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ON THE TIME DEPENDENT BEHAVIOUR OF THE TRUNCATED BIRTH–DEATH PROCESS

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Conditions are obtained for the truncated birth–death process to be stochastically monotone in the long run.

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1. Introduction and summary

A stochastic process $\{X(t): t \geqslant 0\}$ with discrete state space $S \subset \mathbb{R}$ is said to be stochastically increasing (decreasing) on the interval (t_1, t_2) iff for all τ_1, τ_2 with $t_1 \leqslant \tau_1 < \tau_2 < t_2$ and all $k \in S$,

$$\mathbf{P}\{X(\tau_2) \geqslant k\} \geqslant (\leqslant) \mathbf{P}\{X(\tau_1) \geqslant k\} \tag{1.1}$$

with inequality for at least one k. The purpose of this paper is to study the phenomenon of stochastic monotonicity in the context of birth–death processes on a finite state space. Such a truncated birth–death process is defined as a temporally homogeneous Markov process $\{X(t): t \geqslant 0\}$ with state space $S = \{-1, 0, 1, \ldots, N, N+1\}$, say, and transition probabilities

$$p_{ij}(t) = \mathbf{P}\{X(t+s) = j \mid X(s) = i\}$$

which satisfy the conditions

$$p_{-1,j}(t) = \delta_{-1,j} \quad \text{and} \quad p_{N+1,j}(t) = \delta_{N+1,j}, \; j \in S, t \geqslant 0 \tag{1.2}$$

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(δ_{ij} is Kronecker's delta) and, for $i=0,1,\ldots,N$,

$$\begin{aligned} p_{i,i+1}(t) &= \lambda_i t + o(t) \\ p_{i,i}(t) &= 1-(\lambda_i+\mu_i)t + o(t) \\ p_{i,i-1}(t) &= \mu_i t + o(t) \end{aligned} \tag{1.3}$$

as $t\downarrow 0$. Here λ_i and μ_i, $i=0,1,\ldots,N$, are non-negative constants: the birth and death rates. Throughout this paper we assume $\lambda_i>0$ for $i=0,1,\ldots,N-1$, and $\mu_i>0$ for $i=1,2,\ldots,N$.

In Section 3 we will show that if $\mu_0=0$ or $\lambda_N=0$, the truncated birth–death process is stochastically monotone in the long run for almost all initial distributions. The precise conditions are as follows. The process $\{X(t)\}$ is stochastically increasing in the long run if $\mu_0=0$ and $\lambda_N>0$, or, $\mu_0=\lambda_N=0$ and $\sum q_iQ_i(x_2)>0$; it is stochastically decreasing in the long run if $\mu_0>0$ and $\lambda_N=0$, or, $\mu_0=\lambda_N=0$ and $\sum q_iQ_i(x_2)<0$. Here (q_i) is the initial distribution, Q_i, $i=0,1,\ldots,N$, are the birth–death polynomials, and x_2 is the second point in the spectrum of the process.

In comparison to van Doorn [1], where birth–death processes with an infinite state space are analyzed, our approach is rather direct and we do not need the concept of dual processes and the sign variation diminishing property of the transition probability matrix of the process, although it is feasible to obtain our results by making use of them instead of the algebraic tools that are presently used.

2. Preliminaries

Using the conditions (1.2) and (1.3) and the Markovian nature of the truncated birth–death process, it is easy to show that the matrix $P(t)=(p_{ij}(t))$, $i,j\in S$ must satisfy the initial condition

$$P(0)=\mathrm{I}, \tag{2.1}$$

and the differential equations

$$P'(t)=AP(t) \tag{2.2}$$

and

$$P'(t)=P(t)A \tag{2.3}$$

where $A=(a_{ij})$, $i,j\in S$, is the matrix

$$A=\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & \cdot & \cdot \\ \mu_0 & -(\lambda_0+\mu_0) & \lambda_0 & 0 & \cdots & \cdot & \cdot \\ 0 & \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \mu_N & -(\lambda_N+\mu_N) & \lambda_N \\ \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.4}$$

It is well known (see, e.g., [6]) that the differential equations (2.2) and (2.3) (with initial condition (2.1)) have the same unique solution, which is a stochastic semi-group. That is, $P(t)$, $t \geq 0$, has the properties

$$P(t+s) = P(t)P(s), \tag{2.5}$$

$$(P(t))_{ij} \geq 0 \tag{2.6}$$

and

$$P(t)\mathbf{1} = \mathbf{1}, \tag{2.7}$$

$\mathbf{1}$ denoting the column vector consisting of 1's. It is clear now that the truncated birth–death process as introduced in Section 1 is a well-defined temporally homogeneous Markov process on the state space S.

