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Dynamic Input-Output Decoupling of Nonlinear Control Systems

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***Abstract*—In this note we study the problem of dynamic input-output decoupling of nonlinear control systems. Based on an analytic algorithm we obtain necessary and sufficient conditions for the solvability of this problem. The solution of the problem is constructive by applying a series of simple precompensations and linking maps. Some interesting connections with other approaches in nonlinear control theory are discussed. Also we give a few (simple) examples to illustrate the methods used in the note.**

I. Introduction

Over the last decade there has been much interest in the general problem of *input-output decoupling* or *noninteracting* of linear dynamic control systems. By noninteracting, we mean a situation where each (scalar) control variable only affects one (scalar) output variable and none of the other outputs. If the given system does not possess such a property of noninteracting, then we may try to add control loops to the original system such that at the end we have achieved noninteraction. Depending on the sort of control strategy, one can formulate various different decoupling problems. One of the earliest attempts in this area goes back to Morgan in 1964 (see [18]) where static-state feedback in the control loops was allowed. Many other contributions on the question of noninteracting have been produced; see [31] for a very readable survey. Of particular interest here is the contributions in which one achieves decoupling by allowing *dynamic* state feedback in the control loops; see [9], [19] and especially [2], [30].

More recently, the same problem of noninteraction was formulated for nonlinear dynamic control systems, and depending on the sort of permitted control loops, the problem has been solved in particular cases. The first and simplest version in which one allows for static-state feedback, i.e., the nonlinear Morgan's problem, was solved by Porter [24]; see also [1], [8], [14], [28], and [29].

The problem of dynamic input-output decoupling was studied by Singh [25], [26] via a generalization of Hirschorn's nonlinear version [11] of the Silverman structure algorithm [32]. Recently, an interesting extension of linear dynamic decoupling as was used in [30] to nonlinear systems was given by Descusse and Moog [3], who formulated and solved the nonlinear dynamic decoupling problem for strongly left-invertible systems. This solution began an increasing interest in the problem of dynamic feedback for nonlinear systems (compare [13] and [33]).

In the present note we deal with a general solution of the dynamic decoupling problem for affine nonlinear control systems. We give necessary and sufficient conditions for the local solvability of this problem and our tool to do it consists of an analytic algorithm which at each step produces a decoupling matrix of the type introduced in [24]; see also [12]. The idea of our algorithm is like the one used in [3], i.e., to precompensate the inputs that appear "too early." However, both algorithms suggest different feedback laws at every step. Using our algorithm we have a precise procedure for defining a simple precompensator and an iterative composition of linking maps. The resulting extended system possesses q^* decoupled input-output channels and it turns out that q^* is the maximal number of decouplable input-output channels for the original system. Therefore, decoupling is possible if and only if q^* equals the number of output channels. Because our algorithm converges on an open dense submanifold of the state space we have an explicit way of testing the solvability of the dynamic decoupling problem, and moreover we have an explicit way to compute the required compensator and feedback which decouples the system. Although it is true that the algorithm "works" on an open dense submanifold of the state space, it is important to note that the initial state of the compensator also has to be chosen correctly from an open dense submanifold of the state space of the compensator. This fact, which was not emphasized in [3], is essential in our approach and cannot be avoided under additional conditions like in [3]; cf. Example 2.2. Furthermore, similar to the paper of Descusse and Moog [3] our algorithm only locally works in the situation that the nonlinear system is left-invertible; see also Example 2.1 where this notion of *local* dynamic input-output decoupling appears. For the concept of left invertibility we refer to [10], [11], [20], [25]-[27] where definitions and characterizations are given.

The approach we have taken here is purely analytic. No differential geometric tools like foliations, distributions, involutivity, controlled invariance, etc., have been used. Because most often the state space of a nonlinear system is not a Euclidean space, we have chosen to work on manifolds and use concepts like vector fields and Lie derivatives, but if desired one may think of open neighborhoods in $\mathbb{R}^n$, mappings from $\mathbb{R}^n$ into itself, and directional derivatives. It would be interesting to have a differential geometric counterpart to the theory we have developed so far.

Let us note here that very recently Fliess was able to treat the same problem in a differential algebraic context; see [5]-[7]. Some relations with his approach and the solvability of the dynamic decoupling problem are discussed in [35].

