VI. Conclusion

This short paper has argued that strong versions of controllability and observability underlie a range of linear multivariable control problems, and hence are closely related to the existing approaches to these problems. These system concepts have been used to unify the various approaches, by recovering their basic results from a consistent point of view. The simplicity of the proofs involved indicates that this new approach is an effective tool in its own right, and several multivariable control problems are being reworked from this point of view.

References

[1] W. M. Wonham, *Linear Multivariable Control, A Geometric Approach* (Lecture Notes in Economics and Mathematical Systems), vol. 101. Berlin: Springer, 1974.

[2] L. M. Silverman, "Inversion of multivariable linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-14, pp. 270–276, June 1969.

[3] ——, "Input-output structure of linear systems with application to the decoupling problem," *SIAM J. Contr.*, vol. 9, pp. 199–233, May 1971.

[4] D. Rappaport and L. M. Silverman, "Structure and stability of discrete-time optimal systems," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 227–232, June 1971.

[5] H. J. Payne and L. M. Silverman, "On the discrete-time Riccati equation," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 226–234, June 1973.

[6] L. M. Silverman, "Discrete Riccati equations: Alternative algorithms, asymptotic properties and system theory interpretations," in *Advances in Control and Dynamic Systems: Theory and Applications*, vol. 12. New York: Academic, 1975.

[7] M. Aoki and M. T. Li, "Partial reconstruction of state vectors in decentralized dynamic systems," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 289–292, June 1973.

[8] M. Aoki, "On subspaces associated with partial reconstruction of state vectors, the structure algorithm, and the predictable directions of Riccati equations," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 399–400, Aug. 1973.

[9] B. P. Molinari, "Extended controllability and observability for linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 136–137, Feb. 1976.

[10] G. Basile and G. Marro, "Controlled and conditioned invariant subspaces in linear system theory," *J. Optimiz. Theory Appl.*, vol. 3, pp. 306–315, May 1969.

[11] ——, "On the observability of linear, time-invariant systems with unknown inputs," *J. Optimiz. Theory Appl.*, vol. 3, pp. 411–415, 1969.

[12] D. Rappaport, "Constant directions of the Riccati equation," *Automatica*, vol. 8, pp. 179–186, Mar. 1972.

[13] M. Gevers and T. Kailath, "Constant, predictable and degenerate directions of the discrete-time Riccati equation," *Automatica*, vol. 9, pp. 699–711, Nov. 1973.

[14] B. D. O. Anderson, "Output-nulling invariant and controllability subspaces," in *Proc. 1976 IFAC Congr.*

[15] M. Morf and T. Kailath, "Square-root algorithms for least-squares estimation," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 487–497, Aug. 1975.

[16] A. S. Morse, "Structural invariants of linear multivariable systems," *SIAM J. Contr.*, vol. 11, pp. 446–465, Aug. 1973.

Stochastic Linear Differential Game with a Square Integrable Martingale as Noise

ARUNABHA BAGCHI

***Abstract*—The problem of a stochastic linear differential game with any square integrable Martingale as the noise is solved. The solution is obtained by converting the problem to an optimization problem in a Hilbert space.**

I. Introduction

The function space approach of Balakrishnan [1] can be used to extend the well-known stochastic linear regulator problem to any square integrable Martingale as noise [2]. In this short paper we use a similar approach to solve a stochastic differential game problem with a fixed time quadratic cost function, and with perfect observation when the system is corrupted by any square integrable Martingale as noise. In particular, it includes any zero mean, independent increment process, not necessarily sample continuous, as noise affecting the evolution of the system.

Manuscript received December 29, 1975; revised May 28, 1976. Paper recommended by N. R. Sandell, Jr., Chairman of the IEEE S-CS Large Scale Systems, Differential Games Committee.

The author is with the Department of Applied Mathematics, Twente University of Technology, Enschede, The Netherlands.