According to Karlin [2, Theorem 3.3.4] the matrix $\bar{P}(t) = (p_{ij}(t))$, $i, j = 0, 1, \ldots, N$, is strictly totally positive for $t > 0$, which means that each subdeterminant of $\bar{P}(t)$, $t > 0$, is strictly positive. An immediate and useful consequence is

$$p_{ij}(t) > 0, \quad t > 0, \quad i, j = 0, 1, \ldots, N. \tag{2.8}$$

Another consequence which will be used in Section 3 is

$$\det(\bar{P}(t)) \neq 0, \quad t \geq 0. \tag{2.9}$$

Our next step is to give explicit expressions for the probabilities $p_{ij}(t)$, $i, j \in S$. From (2.1) and (2.3) one gets

$$p_{i,-1}(t) = \mu_0 \int_0^t p_{i0}(\tau)\, \mathrm{d}\tau, \quad i = 0, 1, \ldots, N \tag{2.10}$$

and

$$p_{i,N+1}(t) = \lambda_N \int_0^t p_{iN}(\tau)\, \mathrm{d}\tau, \quad i = 0, 1, \ldots, N. \tag{2.11}$$

Thus our attention is focused on the probabilities $p_{ij}(t)$, $i, j = 0, 1, \ldots, N$. Several authors have indicated that these have spectral representations (Ledermann and Reuter [7], Kemperman [6], Keilson [4], Karlin and McGregor [3], and Rosenlund [9]). In what follows our notation will be essentially the same as Karlin and McGregor's. The potential coefficients π_i, $i = 0, 1, \ldots, N$, of $\{X(t)\}$ are defined as

$$\pi_0 = 1, \qquad \pi_i = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \cdots \frac{\lambda_{i-1}}{\mu_i}, \quad i = 1, 2, \ldots, N. \tag{2.12}$$

Associated with the birth and death rates are also the polynomials $Q_i(x)$, $i = 0, 1, \ldots, N$, defined by the recurrence relations

$$\begin{aligned} &Q_0(x) = 1, \\ &\lambda_0 Q_1(x) = (\lambda_0 + \mu_0 - x) Q_0(x), \\ &\lambda_i Q_{i+1}(x) = (\lambda_i + \mu_i - x) Q_i(x) - \mu_i Q_{i-1}(x), \quad i = 1, 2, \ldots, N-1, \end{aligned} \tag{2.13}$$

and the polynomial $Q_{N+1}(x)$ of degree $N+1$, defined by

$$Q_{N+1}(x) = (\lambda_N + \mu_N - x)Q_N(x) - \mu_N Q_{N-1}(x). \tag{2.14}$$

Karlin and McGregor [3] have shown that $Q_{N+1}(x)$ has $N+1$ distinct, real zeros $x_1 < x_2 < \cdots < x_{N+1}$. They give the spectral representation of $p_{ij}(t)$, $i, j = 0, 1, \ldots, N$, as

$$p_{ij}(t) = \pi_j \sum_{k=1}^{N+1} \exp(-x_k t) Q_i(x_k) Q_j(x_k) \rho_k \tag{2.15}$$

where

$$1/\rho_k = \sum_{i=0}^{N} Q_i^2(x_k)\pi_i. \tag{2.16}$$

It follows by induction that for $x < 0$

$$1 = Q_0(x) < Q_1(x) < \cdots < Q_N(x). \tag{2.17}$$

As a consequence of (2.17) and (2.14) one has $Q_{N+1}(x) > \lambda_N Q_N(x) \geqslant 0$ for $x < 0$, whence we conclude $x_1 \geqslant 0$. Considering that

$$Q_i(0) = \begin{cases} 1 & \text{if } i = 0, \\ 1 + \mu_0 \sum_{k=0}^{i-1} 1/\lambda_k \pi_k & \text{if } i = 1, 2, \ldots, N, \end{cases} \tag{2.18}$$

so that

$$Q_{N+1}(0) = \lambda_N + \mu_0/\pi_N + \lambda_N \mu_0 \sum_{k=0}^{N-1} 1/\lambda_k \pi_k, \tag{2.19}$$

it follows that the next, more detailed statement holds.

Lemma 2.1. (i) *If* $\mu_0 = \lambda_N = 0$, *then* $x_1 = 0$.
(ii) *If* $\mu_0 > 0$ *or* $\lambda_N > 0$, *then* $x_1 > 0$.

