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On Characterization of Hamiltonian Graphs

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A necessary and a sufficient condition are derived for a graph to be non-Hamiltonian.

1. INTRODUCTION

We use the terminology of [1]. Following the definition in Chapter 7 of this book where a θ graph is defined to be a block with two nonadjacent points of degree 3 and all other points of degree 2, we shall use the names θ and θ^* for the two graphs depicted in Fig. 1.

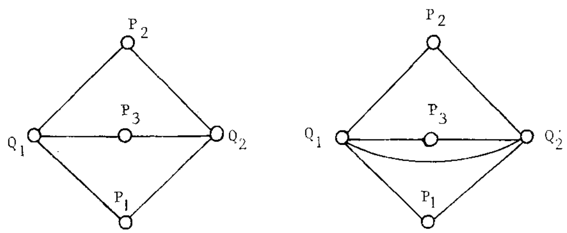

FIGURE 1

Theorem 7.2 in [1] states that every Hamiltonian graph is 2-connected and that every non-Hamiltonian 2-connected graph has a θ subgraph. Because of the contractibility of any θ graph to θ it follows that a necessary condition for a 2-connected graph to be non-Hamiltonian is that it has a subgraph contractible to θ. The main result presented in this paper is that every non-Hamiltonian 2-connected graph is contractible to either θ or θ^*.

Addition of a point to θ and connection of the new point to two points of degree 2 leads to a graph that is Hamiltonian and has θ as a subgraph. Having a θ subgraph is therefore not a sufficient condition for a 2-connected graph to be non-Hamiltonian. An example will be given to show that the

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(more general) necessary condition just mentioned fails to be a sufficient condition too. However, one can state and prove a sufficient condition for a graph to be non-Hamiltonian in terms of contractibility.

2. Preliminaries

We give some characterizations for a graph to be Hamiltonian.

THEOREM 1. *Let G be a 2-connected graph. The following statements are equivalents.*

(i) *G is Hamiltonian.*

(ii) *For any pair of cycles there is a cycle in G that contains all points of the two cycles.*

(iii) *For any pair of cycles C_1 and C_2 and any pair of points P_1 and P_2, contained in C_1 but not in C_2 and in C_2 but not in C_1, respectively, there is a cycle in G that contains the points that C_1 and C_2 have in common as well as P_1 and P_2.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): Obvious from the fact that if G is Hamiltonian, the Hamiltonian cycle contains the points of any pair of cycles.

(iii) $\Rightarrow$ (i): Suppose G satisfies the condition but G is non-Hamiltonian. Let C be a cycle of G, containing a maximum number of points. Then there is a point P' not contained in C and a cycle C' that contains P'. By the way in which C was chosen there is a point P_1 of C that is not contained in C'. According to the condition, there is a cycle C'' that contains P_1, P' and all common points of C and C'. In particular, C'' contains P' and has at least one more point in common with C than does C'.

Consider the cycles C and C''. The point P' is not contained in C and C'' contains P'. By the way in which C was chosen, there is a point P_2 of C that is not contained in C''. By the same reasoning as above, there is a cycle C''' that contains P' and has at least one more point in common with C than does C''. Since the number of points of C is finite, repetition of this procedure leads to a cycle that contains all the points of C and the point P', contradicting the choice of C. ▮

Remark. The original version of the sufficiency of the condition (iii) was considerably shortened by the referee, whose proof has been given here. Another short proof was given by A. M. Cohen who proved both characterizations given here to be equivalent to the intermediate characterization that G is Hamiltonian if and only if for any pair of cycles C_1 and C_2 and any point P in C_2 not contained in C_1 there is a cycle in G that contains P and the points of C_1.

The transition to the next couple of conditions, one necessary and one sufficient, is made by the remark that according to the characterization (iii) if a 2-connected graph is non-Hamiltonian then two cycles C_1 and C_2 can be indicated as well as two points P_1 and P_2, P_1 on C_1 but not on C_2 and P_2 on C_2 but not on C_1, with the property that there is no cycle containing P_1, P_2 and the points of $C_1 \cap C_2$. We have denoted the set of common points by $C_1 \cap C_2$. Likewise we shall denote the union of the points of the two cycles by $C_1 \cup C_2$.

3. A Necessary Condition

Identifying multiple lines whenever they occur, we have

THEOREM 2. *Every non-Hamiltonian 2-connected graph can be contracted to θ or to θ^*.*

Proof. Let G be a non-Hamiltonian 2-connected graph and C_1 and C_2 and P_1 and P_2 be cycles respectively points that do not have the property stated in characterization (iii) of Theorem 1. The two cycles must have at least three points in common. As G is 2-connected there can always be found a cycle containing P_1, P_2 and $C_1 \cap C_2$ whenever $C_1 \cap C_2$ consists of at most two points.