II. Problem Formulation

Let (Ω, F, P) be the basic probability space where F is complete with respect to P, and suppose that we are given an increasing family of σ-algebras $F_t C F$, $0 \leqslant t < \infty$, each containing all P-null sets and $F_t = U_{\epsilon > 0} F_{t+\epsilon}$. Let R^n denote n-dimensional Euclidean space. A right continuous R^n-valued stochastic process $Z(t;\omega)$ is called a square integrable Martingale with respect to F_t if

1) $Z(t;\omega)$ is measurable F_t for each $t \geqslant 0$; $EZ(t;\omega) = 0$;
2) $E(\|Z(t;\omega)\|^2) < \infty$ for every $t \in [0,\infty)$;
3) $E(Z(t;\omega)|F_s) = Z(s;\omega)$ a.s. for every $t \leqslant s$.

If $\beta_Z(t)$ is the smallest σ-algebra containing all P-null sets generated by $Z(s;\omega)$, $0 \leqslant s \leqslant t$, then $Z(t;\omega)$ is a square integrable Martingale with respect to $\beta_Z(t)$ as well. Let the associated increasing $n \times n$ matrix-valued process of $Z(t;\omega)$ that follows from the decomposition theorem of Kunita and Watanabe [3] be denoted by $\langle Z, Z \rangle_t$.

We consider a zero-sum two person differential game with the knowledge of the state of the system up to the present time being available to both players. We use subscript p to denote one player referred to as pursuer and subscript e to denote the other player referred to as evader. The differential equation describing the game is given by

$$x(t;\omega) = \int_0^t A(s)x(s;\omega)\,ds + \int_0^t B_p(s)u_p(s;\omega)\,ds + \int_0^t B_e(s)u_e(s;\omega)\,ds + Z(t;\omega), \qquad 0 \leqslant t \leqslant T \quad (1)$$

where $x(t;\omega)$ is an $n \times 1$ vector denoting the state, $u_p(\cdot;\omega)$ is an $r_1 \times 1$ vector strategy of the pursuer, $u_e(\cdot;\omega)$ is an $r_2 \times 1$ vector strategy of the evader, $Z(t;\omega)$ is an R^n-valued square integrable Martingale defined above, and $A(\cdot), B_p(\cdot), B_e(\cdot)$ are $n \times n$, $n \times r_1$, $n \times r_2$ deterministic matrix functions, respectively, which we assume to be continuous. We want the strategies $u_p(t;\omega)$ and $u_e(t;\omega)$ at time t to depend on the "state" up to time t; that is, on $x(s;\omega)$, $0 \leqslant s \leqslant t$. In this restricted class, our problem is to find

$$\sup_{u_e} \inf_{u_p} E\left\{ [Qx(T;\omega), x(T;\omega)] + \lambda \int_0^T \|u_p(t;\omega)\|^2\,dt - \mu \int_0^T \|u_e(t;\omega)\|^2\,dt \right\}, \qquad \lambda > 0, \mu > 0. \quad (2)$$

We now make precise this restricted class for strategies u_p and u_e. It is implicit that $u_p(t;\omega)$ and $u_e(t;\omega)$ are jointly measurable in t and ω. We seek to solve the sup-inf problem (2) in the space $H_{p,e}$ of the pair of right continuous processes $(u_p(\cdot;\omega), u_e(\cdot;\omega))$, $0 \leqslant t \leqslant T$, such that

$$\int_0^T E\|u_p(t;\omega)\|^2\,dt < \infty, \qquad u_p(t;\omega) \text{ is adapted to } \beta_x(t);$$

similar conditions hold for $u_e(\cdot;\omega)$ and for which (1) has a solution, where $\beta_x(t)$ is the smallest σ-algebra generated by $x(s;\omega)$, $0 \leqslant s \leqslant t$.