Because of (2.1) and (2.15) one has

$$\pi_j \sum_{k=1}^{N+1} Q_i(x_k) Q_j(x_k) \rho_k = \delta_{ij}, \quad i, j = 0, 1, \ldots, N, \tag{2.20}$$

which exhibits the fact that the polynomials $Q_i(x)$, $i = 0, 1, \ldots, N$, are orthogonal polynomials belonging to the mass distribution with masses ρ_k located at the $N+1$ points x_k. Szegö [11] gives in Chapter III of his book a number of general properties of orthogonal polynomials. Although these are formulated in terms of an infinite system, it is easily verified by adapting Szegö's proofs that the next two lemmas hold with regard to our finite system $\{Q_i(x), i = 0, 1, \ldots, N\}$.

Lemma 2.2. *The zeros of the polynomials $Q_i(x)$, $i=1,2,\dots,N$, are real and distinct. They are located in the interval (x_1, x_{N+1}).*

Lemma 2.3.

$$\sum_{n=0}^{i} Q_n^2(x)\pi_n = \begin{cases} \lambda_i\pi_i\{Q_i'(x)Q_{i+1}(x)-Q_{i+1}'(x)Q_i(x)\} & \text{if } 0\leq i<N, \\ \pi_N\{Q_N'(x)Q_{N+1}(x)-Q_{N+1}'(x)Q_N(x)\} & \text{if } i=N. \end{cases}$$

Since $\sum_{n=0}^{i} Q_n^2(x)\pi_n \geq Q_0^2(x)\pi_0 = 1$ for $i=0,1,\dots,N$, we have the following corollary.

Corollary 2.4.

$$Q_i'(x)Q_{i+1}(x) > Q_{i+1}'(x)Q_i(x), \quad i=0,1,\dots,N.$$

We note that as a consequence of this result the polynomials $Q_i(x)$ and $Q_{i+1}(x)$ cannot have common zeros.

In Section 3 we shall encounter the problem to determine whether the sequence $Q_0(x_k)$, $Q_1(x_k),\dots,Q_N(x_k)$ is monotone or not, where $k=1$ if $\mu_0>0$ or $\lambda_N>0$ (exclusively), and $k=1,2,\dots,N+1$ if $\mu_0=\lambda_N=0$. The case $k=1$ is covered by the next theorem.

Theorem 2.5. (i) *If $\mu_0=\lambda_N=0$, then $Q_i(x_1)=1$ for $i=0,1,\dots,N$.*

(ii) *If $\mu_0=0$ and $\lambda_N>0$, then $1=Q_0(x_1)>Q_1(x_1)>\cdots>Q_N(x_1)>0$.*

(iii) *If $\mu_0>0$ and $\lambda_N=0$, then $Q_N(x_1)>Q_{N-1}(x_1)>\cdots>Q_0(x_1)=1$.*

Proof. Considering Lemma 2.1 and (2.18), (i) is evident.

To prove (ii) and (iii) we observe that $x_1Q_i(x_1)>0$ if $\mu_0>0$ or $\lambda_N>0$, in view of Lemma 2.1, Lemma 2.2 and (2.18). Consequently, by (2.13), $Q_1(x_1)<Q_0(x_1)=1$ if $\mu_0=0$ and $\lambda_N>0$, and (ii) subsequently follows by induction. Furthermore, by (2.14) one obtains $Q_N(x_1)>Q_{N-1}(x_1)>0$ if $\mu_0>0$ and $\lambda_N=0$, and then (iii) follows by induction, using (2.13).

Let $\boldsymbol{u}=(u_0,u_1,\dots,u_m)^{\mathrm{T}}$ be a vector of real numbers. We denote by $S(\boldsymbol{u})$ the number of sign changes in the sequence $u_0,u_1,\dots,u_m$ by deleting all zero terms, with the special convention $S(\mathbf{0})=-1$, $\mathbf{0}$ denoting the vector consisting of 0's. The solution of the aforementioned problem for the case $\mu_0=\lambda_N=0$ is now given by the next theorem.

Theorem 2.6. *Let $\mu_0=\lambda_N=0$, $0\leq k\leq N$ and $\boldsymbol{\Delta}(x)=(Q_1(x)-Q_0(x), Q_2(x)-Q_1(x),\dots,Q_N(x)-Q_{N-1}(x))^{\mathrm{T}}$. Then $S(\boldsymbol{\Delta}(x_{k+1}))=k-1$. Moreover, the first component of $\boldsymbol{\Delta}(x_{k+1})$ is negative if $k>0$.*

As a consequence of Theorem 2.5(i) the above theorem is valid for $k=0$. The proof for $k>0$ has been relegated to the appendix.