Finally, we remark that, to our best knowledge, the method described here is new also for linear control systems, where it is known that a system is dynamically decouplable if and only if the rank of the

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corresponding transfer matrix equals the number of output channels (see [30]) although there may exist a connection via the paper [2].

The outline of the note is as follows. In Section II some motivating examples and the problem formulation are given. Then in the next section our algorithm which is essential in the whole solution of the problem is given. Hereafter, we formulate and prove our main theorem on dynamic input–output decoupling in Section IV. Section V contains the conclusions of the note and some comments on related topics in nonlinear control theory.

II. Motivation, Problem Formulation, and Notation

We will consider affine nonlinear control systems of the form

$$\begin{cases} \dot{x}=f(x)+\sum_{i=1}^{m} g_i(x)u_i, \ x(0)=x_0 \in M \\ y=h(x) \end{cases} \tag{2.1}$$

where $x = (x_1, \cdots, x_n)^T$ are analytic local coordinates of an analytic manifold $M, f, g_1, \cdots, g_m$ are analytic vector fields on M, and $h = (h_1, \cdots, h_p)^T$ analytic functions from M into $\mathbb{R}^p$. As stated in the Introduction we will discuss the general problem of dynamic decoupling of the system (2.1). Before giving a precise definition of dynamic compensators, we will first treat a few motivating examples.

Example 2.1 (See [11]):

$$\begin{aligned} \dot{x}_1&=x_1u_1 & y_1&=x_1 \\ \dot{x}_2&=x_3-x_3u_1 & y_2&=x_2 \\ \dot{x}_3&=x_1u_2. \end{aligned} \tag{2.2}$$

It is straightforward to verify

$$\begin{cases} \dot{y}_1=x_1u_1 \\ \dot{y}_2=x_3-x_3u_1 \end{cases}$$

and so the nonlinear decoupling matrix (see the end of this section) of the system (2.2) is

$$\begin{pmatrix} x_1 & 0 \\ -x_3 & 0 \end{pmatrix}$$

which has rank 1, and therefore this system is not decouplable via static-state feedback (see [12]). Now add to the dynamics of (2.2) the precompensator with state $z \in \mathbb{R}$

$$\dot{z}=w_1, \ u_1=z. \tag{2.3}$$

The extended system has decoupling matrix

$$\begin{pmatrix} x_1 & 0 & \\ -x_3 & x_1 & -x_1z \end{pmatrix}$$

and this matrix is full rank if $x_1 \neq 0$ and $z \neq 1$. So if $x_1 \neq 0$ and $z \neq 1$ we can decouple the overall system (2.2), (2.3). Notice that this system is decouplable also by the algorithm given in [3], but the proofs given in [3] fail for this kind of system.

Example 2.2: Let

$$\begin{aligned} \dot{x}_1&=u_1 & y_1&=x_1 \\ \dot{x}_2&=x_3+e^{x_3}u_1 & y_2&=x_2 \\ \dot{x}_3&=u_2. \end{aligned} \tag{2.4}$$

It is easy to verify that there does not exist a singular controlled invariant distribution in the kernel of the output function (cf. [20]), and therefore the system is dynamically decouplable; see [3]. If we take, following [3], the precompensator

$$\dot{z}_1=w_1, \ u_1=z_1 \tag{2.5}$$

then we get as decoupling matrix of (2.4) and (2.5)

$$D(x, z_1)=\begin{pmatrix} 1 & 0 \\ e^{x_3} & 1+e^{x_3}z_1 \end{pmatrix} \tag{2.6}$$

and this matrix is nonsingular provided that $z_1 \neq -e^{-x_3}$ [or see (2.5)] $u_1 = z_1 \neq -e^{-x_3}$. That is, we have to be careful in initializing the precompensator (2.5). The above two simple examples illustrate the difficulties in establishing general results on the dynamic decoupling problem. Before we will formulate our algorithm that is essential in our solution of the dynamic decoupling problem, we will give a detailed problem description.

