $X_8$ | 0 | $-16X_{12}$ | $8X_{17}-32X_{15}$ | $4X_{16}$ |
| $X_9$ | $X_{18}/4$ | $4X_{19}$ | | $-X_{14}X_{16}+X_{12}X_{17}/4$ |
| $X_{10}$ | $2X_{19}$ | $X_{18}$ | $-16X_{14}X_{16}+2X_{12}X_{17}$ | $X_9X_{18}/8$ |
| $X_{11}$ | $-4X_{16}$ | $16X_{15}$ | 0 | 0 |
| $X_{12}$ | 0 | | 0 | $-X_{16}X_{17}/4$ |
| $X_{13}$ | $-4X_{19}$ | 0 | $-4X_{12}X_{17}$ | $-X_9X_{18}/4$ |
| $X_{14}$ | | $X_9X_{18}/4$ | 0 | |
| $X_{15}$ | $-X_{16}X_{17}/4$ | $16X_7X_{15}$ | | |
| $X_{16}$ | 0 | | $2X_{17}X_{19}-8X_{15}X_{19}$ | |
| $X_{17}$ | | 0 | $32X_{16}X_{19}+4X_{15}X_{18}$ | |
| $X_{18}$ | | | 0 | |
| $X_{19}$ | | | | 0 |

Table B. Basistransformation

| | |
|---|---|
| $X'_1 = 2X_4/3$ | $X_1 = (-X'_{12}-2\sqrt{2}X'_{11}-2\sqrt{2}X'_8+4X'_4)/4$ |
| $X'_2 = (X_8+2X_2)/2$ | $X_2 = -\sqrt{2}X'_9-\sqrt{2}X'_6+X'_2$ |
| $X'_3 = (X_{11}+2X_6)/2$ | $X_3 = -\sqrt{2}X'_9+\sqrt{2}X'_6$ |
| $X'_4 = (X_{13}+4X_1)/4$ | $X_4 = 3X'_1/2$ |
| $X'_5 = (X_{17}+16X_7)/16$ | $X_5 = -X'_{10}+X'_7$ |
| $X'_6 = (X_8+2X_3)/(\sqrt{4})$ | $X_6 = -X'_{10}-X'_7+X'_3$ |
| $X'_7 = (X_{11}+2X_5)/4$ | $X_7 = (-X'_{17}-X'_{16}+4\surd X_{15}+16\sqrt{2}X'_5)/(16\sqrt{2})$ |
| $X'_8 = (X_{13}-2X_{10}+4X_9)/(8\surd)$ | $X_8 = 2\sqrt{2}X'_9+2\sqrt{2}X'_6$ |
| $X'_9 = (X_8-2X_3)/(4\sqrt{2})$ | $X_9 = -\sqrt{2}X'_{11}+\sqrt{2}X'_8$ |
| $X'_{10} = (X_{11}-2X_5)/4$ | $X_{10} = (X'_{12}-2\sqrt{2}X'_{11}-2\sqrt{2}X'_8)/2$ |
| $X'_{11} = (X_{13}-2X_{10}-4X_9)/(8\sqrt{2})$ | $X_{11} = 2X'_{10}+2X'_7$ |
| $X'_{12} = (X_{13}+2X_{10})/2$ | $X_{12} = (-X'_{14}-X'_{13})/\sqrt{2}$ |
| $X'_{13} = (-2\sqrt{2}X_{14}-\sqrt{2}X_{12})/2$ | $X_{13} = X'_{12}+2\sqrt{2}X'_{11}+2\sqrt{2}X'_8$ |
| $X'_{14} = (2\sqrt{2}X_{14}-\sqrt{2}X_{12})/2$ | $X_{14} = (X'_{14}-X'_{13})/(2\sqrt{2})$ |
| $X'_{15} = (-X_{17}-4X_{15})/8$ | $X_{15} = (-X'_{17}-X'_{16}-4\sqrt{2}X'_{15})/(4\sqrt{2})$ |
| $X'_{16} = (\sqrt{2}X_{17}-4\sqrt{2}X_{16}-4\sqrt{2}X_{15})/4$ | $X_{16} = (X'_{17}-X'_{16})/(2\sqrt{2})$ |
| $X'_{17} = (\sqrt{2}X_{17}+4\sqrt{2}X_{16}-4\sqrt{2}X_{15})/4$ | $X_{17} = (X'_{17}+X'_{16}-4\sqrt{2}X'_{15})/\sqrt{2}$ |
| $X'_{18} = (-8X_{19}+X_{18})/4$ | $X_{18} = 2X'_{19}+2X'_{18}$ |
| $X'_{19} = (8X_{19}+X_{18})/4$ | $X_{19} = (X'_{19}-X'_{18})/4$ |