Let the points of $C_1 \cap C_2$ be ordered according to their occurrence in one of the cycles, say the cycle C_1, and consider the first and the last point in $C_1 \cap C_2$ when C_1 is traversed with starting point P_1. These points are not necessarily first and last points of $C_1 \cap C_2$ when C_2 is traversed with starting point P_2. Three cases may be distinguished:

(a) First and last points of $C_1 \cap C_2$, along C_2, coincide with those along C_1.

(b) First and last points of $C_1 \cap C_2$, along C_2, have one point in common with those along C_1.

(c) Both the first and last points of $C_1 \cap C_2$, along C_2, differ from those along C_1.

It is immaterial for our considerations in which of the two possible directions the cycles are traversed. We choose a direction for each of the two cycles. In all three cases the first and the last point of $C_1 \cap C_2$, along C_2, will be called Q_1 and Q_2. The three essentially different ways of intertwining of the cycles C_1 and C_2 are schematically indicated in Fig. 2, where all points of $C_1 \cap C_2$, in their ordering along C_1, are drawn. The ordering of these points along C_2 depends on the structure of the graph G. Therefore the part

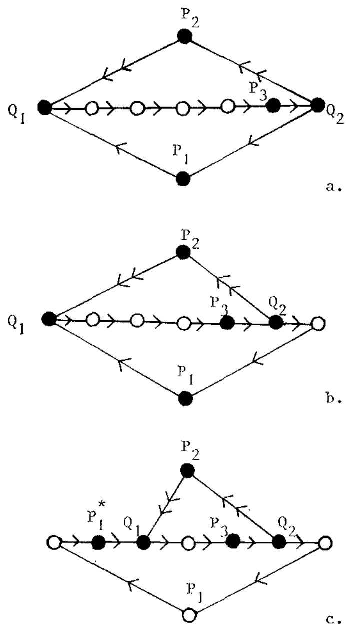

FIGURE 2

of C_2 between Q_1 and Q_2 has not been drawn. Of course, the drawn lines in general represent paths containing points of the cycles.

In each of the three figures five points have been indicated by solid points: Q_1, Q_2, P_2, the point P_3 of $C_1 \cap C_2$ preceding Q_2 along C_1 and the point P_1 in the first two cases respectively the point $P_1{}^*$ of $C_1 \cap C_2$ preceding Q_1 along C_1 in the third case. We want to show that all points of G different from these five special points can be contracted to either point Q_1 or point Q_2 and that the resulting graph is θ or θ^*. The essential part is the proof that no two points P are connected after the contraction procedure. For this we consider the points in G, if any, that are not contained in $C_1 \cup C_2$. Two points P cannot be connected by a path containing only points like these nor can they be adjacent. This follows from the fact that in all nine cases (three combinations of two points P for each of the three cases (a), (b), and (c)) a cycle can be indicated containing all points of $C_1 \cap C_2 \cup \{P_1, P_2\}$, contradictory to the choice of C_1, C_2, P_1, and P_2.

In Fig. 3 the connection between two points P is indicated by a single line added to the graphs in Fig. 2 and the cycles are indicated by arrows.

Let the subgraph on the points of $C_1 \cup C_2$ be called H. Every point of G not contained in H is connected by at least two disjoint paths to different points of H because G is 2-connected. These paths may consist of a single line or may contain other points not contained in H. As we have just shown only

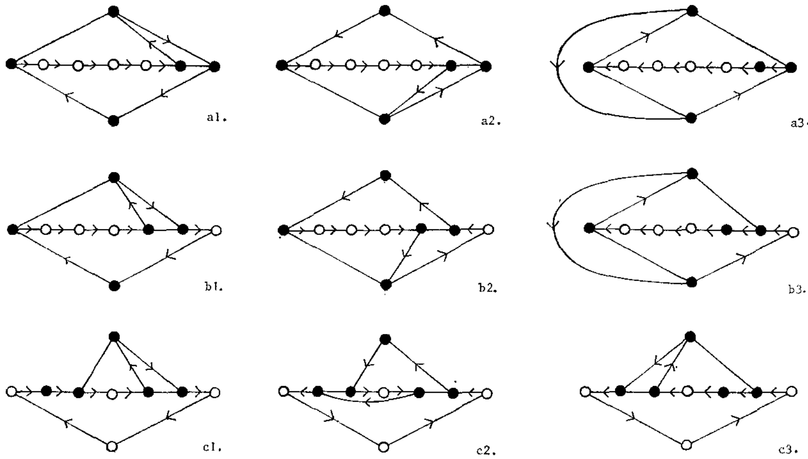

FIGURE 3

one of the end points in H can be a point P. Thus every point of G not contained in H is connected by such a path to a point in H that is not a point P. It is now clear that the following three sets of contractions lead to θ or θ^*.