3. Stochastic monotonicity

Let $\boldsymbol{q} = (q_{-1}, q_0, \ldots, q_N, q_{N+1})^{\mathrm{T}}$ be the initial distribution vector of the truncated birth–death process $\{X(t): t \geqslant 0\}$. To avoid inessential difficulties we will assume $q_{-1} = q_{N+1} = 0$. Furthermore let $p_i(t) = \mathbf{P}\{X(t) = i \mid \boldsymbol{q}\}$, $i = -1, 0, \ldots, N, N+1$. Let $\boldsymbol{p}(t) = (p_{-1}(t), p_0(t), \ldots, p_N(t), p_{N+1}(t))^{\mathrm{T}}$, then

$$\boldsymbol{p}^{\mathrm{T}}(t) = \boldsymbol{q}^{\mathrm{T}}P(t) \quad \text{and} \quad \boldsymbol{p}^{\mathrm{T}}(t)\mathbf{1} = \boldsymbol{q}^{\mathrm{T}}\mathbf{1} = 1. \tag{3.1}$$

We define the vector $\boldsymbol{e}(t) = (e_{-1}(t), e_0(t), \ldots, e_N(t), e_{N+1}(t))^{\mathrm{T}}$ for $t \geqslant 0$ as

$$\boldsymbol{e}^{\mathrm{T}}(t) = (\mathrm{d}/\mathrm{d}t)\boldsymbol{p}^{\mathrm{T}}(t)T \tag{3.2}$$

where T is the lower triangular matrix with entries $t_{ij} = 1$ if $i \geqslant j$ and $t_{ij} = 0$ otherwise. Then T^{-1} is given by $(T^{-1})_{ij} = 1$ if $i = j$, $(T^{-1})_{ij} = -1$ if $i = j+1$ and 0 otherwise. From (3.1), (2.2) and (2.5) we find

$$\begin{aligned}\boldsymbol{e}^{\mathrm{T}}(t+s) &= \boldsymbol{q}^{\mathrm{T}}P'(t+s)T = \boldsymbol{q}^{\mathrm{T}}AP(t+s)T \\ &= \boldsymbol{q}^{\mathrm{T}}AP(t)P(s)T = \boldsymbol{q}^{\mathrm{T}}AP(t)TT^{-1}P(s)T = \boldsymbol{e}^{\mathrm{T}}(t)T^{-1}P(s)T,\end{aligned}$$

whence

$$\boldsymbol{e}(t+s) = (T^{-1}P(s)T)^{\mathrm{T}}\boldsymbol{e}(t). \tag{3.3}$$

From [5, Theorem 2.1] we obtain

$$(T^{-1}P(s)T)_{ij} \geqslant 0, \tag{3.4}$$

a result that we will use in the proof of the next lemma. Vector inequality is defined as

$$\boldsymbol{a} \leqslant \boldsymbol{b} \text{ iff } a_i \leqslant b_i \text{ for all } i.$$

Lemma 3.1. *Let* $\varepsilon > 0$. *In* (a) *and* (b) *below, the statements* (i), (ii) *and* (iii) *are equivalent.*

(a) (i) $\{X(t)\}$ *is stochastically increasing on* $(t_1, t_1 + \varepsilon)$.
  (ii) $\{X(t)\}$ *is stochastically increasing on* (t_1, ∞).
  (iii) $\mu_0 = 0$, $\boldsymbol{e}(t_1) \geqslant \mathbf{0}$ *and* $\boldsymbol{e}(t_1) \neq \mathbf{0}$.

(b) (i) $\{X(t)\}$ *is stochastically decreasing on* $(t_1, t_1 + \varepsilon)$.
  (ii) $\{X(t)\}$ *is stochastically decreasing on* (t_1, ∞).
  (iii) $\lambda_N = 0$, $\boldsymbol{e}(t_1) \leqslant \mathbf{0}$ *and* $\boldsymbol{e}(t_1) \neq \mathbf{0}$.