In the following analysis it turned out to be very comfortable to use the following $Z^1$-grading, which can be derived by assuming all relators to be homogeneous with respect to the grading.

$$\{\text{degree; element}\}: \quad \{-3;X_4\}, \quad \{-1;X_2,X_3,X_8\}, \quad \{0;X_5,X_6\}, \quad \{1;X_1,X_9,X_{10}\}, \quad \{3;X_7\}. \quad (6)$$

Introducing new elements and using only Jacobi identities we arrive at the following uncomplete Lie commutator table, see table A. Continuing in the same way (up to dimension 27) and constructing as well as analyzing finite closures of the uncomplete algebra, the structure became transparent. This becomes even more clear, by applying the transformation of the basis, defined in table B. The result of this transformation is given in table C. Obviously by $X'_1, X'_2, X'_3, X'_4$ and $X'_5$ a 5-dimensional nilpotent Heisenberg algebra $H_5$ is generated. Furthermore it is a direct factor of the complete algebra. This may be proved by mathematical induction: Take for instance one example of an induction step. Let $X'_{20} = X'_8X'_{18}$ and use the Jacobi identity

$$J(i,8,18) = (X'_iX'_8)X'_{18} + (X'_8X'_{18})X'_i - (X'_iX'_{18})X'_8 = 0.$$

This formula reduces by hypothesis to $X'_i(X'_8X'_{18}) = X'_iX'_{20} = 0$ for $i = 1,2,3,4,5$. Therefore we now know, $L = H_5 \times S$ (direct product).

To investigate the structure of the remaining part S we proceed as follows: Construct the $C_2$-module $L' = C_2 \otimes C\,[t]$, an algebra of $C_2$-Kac–Moody type. An element of $L'$ is a polynomial in $t$, with coeffi-

TABLE C transformed table A

| | $X'_1$ | $X'_2$ | $X'_3$ | $X'_4$ | $X'_5$ | $X'_6$ | $X'_7$ | $X'_8$ | $X'_9$ | $X'_{10}$ | $X'_{11}$ | $X'_{12}$ | $X'_{13}$ | $X'_{14}$ | $X'_{15}$ | $X'_{16}$ | $X'_{17}$ | $X'_{18}$ | $X'_{19}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X'_1$ | 0 | 0 | 0 | 0 | $-X'_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X'_2$ | | 0 | 0 | $-X'_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X'_3$ | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X'_4$ | | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X'_5$ | | | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X'_6$ | | | | | | 0 | $2X'_6$ | $X'_7$ | 0 | 0 | 0 | 0 | 0 | $-X'_{12}$ | $-X'_{13}$ | 0 | 0 | $2X'_{16}$ | 0 |
| $X'_7$ | | | | | | | 0 | $2X'_8$ | 0 | 0 | 0 | $-X'_{12}$ | $-X'_{13}$ | $X'_{14}$ | $X'_{15}$ | $-2X'_{16}$ | 0 | 0 | 0 |
| $X'_8$ | | | | | | | | 0 | 0 | 0 | 0 | $X'_{14}$ | $X'_{15}$ | 0 | 0 | $-X'_{18}$ | 0 | $-2X'_{14}X'_{15}$ | 0 |
| $X'_9$ | | | | | | | | | 0 | $2X'_9$ | $X'_{10}$ | 0 | $-X'_{12}$ | 0 | $-X'_{14}$ | 0 | 0 | 0 | $2X'_{17}$ |
| $X'_{10}$ | | | | | | | | | | 0 | $2X'_{11}$ | $-X'_{12}$ | $X'_{13}$ | $-X'_{14}$ | $X'_{15}$ | 0 | $-2X'_{17}$ | 0 | 0 |
| $X'_{11}$ | | | | | | | | | | | 0 | $X'_{13}$ | 0 | $X'_{15}$ | 0 | 0 | $-X'_{19}$ | 0 | $-2X'_{13}X'_{15}$ |