If there exists a pair $(u_p(\cdot;\omega), u_e(\cdot;\omega))$, $0 \leqslant t \leqslant T$, in $H_{p,e}$, the solution $x(t;\omega)$ of (1) is adapted to $\beta_Z(t)$ and consequently, both $u_p(t;\omega)$ and $u_e(t;\omega)$ are adapted to $\beta_Z(t)$. Let H_Z^i, $i = 1,2$, be classes of R^{r_i}-valued stochastic processes $y_i(t;\omega)$, jointly measurable in t and ω and right continuous, for which

$$\int_0^T E\|y_i(t;\omega)\|^2\,dt < \infty$$

and

$$y_i(t;\omega) \text{ is adapted to } \beta_Z(t).$$

It follows immediately that $H_{p,e} C H_Z^1 \times H_Z^2$. We solve the sup-inf problem (2) in the broader class $H_Z^1 \times H_Z^2$ of admissible strategies. We shall see that the solution belongs to the class $H_{p,e}$ and therefore, solves the problem in the class $H_{p,e}$ as well.

III. Optimal Strategies

We first state the following lemma.

Lemma 1: H_Z^i, $i=1,2$, are Hilbert spaces with inner product

$$[y_1^i, y_2^i] = \int_0^T E\left([y_1^i(t;\omega), y_2^i(t;\omega)]\right) dt.$$

The space H_Z^1 contains processes $u_p(\cdot;\omega)$ of the form

$$u_p(t;\omega) = \int_0^t k_p(t,s)\, dZ(s;\omega) \tag{3a}$$

$$u_p(t;\omega) = \int_0^t k_p(t,s) x(s;\omega)\, ds \tag{3b}$$

where k_p is Lebesgue measurable function on the triangle $0 \leqslant s \leqslant t \leqslant T$ such that

$$\int_0^T \int_0^t \operatorname{tr} E\left(k_p(t,s)\, d\langle Z, Z\rangle_s k_p(t,s)^*\right) dt < \infty \text{ a.s.} \tag{4}$$

where * denotes the transpose and tr denotes trace of a matrix, and similar result holds for H_Z^2.

Proof: The proof is obvious and thus omitted.

Corollary: Processes in H_Z^1 of form (3a), form a closed subspace of H_Z^1 which we denote by L_Z^1. A similar subspace of H_Z^2 we denote by L_Z^2.

Definition: Let H_n be the Hilbert space of R^n-valued random vectors $x(\omega)$ such that $E\|x(\omega)\|^2 < \infty$ under the inner product

$$[x,y] = E([Qx(\omega), y(\omega)]).$$

Lemma 2: Let L_1 be a bounded linear transformation from H_Z^1 into H_n defined by

$$L_1 u = x, \qquad x(\omega) = \int_0^T L_1(s) u(s;\omega)\, ds$$

where $L_1(s)$ is continuous in $0 \leqslant s \leqslant T$. Let

$$r_1(t;\omega) = E\left(L_1(t)^* Q y(\omega) \mid \beta_Z(t)\right)$$

where * denotes the transpose and y is in H_n. Suppose that $r_1(t;\omega)$ is of the form (3a) or (3b). Then $r_1 = L_1^* y$ where L_1^*, the adjoint of L_1, maps H_n into H_Z^1.

Proof: For $u(t;\omega)$ in H_Z^1 and $y(\omega)$ in H_n,

$$\begin{aligned}[L_1 u, y] &= E\left(\left[Q\int_0^T L_1(t) u(t;\omega)\, dt, y(\omega)\right]\right) \\ &= \int_0^T E\left(E\left([u(t;\omega), L_1(t)^* Q y(\omega)] \mid \beta_Z(t)\right)\right) dt \\ &= [u, r_1], \qquad \text{since } u(t;\omega) \text{ is adapted to } \beta_Z(t).\end{aligned}$$

We use the same notation for inner product in different spaces but they are obvious from the context. Now, if $r_1(t;\omega)$ is of the form (3a) or (3b), r_1 is in H_Z^1 and the lemma follows.