Proof. (a) Let $\{X(t)\}$ be stochastically increasing on $(t_1, t_1 + \varepsilon)$. Then in particular, $\mathbf{P}\{X(\tau_2) \geqslant 0\} \geqslant \mathbf{P}\{X(\tau_1) \geqslant 0\}$, i.e., $p_{-1}(\tau_2) \leqslant p_{-1}(\tau_1)$, for $t_1 \leqslant \tau_1 < \tau_2 < t_1 + \varepsilon$, so that $\mu_0 = 0$ in view of (2.10) and (2.8). Furthermore for $j = -1, 0, \ldots, N, N+1$,

$$e_j(t_1) = [(\mathrm{d}/\mathrm{d}t)\boldsymbol{p}^{\mathrm{T}}(t)T]_{t=t_1} = [(\mathrm{d}/\mathrm{d}t)\mathbf{P}\{X(t) \geqslant j\}]_{t=t_1},$$

which is evidently non-negative. Finally $\boldsymbol{e}(t_1) \neq \mathbf{0}$, for in view of (3.3) the opposite would imply equality in (1.1) for all k and $\tau_2 > \tau_1 \geqslant t_1$.

Now let $\mu_0 = 0$, $e(t_1) \geqslant \mathbf{0}$ and $e(t_1) \neq \mathbf{0}$. As a consequence of (2.9) and (1.2), $P(s)$, $s \geqslant 0$, is regular, whence $T^{-1}P(s)T$ is regular for $s \geqslant 0$. $T^{-1}P(s)T$ therefore has no zero rows. It follows by (3.3) and (3.4) that $e(t_1+s) \geqslant \mathbf{0}$ and $e(t_1+s) \neq \mathbf{0}$ for all $s \geqslant 0$, whence $\{X(t)\}$ is stochastically increasing on (t_1, ∞) and a fortiori on $(t_1, t_1+\varepsilon)$.

(b) is proven similarly.

Remark 3.2. The equivalence of (i) and (ii) in (a) and (b) is well known (see, e.g., [10]).

Obviously we may write

$$e^{\mathrm{T}}(t) = q^{\mathrm{T}}P'(t)T = q^{\mathrm{T}}P(t)AT = p^{\mathrm{T}}(t)AT, \tag{3.5}$$

so that in particular

$$e^{\mathrm{T}}(0) = q^{\mathrm{T}}AT. \tag{3.6}$$

It follows with Lemma 3.1 that $\{X(t)\}$ is stochastically increasing (decreasing) on the whole positive time axis iff $q^{\mathrm{T}}AT \neq \mathbf{0}$ and $q^{\mathrm{T}}AT \geqslant (\leqslant)\mathbf{0}$, a result that has been derived previously by Keilson and Kester [5, Theorem 3.4]. In this paper we are concerned with the question whether $\{X(t)\}$ is stochastically monotone in the long run or not. We proceed by observing that (3.5), (3.1) and (2.4) imply

$$e_j(t) = \begin{cases} 0 & \text{if } j=-1, \\ -\mu_0 p_0(t) & \text{if } j=0, \\ \lambda_{j-1}p_{j-1}(t) - \mu_j p_j(t) & \text{if } j=1,2,\ldots,N, \\ \lambda_N p_N(t) & \text{if } j=N+1. \end{cases} \tag{3.7}$$

Substitution of the spectral representation (2.15) of $p_{ij}(t)$ in (3.7) yields for $j=0,1,\ldots,N$

$$e_{j+1}(t) = \lambda_j \pi_j \sum_{k=1}^{N+1} \exp(-x_k t)\{Q_j(x_k) - Q_{j+1}(x_k)\} \sum_{i=0}^{N} q_i Q_i(x_k). \tag{3.8}$$

Considering that the first non-zero term in the above sum becomes dominant as t grows, the next theorem is readily obtained as a result of Theorem 2.5 and Lemma 3.1.

Theorem 3.3. (i) *If $\mu_0 > 0$ and $\lambda_N = 0$, then $\{X(t)\}$ is stochastically decreasing in the long run.*

(ii) *If $\mu_0 = 0$ and $\lambda_N > 0$, then $\{X(t)\}$ is stochastically increasing in the long run.*

In case $\mu_0 = \lambda_N = 0$ we have $Q_i(x_1) = 1$ for all $i = 0, 1, \ldots, N$ by Theorem 2.5. Consequently (3.8) reduces to

$$e_{j+1}(t) = \lambda_j \pi_j \sum_{k=1}^{N} \exp(-x_{k+1}t)\{Q_j(x_{k+1}) - Q_{j+1}(x_{k+1})\} \sum_{i=0}^{N} q_i Q_i(x_{k+1}) \tag{3.9}$$