| | $X'_{12}$ | $X'_{13}$ | $X'_{14}$ | $X'_{15}$ | $X'_{16}$ | $X'_{17}$ | $X'_{18}$ | $X'_{19}$ |
|---|---|---|---|---|---|---|---|---|
| $X'_{12}$ | 0 | $X'_{16}$ | $X'_{17}$ | $(-X'_{19}-X'_{18})/2$ | 0 | | $X'_{14}X'_{16}$ | $X'_{14}X'_{16}$ |
| $X'_{13}$ | | 0 | $(-X'_{19}+X'_{18})/2$ | $X'_{13}X'_{15}$ | 0 | $X'_{14}X'_{16}$ | $X'_{14}X'_{19}+X'_{15}X'_{16}-X'_{15}X'_{17}$ | $-X'_{14}X'_{19}-X'_{15}X'_{16}+X'_{15}X'_{17}$ |
| $X'_{14}$ | | | 0 | | | 0 | $-X'_{14}X'_{19}$ | |
| $X'_{15}$ | | | | 0 | | | | |
| $X'_{16}$ | | | | | | 0 | | 0 |
| $X'_{17}$ | | | | | | | 0 | |
| $X'_{18}$ | | | | | | | | |
| $X'_{19}$ | | | | | | | | 0 |

cients in $C_2$. The product "$*$" is defined on the monomials as $(at^j)*(bt^k) = [a,b]\,t^{j+k}$, for $j$ and $k$ positive integers and $a$ and $b$ elements of $C_2$, and bilinearly extended in the usual way.

Let D be the subalgebra of L′, generated by $Y_1 = X''_1$, $Y_2 = X''_2$, $Y_3 = X''_3$, $Y_4 = X''_4$, $Y_5 = X''_5$, $Y_6 = X''_6$, $Y_7 = X''_7 t$, $Y_8 = X''_8 t$, $Y_9 = X''_9 t$, $Y_{10} = X''_{10} t$. Here the $X''_i$ ($i = 1, ..., 10$) are generators of the algebra $C_2$. Their commutator relations are given in table D. The algebra S can now be described as follows: Let $F = F(X'_6, X'_7, X'_8, X'_9, X'_{10}, X'_{11}, X'_{12}, X'_{13}, X'_{14}, X'_{15})$ the free Lie algebra generated by 10 letters, and interprete table C as a set of relators between these letters, which generate an ideal R in F. Then S = F/R. Indeed, consider the homomorphism $f$: F → D, generated by: $Y_1 = f(X'_6)$, $Y_2 = f(X'_7)$, $Y_3 = f(X'_8)$, $Y_4 = f(X'_9)$, $Y_5 = f(X'_{10})$, $Y_6 = f(X'_{11})$, $Y_7 = f(X'_{12})$, $Y_8 = f(X'_{13})$, $Y_9 = f(X'_{14})$, $Y_{10} = f(X'_{15})$. It is easy to prove that each relator of table C is transformed into 0.

As an example take the following relation using tables C and D:

$$X'_{14}X'_{15} = -\tfrac{1}{2}X'_8X'_{18} = -\tfrac{1}{2}X'_8(X'_{13}X'_{14} - X'_{12}X'_{15})$$

using $X'_{18} = (X'_{13}X'_{14} - X'_{12}X'_{15})X'_8X'_{18} = -2X'_{14}X'_{15}$. Now look at $f(X'_{14}X'_{15})$ as well as at $f(-\frac{1}{2}X'_8(X'_{13}X'_{14} - X'_{12}X'_{15}))$:

$$f(X'_{14}X'_{15}) = Y_9Y_{10} = X''_9 t\, X''_{10} t = X''_3 t^2 ,$$

$$f(-\tfrac{1}{2}X'_8(X'_{13}X'_{14} - X'_{12}X'_{15})) = -\tfrac{1}{2}Y_3(Y_8Y_9 - Y_7Y_{10})$$

$$= -\tfrac{1}{2}X''_3(X''_8 t\, X''_9 t - X''_7 t\, X''_{10} t)$$

$$= -\tfrac{1}{2}X''_3[\tfrac{1}{2}(-X''_5 + X''_2) - \tfrac{1}{2}(X''_5 - X''_2)]\, t^2$$

$$= -\tfrac{1}{2}X''_3X''_2 t^2 = X''_3 t^2 .$$

Table D . A $C_2$ algebra.