Problem Formulation

Suppose the system (2.1) is given. Then the dynamic input–output decoupling problem, or shortly decoupling problem, can be formulated as follows. First we introduce the notion of a dynamic compensator which is defined as a nonlinear system on $\mathbb{R}^\nu$ of the form

$$\dot{z}=\phi(x, z)+\psi(x, z)w \tag{2.7a}$$

together with a closing loop

$$u=\alpha(x, z)+\beta(x, z)w \tag{2.7b}$$

where we assume that all the data occurring in (2.7a), (2.7b) are analytic; so the parametrized vector fields $\phi, \psi_1, \cdots, \psi_m: M \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ are analytic; the functions $\alpha, \beta_1, \cdots, \beta_m: M \times \mathbb{R}^\nu \to \mathbb{R}^m$ are analytic. Together with the compensator (2.7a), (2.7b) we have to specify an initial state, say

$$z(0)=z_0 \in \mathbb{R}^\nu. \tag{2.8}$$

Now the dynamic input–output decoupling problem can be stated as follows.

Find—if possible—a nonlinear compensator of the form (2.7a) together with a proper initial state (2.8) and a feedback law (2.7b) such that the overall system, i.e., (2.1) together with (2.7), (2.8) is input–output decoupled. That is the first p components of the new control $w, w_1, \cdots, w_p$, effectuate independently the p outputs $y_1, \cdots, y_p$ and all the other components $w_{p+1}, \cdots, w_m$ affect none of the outputs.

In this note we solve the above problem in a local fashion, i.e., we find conditions which guarantee that we can solve the dynamic decoupling problem in a neighborhood of the initial state (x_0, z_0) of the system (2.1) and (2.7a). It will be clear that in general the map β is not full rank (in fact as will be seen later, this is true if and only if the decoupling of the system could be achieved with static-state feedback).

Notation

Consider the system (2.1)

$$\begin{cases} \dot{x}=f(x)+g(x)u=f(x)+\sum_{i=1}^{m} u_ig_i(x) \\ y=h(x)=(h_1(x), \cdots, h_p(x)). \end{cases}$$

Throughout the note we use a vector notation, upper indexes denote vectors, e.g., $h^l = (h_{ql-1+1}, \cdots, h_{ql})^T$ and similarly for y^l and u^l, where the index q_l is specified in the context. The remaining components will be denoted as $\bar{h}^l$ as follows $h = (h^1, h^2, \cdots, h^l, \bar{h}^l)^T$ and similarly for $\bar{y}^l$ and $\bar{u}^l$. Moreover, $U^l = (u^1, \cdots, u^l)$ and so $u = (U^l, \bar{u}^l)$. Time derivatives will be denoted as $u_i^{(k)} = d^ku_i/dt^k$ and similarly for u, y_i, and y. For the multiindex $\rho^l = (\rho_{q1}^l, \cdots, \rho_{ql}^l)(y^l)^{(\rho^l)}$ denotes the vector with components $(y_i^l)^{(\rho_i^l)} = d^{\rho_i^l}/dt^{\rho_i^l}(y_i^l)$.

The following simple lemma is essential in what follows in the next sections.

Lemma 2.3: Consider the analytic system (2.1). Then we have the following.

i) $y_i^{(k)} = A_i^k(x, u, \cdots, u^{(k-1)}) = B_i^k(x, u, \cdots, u^{(k-2)}) + C_i^k(x)u^{(k-1)}$, where $C_i^k(x) = L_gh_i(x)$ and the function B_i^k is polynomial in the components of $u, \cdots, u^{(k-2)}$.

ii) If $A^1, \cdots, A^k$ do not depend on u_j, then also A^l, $1 \leq l \leq k$ do not depend on $u_j, \cdots, u_j^{(l-1)}$.

Proof: Straightforward by induction.

Static-State Feedback Decoupling

We conclude this section with a brief review of the case that we can achieve input–output decoupling with applying only static-state feedback, see, e.g., [1], [8], [12], [24], [29]. Define integers $\rho_1, \cdots, \rho_p$ as the smallest numbers for which the ρ_ith time derivative of y_i, i.e., $A_i^{\rho_i}$ depends explicitly on the input u. The $(\rho_i - 1)$'s are the so-called characteristic numbers. Now form the (p, m)-decoupling matrix $D(x)$ of which the (i, j)th entry is defined as $\langle A_i^{\rho_i - 1}(x), g_j(x)\rangle$. The static-state feedback input–output decoupling problem is solvable if and only if rank $D(x) = p$ (= constant), i.e., if this condition holds we can locally around the initial state find a feedback law $u = \alpha(x) + \beta(x)v$, $\beta: M \to \mathbb{R}^{m \times m}$ being analytic maps, which applied to (2.1) achieves the input–output decoupling.