with $j = 0, 1, \dots, N$. If $\sum_{i=0}^{N} q_i Q_i(x_{k+1}) = 0$ for $k = 1, 2, \dots, N$ (which can be shown to be the case iff the initial distribution is the stationary distribution), then $\boldsymbol{e}(t) = \boldsymbol{0}$, so that the process is not stochastically monotone. Next suppose that $\hat{x}$, the smallest of the x_{k+1}, $k = 1, 2, \dots, N$, for which $\sum_{i=0}^{N} q_i Q_i(x_{k+1}) \neq 0$, exists. If $\hat{x} > x_2$, it follows from (3.9) and Theorem 2.6 that for t sufficiently large (and hence for all $t > 0$ by Lemma 3.1) there will be components of $\boldsymbol{e}(t)$ with opposite sign, whence $\{X(t)\}$ is nowhere stochastically monotone. If $\hat{x} = x_2$, however, then the non-zero components of $\boldsymbol{e}(t)$ will have the same sign for t sufficiently large by Theorem 2.6. Thus we have the next theorem.

Theorem 3.4. *Let* $\mu_0 = \lambda_N = 0$.

(i) $\{X(t)\}$ *is stochastically increasing in the long run iff* $\sum_{i=0}^{N} q_i Q_i(x_2) > 0$.
(ii) $\{X(t)\}$ *is stochastically decreasing in the long run iff* $\sum_{i=0}^{N} q_i Q_i(x_2) < 0$.
(iii) $\{X(t)\}$ *is nowhere stochastically monotone iff* $\sum_{i=0}^{N} q_i Q_i(x_2) = 0$.

Appendix. Proof of Theorem 2.6 for $k>0$

For the proof of Theorem 2.6 for $k > 0$ we will resort to Sturm's theorem. Before we can state and apply this theorem, however, we need some preliminaries.

Lemma A.1. *Let* $\boldsymbol{u} = (u_0, u_1, \dots, u_m)^{\mathrm{T}}$, *with* $m > 0$, *be a vector of real numbers with the properties* (i) $u_0 \neq 0$, (ii) $u_m \neq 0$ *and* (iii) *if* $u_i = 0$ $(0 < i < m)$, *then* $u_{i-1}u_{i+1} < 0$. *With* $\tilde{\boldsymbol{u}} = (\tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_m)^{\mathrm{T}}$ *denoting the vector having components* $\tilde{u}_i = (-1)^i u_i$, *one has* $S(\boldsymbol{u}) + S(\tilde{\boldsymbol{u}}) = m$.

Proof. Let X_m $(m > 0)$ be the set of vectors $\boldsymbol{u} = (u_0, u_1, \dots, u_m)^{\mathrm{T}}$ satisfying the conditions (i), (ii) and (iii), and let Prop($\boldsymbol{u}$), with $\boldsymbol{u} = (u_0, u_1, \dots, u_m)^{\mathrm{T}}$, denote the proposition $S(\boldsymbol{u}) + S(\tilde{\boldsymbol{u}}) = m$. The proof is readily established by induction through verification of the next four statements, where $\boldsymbol{u} \in X_m$.

(I) If $m = 1$, then Prop($\boldsymbol{u}$).
(II) If $m = 2$ and $u_1 = 0$, then Prop($\boldsymbol{u}$).
(III) If $m > 1$ and $u_{m-1} \neq 0$ and (if $\boldsymbol{v} \in X_{m-1}$, then Prop($\boldsymbol{v}$)), then Prop($\boldsymbol{u}$).
(IV) If $m > 2$ and $u_{m-1} = 0$ and (if $\boldsymbol{v} \in X_{m-2}$, then Prop($\boldsymbol{v}$)), then Prop($\boldsymbol{u}$).

Definition A.2. A sequence of $m + 1 > 1$ polynomials $P_0, P_1, \dots, P_m$ is called a *Sturmian sequence* on the interval (a, b) iff the following four conditions are satisfied:

(i) $P_m(x) \neq 0$ for $x = a, b$.
(ii) $P_0(x) \neq 0$ for all $x \in [a, b]$.
(iii) If $P_i(x) = 0$ $(0 < i < m)$ and $x \in [a, b]$, then $P_{i-1}(x)P_{i+1}(x) < 0$.
(iv) If $P_m(x) = 0$ and $x \in [a, b]$, then $P_{m-1}(x)P'_m(x) > 0$.

The next theorem holds [8, Satz 7].