Main theorem: For the functional (2) there exists two constants $\mu_1 > \mu_0 \geqslant 0$ such that for $\mu \leqslant \mu_0$, the value of the game is infinite, while for $\mu_0 < \mu \leqslant \mu_1$, the game has no value. For $\mu > \mu_1$, the game has a finite value and a unique saddle point exists. It is given by

$$\begin{cases} u_{op}(t;\omega) = B_p^*(t) P_g(t) x(t;\omega)/\lambda \\ u_{oe}(t;\omega) = B_e^*(t) P_g(t) x(t;\omega)/\mu \end{cases} \tag{5}$$

where $P_g(t)$ satisfies

$$\dot{P}_g(t) + P_g(t)A(t) + A(t)^* P_g(t) + P_g(t) \cdot \left[\frac{B_e(t)B_e(t)^*}{\mu} - \frac{B_p(t)B_p(t)^*}{\lambda}\right] P_g(t) = 0, \qquad P_g(T) = Q. \tag{6}$$

Proof: Let $\Phi(t)$ be a fundamental matrix for the linear system $\dot{x}(t) = A(t)x(t)$. Let L_p be a bounded linear operator from H_Z^1 into H_n and L_e be a bounded linear operator from H_Z^2 into H_n defined, respectively, by

$$L_p u_p = f; \quad f(\omega) = \int_0^T \Phi(T)\Phi(s)^{-1} B_p(s) u_p(s;\omega)\, ds \tag{7}$$

and

$$L_e u_e = g; \quad g(\omega) = \int_0^T \Phi(T)\Phi(s)^{-1} B_e(s) u_e(s;\omega)\, ds. \tag{8}$$

Let $w(\omega) = \int_0^T \Phi(T)\Phi(s)^{-1}\, dZ(s;\omega)$. Then the functional (2) can be written as

$$F(u_p; u_e) = \|L_p u_p + L_e u_e + w\|^2 + \lambda \|u_p\|^2 - \mu \|u_e\|^2. \tag{9}$$

Here again we have used same symbol for norms in different spaces but they are apparent from the context. Let

$$R = L_p(L_p^* L_p + \lambda 1)^{-1} L_p^*$$

and let

$$\inf_{u_e} [L_e^*(I-R)L_e u_e, u_e]/[u_e, u_e] = \mu_0$$

$$\sup_{u_e} \|L_e u_e\|^2 / \|u_e\|^2 = \mu_1.$$

Then, following Balakrishnan [1, p. 189], $0 \leqslant \mu_0 < \mu_1$ and for $\mu \leqslant \mu_0$, the value of the game is infinite while for $\mu_0 < \mu \leqslant \mu_1$, the game has no value. On the other hand, for $\mu > \mu_1$, the functional $F(u_p; u_e)$ is strictly convex in u_p and strictly concave in u_e and hence the game has a finite value attained at a saddle point which is unique, denoted by u_{op} and u_{oe}. u_{op} is obtained by minimizing the functional $F(u_p; u_{oe})$ and, by a routine variation, is given by

$$u_{op} = -(1/\lambda) L_p^*(L_p u_{op} + L_e u_{oe} + w) \tag{10a}$$

and also by

$$u_{op} = -(L_p^* L_p + \lambda I)^{-1} L_p^*(L_e u_{oe} + w). \tag{10b}$$

Similarly, u_{oe} is obtained by maximizing $F(u_{op}; u_e)$ and is given by

$$u_{oe} = (1/\mu) L_e^*(L_p u_{op} + L_e u_{oe} + w) \tag{11a}$$

and also by

$$u_{oe} = (\mu I - L_e^* L_e)^{-1} L_e^*(L_p u_{op} + w). \tag{11b}$$

In the forms (10a) and (11a), $L_p u_{op} + L_e u_{oe} + w$ is the optimal final state $x(T;\omega)$ and by Lemma 2, $L_p^*(L_p u_p + L_e u_e + w)$ and $L_e^*(L_p u_p + L_e u_e + w)$ are given, respectively, by

$$B_p^* E\left(\Phi(t)^{*-1}\Phi(T)^* Q x(T;\omega) \mid \beta_Z(t)\right) \tag{12}$$

and

$$B_e^* E\left(\Phi(t)^{*-1}\Phi(T)^* Q x(T;\omega) \mid \beta_Z(t)\right) \tag{13}$$