III. The Algorithm

We now come to the algorithm that is essential in the dynamic decoupling problem. So we consider the analytic system on an analytic manifold M described in local coordinates.

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + g(x)u \\ y = h(x) = (h_1(x), \cdots, h_p(x))^T \end{cases} \tag{3.1}$$

initialized at $x_0 \in M$. Recall the notations and conventions given at the end of Section II. In the first step we are dealing with the usual nonlinear version of the Falb–Wolovich matrix; see, e.g., [1], [8], [12], [24], [29].

Step 1: Define the integers $\rho_1^1, \cdots, \rho_p^1$ as the smallest numbers such that $(y_i)^{(\rho_i^1)}$ depends explicitly on u, i.e., $\rho_i^1 - 1$ is the characteristic number of the ith output channel. We have (compare Lemma 2.3)

$$\begin{pmatrix} (y_1)^{(\rho_1^1)} \\ \vdots \\ (y_p)^{(\rho_p^1)} \end{pmatrix} = E^1(x) + D^1(x)u \tag{3.2}$$

for a $(p, 1)$-vector $E^1(x)$ and a (p, m)-matrix $D^1(x)$. Let

$$r_1(x) = \text{rank } D^1(x). \tag{3.3}$$

Clearly, $r_1(x)$ is constant on an open dense submanifold M_1 of M, say $r_1(x) = r_1$ for $x \in M_1$. Assume we are working on an open neighborhood in M_1. Reorder the output functions $h_1, \cdots, h_p$ such that the first r_1 rows of the matrix D^1 are linearly independent and write $h = (h^1, \bar{h}^1)$, where $h^1 = (h_1, \cdots, h_{r_1})^T$ and $\bar{h}^1 = (h_{r_1+1}, \cdots, h_p)^T$. Denote in the corresponding way y as $(y^1, \bar{y}^1)^T$ and $(\rho_1^1, \cdots, \rho_p^1)^T$ as $(\rho^1, \bar{\rho}^1)^T$. Choose an $(m, 1)$-vector $\alpha_1(x)$ and an invertible (m, m)-matrix $\beta_1(x)$ on a neighborhood in M_1 such that after applying

$$u = \alpha_1(x) + \beta_1(x)\begin{pmatrix} u^1 \\ \bar{u}^1 \end{pmatrix} \tag{3.4}$$

we arrive at

$$\begin{pmatrix} (y^1)^{(\rho^1)} \\ (\bar{y}^1)^{(\bar{\rho}^1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda^1(x) \end{pmatrix} + \begin{pmatrix} I_{r_1} & 0 \\ \mu^1(x) & 0 \end{pmatrix}\begin{pmatrix} u^1 \\ \bar{u}^1 \end{pmatrix} \tag{3.5}$$

Now define the modified vector fields

$$f^1(x) = f(x) + g(x)\alpha_1(x) \tag{3.6a}$$

$$g^1(x) = g(x)\beta_1(x) \tag{3.6b}$$

and consider the dynamics

$$\dot{x} = f^1(x) + g^1(x)\begin{pmatrix} u^2 \\ \bar{u}^2 \end{pmatrix}. \tag{3.7}$$

What is done so far is nothing else as applying static-state feedback to achieve decoupling of r_1 input–output channels.