Theorem A.3. (Sturm's theorem). *If the sequence of polynomials $P_0, P_1, \ldots, P_m$ is a Sturmian sequence on the interval (a, b), then the number of zeros of P_m in the interval (a, b) equals $S(\boldsymbol{P}(a)) - S(\boldsymbol{P}(b))$ where $\boldsymbol{P}(x) = (P_0(x), P_1(x), \ldots, P_m(x))^{\mathrm{T}}$.*

We define the polynomials $U_i(x)$, $i = 0, 1, \ldots, N, N+1$, as

$$\begin{aligned} U_0(x) &= 1, \\ U_{i+1}(x) &= \lambda_i \pi_i (Q_{i+1}(x) - Q_i(x)), \quad i = 0, 1, \ldots, N-1, \\ U_{N+1}(x) &= \pi_N (Q_{N+1}(x) - \lambda_N Q_N(x)). \end{aligned} \tag{A.1}$$

Since $\lambda_i \pi_i = \mu_{i+1} \pi_{i+1}$ for $i = 0, 1, \ldots, N-1$, we obtain from (2.13) and (2.14) the relations

$$\begin{aligned} U_1(x) &= \mu_0 - x, \\ U_{i+1}(x) &= U_i(x) - x Q_i(x) \pi_i, \quad i = 1, 2, \ldots, N. \end{aligned} \tag{A.2}$$

Lemma A.4. *Let $b > a > 0$ be such that $U_{N+1}(a) \neq 0$ and $U_{N+1}(b) \neq 0$, then the sequence of polynomials $U_0(x), -U_1(x), U_2(x), \ldots, (-1)^{N+1} U_{N+1}(x)$ is a Sturmian sequence on the interval (a, b).*

Proof. The conditions (i) and (ii) of Definition A.2 are clearly satisfied.

To prove (iii) suppose that $U_i(\hat{x}) = 0$, with $0 < i \leqslant N$ and $x \in [a, b]$. If $i = 1$ we have $Q_1(\hat{x}) = Q_0(\hat{x}) = 1$, and from (A.2) it is seen that $U_2(\hat{x}) = -\hat{x} Q_1(\hat{x}) \pi_1$, whence $U_0(\hat{x}) U_2(\hat{x}) = -\hat{x} \pi_1 < 0$, considering that $\hat{x} \geqslant a > 0$. If $i > 1$ we have $Q_i(\hat{x}) = Q_{i-1}(\hat{x})$, and from (A.2) we obtain $U_{i-1}(\hat{x}) = \hat{x} Q_{i-1}(\hat{x}) \pi_{i-1}$ and $U_{i+1}(\hat{x}) = -\hat{x} Q_i(\hat{x}) \pi_i$. Consequently,

$$(-1)^{i-1} U_{i-1}(\hat{x}) (-1)^{i+1} U_{i+1}(\hat{x}) = U_{i-1}(\hat{x}) U_{i+1}(\hat{x}) = -\hat{x}^2 Q_i^2(\hat{x}) \pi_i \pi_{i-1}.$$

The latter is strictly negative since $Q_i(\hat{x}) = Q_{i-1}(\hat{x})$, and we know from Corollary 2.4 that Q_i and Q_{i-1} do not have common zeros. So condition (iii) is satisfied.

Finally suppose $\hat{x} \in [a, b]$ and $U_{N+1}(\hat{x}) = 0$. From (A.1) we see $U'_{N+1}(\hat{x}) = \pi_N (Q'_{N+1}(\hat{x}) - \lambda_N Q'_N(\hat{x}))$ and from (A.2), $U_N(\hat{x}) = \hat{x} Q_N(\hat{x}) \pi_N$. Furthermore $Q'_N(\hat{x}) Q_{N+1}(\hat{x}) > Q_N(\hat{x}) Q'_{N+1}(\hat{x})$ by Corollary 2.4. Combining these results we have

$$\begin{aligned} (-1)^N U_N(\hat{x}) (-1)^{N+1} U'_{N+1}(\hat{x}) &= -\hat{x} Q_N(\hat{x}) \pi_N^2 (Q'_{N+1}(\hat{x}) - \lambda_N Q'_N(\hat{x})) \\ &> -\hat{x} Q'_N(\hat{x}) \pi_N^2 (Q_{N+1}(\hat{x}) - \lambda_N Q_N(\hat{x})) = -\hat{x} Q'_N(\hat{x}) \pi_N U_{N+1}(\hat{x}) = 0 \end{aligned}$$

Hence condition (iv) is satisfied too.

As a result of the above lemma, Sturm's theorem and Lemma A.1 the next lemma holds.