if we can show that they are of the form (3a). To prove this, we consider forms (10b) and (11b). Putting the expression for u_{oe} from (11b) into (10b) and simplifying, we get

$$u_{op} = K(L_p^* L_p + \lambda I)^{-1} L_p^*(L_e(\mu I - L_e)^{-1} L_e^* w + w)$$

where K is an operator involving L_p, L_p^*, L_e and L_e^* that map H_Z^1 into itself. Since $v \equiv L_e(\mu I - L_e L_e^*)^{-1} L_e^* w + w$ is of the form

$$\int_0^T h(T,s)\, dZ(s;\omega)$$

it is clear that

$$B_p^*(t) E\left(\Phi(t)^{*-1}\Phi(T)^* Q v(\omega) \mid \beta_Z(t)\right)$$

is of the form of (3a) and hence is the element $L_p^* v$. The operator $L_p^* L_p$ maps the closed subspace L_Z^1 into itself and therefore, $K(L_p^* L_p + \lambda I)^{-1}$ also maps L_Z^1 into itself. The same argument shows that u_{oe} belongs to L_Z^2. It follows then that $x(t;\omega)$ has the form $\int_0^t k(t,s)\,dZ(s;\omega)$, and consequently, (12) and (13) are of the form of (3a). Therefore,

$$u_{op}(t;\omega) = -(1/\lambda) B_p^*(t) \hat{q}(t;\omega)$$
$$u_{oe}(t;\omega) = (1/\mu) B_e^*(t) \hat{q}(t;\omega)$$

where $\hat{q}(t;\omega) = E(q(t;\omega) | \beta_Z(t))$ and $q(t;\omega)$ is the unique solution of

$$\dot{q}(t;\omega) + A^*(t) q(t;\omega) = 0, \qquad q(T;\omega) = Qx(T;\omega).$$

Thus,

$$dq(t;\omega) = -A^* q(t;\omega)\,dt$$
$$dx(t;\omega) = Ax(t;\omega)\,dt - 1/\lambda B_p B_p^* q(t;\omega)\,dt + 1/\mu B_e B_e^* \hat{q}(t;\omega)\,dt + dZ(t;\omega)$$

and from (6),

$$dP_g(t) = \left[-P_g(t)A - A^* P_g(t) + P_g(t)\left(1/\lambda B_p B_p^* - 1/\mu B_e B_e^*\right) P_g(t) \right] dt.$$

After algebraic manipulation,

$$d\left(q(t;\omega) - P_g(t)x(t;\omega)\right) = -\left(A^* - \frac{P_g(t) B_p B_p^*}{\lambda} + \frac{P_g(t) B_e B_e^*}{\mu} \right) \cdot \left(q(t;\omega) - P_g(t)x(t;\omega)\right) dt + P_g(t) \left(\frac{B_p B_p^*}{\lambda} - \frac{B_e B_e^*}{\mu} \right) \cdot \left(\hat{q}(t;\omega) - q(t;\omega)\right) dt + P_g(t)\,dZ(t;\omega)$$

with $q(T;\omega) - P_g(T)x(T;\omega) = 0$. Since

$$E\left(q(s;\omega) - \hat{q}(s;\omega) | \beta_Z(t)\right) = 0, \qquad \text{for } s \geqslant t$$

and

$$E\left(\int_t^T P_g(s)\,dZ(s;\omega) | \beta_Z(t) \right) = 0,$$

we get $E(q(t;\omega) - P_g(t)x(t;\omega) | \beta_Z(t)) = 0$. But u_{op} and u_{oe} being adapted to $\beta_Z(t)$, $x(t;\omega)$ is adapted to $\beta_Z(t)$ and

$$u_{op} = -B_p^*(t) P_g(t) x(t;\omega)/\lambda$$
$$u_{oe} = B_e^*(t) P_g(t) x(t;\omega)/\mu.$$