The first set of contractions consists of contracting the points of G not contained in H to points of H not equal to any of the three points P. We use the lines of the paths of the kind considered above that connect the points to these points of H. The resulting graph is the graph H but for lines added in the contraction procedure. In it any path connecting two points P contains at least one point that is not a point P.

The second set of contractions consists of contracting the points of C_2 not contained in $C_1 \cap C_2$, different from P_2, to those points of $C_1 \cap C_2$, different from P_3, to which they are connected by parts of C_2, using the lines of C_2. The resulting graph contains point P_2 and the points of C_1. Again any path connecting two points P contains at least one point that is not a point P.

The third set of contractions consists of contracting the points of C_1, different from Q_1, O_2, P_3, and P_1, respectively $P_1{}^*$, to Q_1 or to Q_2 whichever can be reached along C_1 without passing a point P, using the lines of C_1. Any path connecting two points P contains Q_1 or Q_2. The resulting graph is θ^* whenever Q_1 and Q_2 were adjacent in G or have become adjacent in the contraction procedure and θ otherwise. ∎

One might hope that this necessary condition for G to be non-Hamiltonian is also sufficient. The Hamiltonian graph, given in Fig. 4 however is contractible to θ: The contracted points are enlarged.

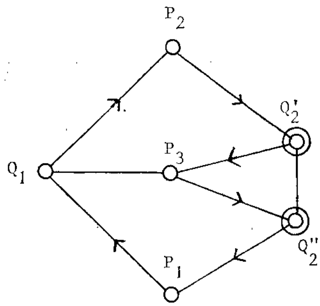

FIGURE 4

4. A Sufficient Condition

In order to find a necessary and sufficient condition one may attempt to pose extra conditions on the contraction procedure so that a case as given in Fig. 4 is not a counterexample. One such attempt is the following. A contraction of a graph G is called sharp whenever none of the final lines arises from identification of two or more lines. The following non-Hamiltonian graph is contractible to θ or θ^* but is not sharply contractible to θ or θ^*.

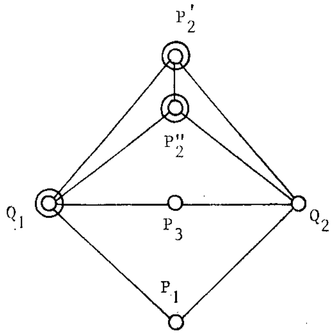

FIGURE 5

The example given in Fig. 6 is one of a Hamiltonian graph that is sharply contractible to θ^*. Thus in this way no stronger necessary condition is arrived at, nor can one state that sharp contractibility to θ^* is sufficient.

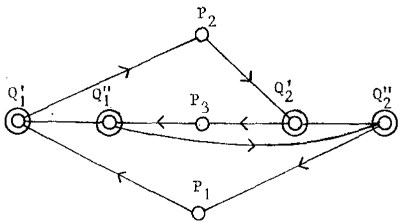

FIGURE 6

However, one can prove

THEOREM 3. *A graph G is nonhamiltonian if G is sharply contractible to θ.*

Proof. We use induction on the number of points. If G has five points there is nothing to prove.

Our induction hypothesis is that if G has $n - 1$ points ($n \geqslant 6$) and is sharply contractible to θ G is non-Hamiltonian. Suppose G has n points and is sharply contractible to θ. As $n \geqslant 6$ there always is a group of points that is contracted to one of the final points of θ consisting of a least two points. Suppose G is Hamiltonian, then the subgraph on this group of points will contain at least one line of the Hamiltonian cycle because the contraction to θ is sharp. Contraction of this line give a graph on $n - 1$ points that is sharply contractible to θ and that is Hamiltonian, contradicting the induction hypothesis. ∎

CONCLUSION

Although necessary and sufficient condition look rather alike the gap between both conditions is considerable. It is disappointing that this turns out to be so for our conditions in terms of contractibility as θ and θ^* play a role reminiscent of that played by K_5 and $K_{3,3}$ in the characterization of nonplanar graphs (see [1, Theorem 11.14]).

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REFERENCE

1. F. HARARY, "Graph Theory," Addison–Wesley, Reading, Mass., 1969.