| | $X''_1$ | $X''_2$ | $X''_3$ | $X''_4$ | $X''_5$ | $X''_6$ | $X''_7$ | $X''_8$ | $X''_9$ | $X''_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $X''_1$ | 0 | $2X''_1$ | $X''_2$ | 0 | 0 | 0 | 0 | 0 | $-X''_7$ | $-X''_8$ |
| $X''_2$ | | 0 | $2X''_3$ | 0 | 0 | 0 | $-X''_7$ | $-X''_8$ | $X''_9$ | $X''_{10}$ |
| $X''_3$ | | | 0 | 0 | 0 | 0 | $X''_9$ | $X''_{10}$ | 0 | 0 |
| $X''_4$ | | | | 0 | $2X''_4$ | $X''_5$ | 0 | $-X''_7$ | 0 | $-X''_9$ |
| $X''_5$ | | | | | 0 | $2X''_6$ | $-X''_7$ | $X''_8$ | $-X''_9$ | $X''_{10}$ |
| $X''_6$ | | | | | | 0 | $X''_8$ | 0 | $X''_{10}$ | 0 |
| $X''_7$ | | | | | | | 0 | $X''_1$ | $X''_4$ | $(-X''_5-X''_2)/2$ |
| $X''_8$ | | | | | | | | 0 | $(-X''_5+X''_2)/2$ | $X''_6$ |
| $X''_9$ | | | | | | | | | 0 | $X''_3$ |
| $X''_{10}$ | | | | | | | | | | 0 |

Table E

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $t^4$ | $X''_1$ | $X''_2$ | $X''_3$ | $X''_4$ | $X''_5$ | $X''_6$ | | | | |
| $t^3$ | | | | | | | $X''_7$ | $X''_8$ | $X''_9$ | $X''_{10}$ |
| $t^2$ | $X''_1$ | $X''_2$ | $X''_3$ | $X''_4$ | $X''_5$ | $X''_6$ | | | | |
| $t^1$ | | | | | | | $X''_7$ | $X''_8$ | $X''_9$ | $X''_{10}$ |
| $t^0$ | $X''_1$ | $X''_2$ | $X''_3$ | $X''_4$ | $X''_5$ | $X''_6$ | | | | |
| | −2 | 0 | 2 | 0 | 0 | 0 | −1 | −1 | 1 | 1 |
| | 0 | 0 | 0 | −2 | 0 | 2 | −1 | 1 | −1 | 1 |

Therefore $f$ generates a homomorphism of S = F/R onto D. Now consider D as a free Lie algebra generated by the letters $Y_1, ..., Y_{10}$ with the relations considering them as elements of L′. Then any such relation is contained in $f$(R). For instance, using tables C and D: $Y_6Y_7 = Y_8$, $Y_6 = f(X'_{11})$, $Y_7 = f(X'_{12})$, $Y_8 = f(X'_{13})$, $f(X'_{11}X'_{12}) = f(X'_{13})$, or a second one: $Y_1Y_2 = 2Y_1$, $Y_1 = f(X'_6)$, $Y_2 = f(X'_7)$, $f(X'_6X'_7) = f(2X'_6)$. It follows that S is isomorphic with D.

The general structure of the algebra D is illustrated in table E, where the numbers on the bottom indicate the weights of the Lie algebra $C_2$.

*3. Conclusion and remarks.* We have been able to determine the prolongation algebra of the coupled system of KdV equations for a special choice of the parameters. The first two columns of the algebra reported in table E make thinking of two quantised particles, whereas the third column describes their interaction, reflecting the fact, that (1) is a system of coupled KdV equations.

## *References*


[1] R. Hirota and J. Satsuma, Phys. Lett. 85A (1981) 407.
[2] R. Dodd and A. Fordy, Phys. Lett. 89A (1982) 168.
[3] H.D. Wahlquist and F.B. Estabrook, J. Math. Phys. 16 (1976) 1.
[4] G. Wilson, Phys. Lett. 89A (1982) 332.
[5] V.G. Drinfel'd and V.V. Sokolov, Soviet Math. Dokl. 23 (1981) 457.
[6] H.N. van Eck, Proc. Kon. Ned. Akad. Wetensch. Ser A86 (1983) 149.
[7] F.B. Estabrook, Invariant differential systems and intrinsic coordinates (Math. Sci. Research Inst. Berkeley, California, 1983) pp. 1–12.
[8] H.N. van Eck, P.K.H. Gragert and R. Martini, Proc. Kon. Ned. Akad. Wetensch. Ser A86 (1983) 165.