Lemma A.5. *Let $b > a > 0$ and $U_{N+1}(x) \neq 0$ for $x = a, b$. The number of zeros of $U_{N+1}(x)$ in the interval (a, b) equals $S(\boldsymbol{U}(b)) - S(\boldsymbol{U}(a))$ where $\boldsymbol{U}(x) = (U_0(x), U_1(x), \ldots, U_N(x), U_{N+1}(x))^{\mathrm{T}}$.*

We recall that we must determine the number of sign changes in the sequence

$$Q_1(x_{k+1}) - Q_0(x_{k+1}),\ Q_2(x_{k+1}) - Q_1(x_{k+1}),\ \ldots,\ Q_N(x_{k+1}) - Q_{N-1}(x_{k+1}),$$

i.e. the number of sign changes in the sequence

$$U_1(x_{k+1}),\ U_2(x_{k+1}),\ \ldots,\ U_N(x_{k+1})$$

for $k > 0$. A few more steps must be taken to settle the problem.

Lemma A.6. *If $\mu_0 = 0$, then $S(\boldsymbol{U}(\varepsilon)) = 1$ for $\varepsilon > 0$ sufficiently small.*

Proof. We find from (A.2),

$$U_0(0) = 1, \qquad U_{i+1}(0) = \mu_0, \quad i = 0, 1, \ldots, N. \tag{A.3}$$

If $\mu_0 = 0$, then $Q_i'(0) > Q_{i+1}'(0)$, $i = 0, 1, \ldots, N-1$, and $\lambda_N Q_N'(0) > Q_{N+1}'(0)$ by (2.18), (2.19) and Corollary 2.4. Consequently we obtain from (A.1):

$$\text{If } \mu_0 = 0 \text{ and } 0 \leqslant i \leqslant N, \text{ then } U_{i+1}'(0) < 0. \tag{A.4}$$

The lemma follows readily from (A.3) and (A.4).

Lemma A.7. *Let $\lambda_N = 0$, then $S(\boldsymbol{U}(x_k - \varepsilon)) = S(\boldsymbol{U}(x_k))$ for $\varepsilon > 0$ sufficiently small.*

Proof. When $\lambda_N = 0$, then $U_{N+1}(x_k) = \pi_N Q_{N+1}(x_k) = 0$ by (A.1). We also have, by Lemma A.4 and Definition A.2, that $U_N(x_k) U_{N+1}'(x_k) < 0$. Consequently
(I) $U_N(x_k - \varepsilon) U_{N+1}(x_k - \varepsilon) \geqslant 0$ for $0 \leqslant \varepsilon < \delta_{N+1}$, say.

If $0 \leqslant m \leqslant N$ and $U_m(x_k) = 0$, then, by Lemma A.4 and Definition A.2, $U_{m-1}(x_k) U_{m+1}(x_k) < 0$, whence
(IIa) $U_{m-1}(x_k - \varepsilon) U_{m+1}(x_k - \varepsilon) < 0$ for $0 \leqslant \varepsilon < \delta_m$, say.

Finally, if $0 \leqslant m \leqslant N$ and $U_m(x_k) > (<) 0$, then
(IIb) $U_m(x_k - \varepsilon) > (<) 0$ for $0 \leqslant \varepsilon < \delta_m$, say.

(I), (IIa) and (IIb) are easily seen to imply that $S(\boldsymbol{U}(x_k - \varepsilon)) = S(\boldsymbol{U}(x_k))$ for $0 < \varepsilon < \delta = \min \delta_m$.

Now let $\mu_0 = \lambda_N = 0$, $0 < k \leqslant N$ and $\varepsilon > 0$ so small that $\varepsilon < x_2$, $\varepsilon < x_{k+1} - x_k$, $S(\boldsymbol{U}(\varepsilon)) = 1$ and $S(\boldsymbol{U}(x_{k+1} - \varepsilon)) = S(\boldsymbol{U}(x_{k+1}))$. Obviously the number of zeros of $U_{N+1}(x) = \pi_N Q_{N+1}(x)$ in the interval $(\varepsilon, x_{k+1} - \varepsilon)$ equals $k-1$. Therefore Lemma A.5 implies $S(\boldsymbol{U}(x_{k+1} - \varepsilon)) - S(\boldsymbol{U}(\varepsilon)) = k - 1$. Thus, by the Lemmas A.6 and A.7,

$$S(\boldsymbol{U}(x_{k+1})) = k. \tag{A.5}$$

We further note that $U_0(x_{k+1}) = 1$ and, by (A.2), $U_1(x_{k+1}) = -x_{k+1} < 0$, so that one of the k sign changes in $\boldsymbol{U}(x_{k+1})$ occurs between U_0 and U_1. Finally $U_{N+1}(x_{k+1}) =$

$\pi_N Q_{N+1}(x_{k+1}) = 0$. These observations and (A.5) complete the proof of Theorem 2.6 for $k > 0$.

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