IV. Conclusion

In this short paper, we have solved the stochastic linear differential game problem with any square integrable Martingale as noise. The solution is obtained by transforming the problem to an optimization problem in the Hilbert space setting. This method of solution brings out an important feature of the problem. The only property of stochastic integral needed in the solution is the fact that

$$E\left(\int_t^T P_g(s)\,dZ(s;\omega) | \beta_Z(t) \right) = 0$$

and this property holds for any square integrable Martingale $Z(t;\omega)$. Formulating the problem with Wiener process as noise is, therefore, unnecessarily restrictive. Specifically, the sample continuity of a Wiener path is completely redundant for this problem, this being an important conclusion of the present work.

References

[1] A. V. Balakrishnan, *Stochastic Differential Systems* (Lecture Notes in Economics and Mathematical Systems). Berlin: Springer, 1973.

[2] A. Bagchi, "Stochastic linear regulator problem with a square integrable Martingale as noise," Dep. Appl. Math., Twente Univ. Technol., Enschede, The Netherlands, Memo 62, 1975.

[3] H. Kunita and S. Watanabe, "On square integrable Martingales," *Nagoya Math. J.*, vol. 30, pp. 209–245, 1967.

[4] H. Kwakernaak, "An extension of the stochastic linear regulator problem," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 121–123, 1974.

Minimal Dynamic Inverses for Linear Systems with Arbitrary Initial States

EROL EMRE, STUDENT MEMBER, IEEE, AND LEONARD M. SILVERMAN, MEMBER, IEEE

Abstract—**In this short paper the problem of finding a minimal left inverse of a linear time-invariant system for nonzero initial conditions is considered. It is shown that this problem is equivalent to finding a minimal dynamical cover. As a result of this, the minimal inverse problem can be solved immediately using the previous results on dynamic covers. No restriction other than invertibility is assumed on the original system.**

I. Introduction

In this short paper we consider the problem of finding a minimal order dynamical inverse for a linear system, a problem of interest in control, filtering, and coding theory. Most previous work on this subject has concentrated on properties of zero-initial state (transfer function matrix) inverses [1]–[4], [6]–[8], particularly with respect to minimality. The first characterization of such minimal inverses was given by Wang and Davison [7]. Forney [8] subsequently gave a more elegant treatment of the problem using the concept of minimal bases for rational vector spaces [8]. Both of these solutions are based on the transfer function matrix representation of the original system. Recently, Morse [6] has shown that the problem of finding a zero-initial state minimal inverse can be solved by a geometric approach, and the problem can be reduced to one of finding a minimal dynamical cover [5], for which two algorithms already exist [5], [15].

A more general problem is that of constructing a dynamical left inverse which will operate as an inverse for arbitrary initial states of the original system. Yuan [9] recently presented an algorithm for finding a minimal inverse of this type, using the properties of L-delay elementary null sequences [9]. However, his results are applicable to only nondegenerate [9] systems. Minimal inverses are also considered by Bengtsson [16]. However, they are not dynamic in general. In this short paper we present a new approach to this problem. In particular, it is established that the general dynamic inverse problem can also be reduced to a dynamic cover problem which leads to an immediate solution of the problem. Using this characterization, certain results on the characteristic polynomials of dynamical inverses are obtained. Also, a further characterization of the minimal dynamic inverses in terms of output injection feedback is given. Unlike [9], the results of this short paper allow the construction of a minimal inverse for any invertible system with an arbitrary initial state not requiring it to be nondegenerate as in [9]. This is achieved by initializing the inverse α units of time later.

II. Characterization of Inverse Systems

Let $S = (A, B, C, D)$ denote the linear time-invariant system

$$x_{k+1} = Ax_k + Bu_k \tag{1a}$$

Manuscript received December 9, 1975; revised April 5, 1976. Paper recommended by J. B. Pearson, Chairman of the IEEE S-CS Linear Systems Committee. This research was supported by the National Science Foundation under Grant ENG 75-03423.

The authors are with the Department of Electrical Engineering, University of Southern California, Los Angeles, CA 90007.