Step 2: In this step we are only concerned about the remaining outputs $\bar{y}^1 = \bar{h}^1(x)$ and we want to examine their dependence on the remaining inputs $\bar{u}^1$. In order to do this we differentiate these outputs with respect to (3.7) to see when $\bar{u}^1$ appears for the first time. Let for $i > r_1$ the integer ρ_i^2 denote the smallest number such that the ρ_i^2th time derivative of y_i explicitly depends on $\bar{u}^1$. Observe that such a time derivative possibly also depends on components of u^1 and their time derivatives. We have

$$\begin{pmatrix} (y_{r_1+1})^{(\rho_{r_1+1}^2)} \\ \vdots \\ (y_p)^{(\rho_p^2)} \end{pmatrix} = E^2(x, \bar{U}^1) + D^2(x, \bar{U}^1)\bar{u}^1 \tag{3.8}$$

for a $(p - r_1, 1)$ vector $E^2(x, \bar{U}^1)$ and a $(p - r_1, m - r_1)$ matrix $D^2(x, \bar{U}^1)$, where $\bar{U}^1$ consists of all components of u^1 and their time derivatives $u_i^{(j)}$, $1 \le i \le r_1$, $j \ge 0$ which occur in (3.8). We emphasize that the highest derivative of u^1 appearing in (3.8) is of order $n - 1$; so $j \le n - 1$ is a natural upper bound in $\bar{U}^1$; see [34]. So $\mu_1 = \dim \bar{U}^1 \le (n - 1)r_1$ when interpreting u^1 and time derivatives $(u^1)^{(j)}$ as independent variables. Let

$$r_2(x, \bar{U}^1) = \text{rank } D^2(x, \bar{U}^1) \tag{3.9}$$

then $r_2(\cdot, \cdot)$ is constant on an open and dense submanifold M_2 of $M_2^1 = M_1 \times \mathbb{R}^{\mu_1}$, say $r_2(\cdot, \cdot) = r_2$. Let

$$q_2 = r_1 + r_2. \tag{3.10}$$

Assume we are working on an open neighborhood in M_2. Reorder the output functions $\bar{h}^1$ such that the first r_2 rows of the matrix $D^2(x, \bar{U}^1)$ are linearly independent and write $\bar{h}_1 = (h^2, \bar{h}^2)$ with $h^2 = (h_{r_1+1}, \cdots, h_{q_2})^T$, $\bar{h}_2 = (h_{q_2+1}, \cdots, h_p)^T$. Accordingly, we write $\bar{y}^1 = (y^2, \bar{y}^2)^T$ and $(\rho_{r_1+1}, \cdots, \rho_p^2)^T = (\rho^2, \bar{\rho}^2)^T$. Choose an $(m - r_1, 1)$-vector $\alpha_2(x, \bar{U}^1)$ and an invertible $(m - r_1, m - r_1)$-matrix $\beta_2(x, \bar{U}^1)$ on a neighborhood in M^2 such that after applying

$$\bar{u}^1 = \alpha_2(x, \bar{U}^1) + \beta_2(x, \bar{U}^1)\begin{pmatrix} u^2 \\ \bar{u}^2 \end{pmatrix} \tag{3.11}$$

we obtain

$$\begin{pmatrix} (y^2)^{(\rho^2)} \\ (\bar{y}^2)^{(\bar{\rho}^1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda^2(x, \bar{U}^1) \end{pmatrix} + \begin{pmatrix} I_{r_2} & 0 \\ \mu^2(x, \bar{U}^1) & 0 \end{pmatrix}\begin{pmatrix} u^2 \\ \bar{u}^2 \end{pmatrix} \tag{3.12}$$

Define the modified parametrized vector fields

$$f^2(x, \bar{U}^1) = f^2(x) + g^1(x)\begin{pmatrix} 0 \\ \alpha_2(x, \bar{U}^1) \end{pmatrix},$$

$$g^2(x, \bar{U}^1) = g^1(x)\begin{pmatrix} I_{r_1} & 0 \\ 0 & \beta_2(x, \bar{U}^1) \end{pmatrix} \tag{3.13}$$

and consider the dynamics

$$\dot{x} = f^2(x, \bar{U}^1) + g^2(x, \bar{U}^1)\begin{pmatrix} U^2 \\ \bar{u}^2 \end{pmatrix} \tag{3.14}$$

where $U^2 = (u^1, u^2)^T$. We will consider the controls u^1 and their time derivatives (occurring in $\bar{U}^1$) as parameters. Alternatively—and this will be crucial in the proof of Section IV—we can interpret them as additional state variables and new controls for the extended system. From the foregoing reasoning the general step is easily established.

Step l + 1: Assume we have defined a sequence of integers $r_1, \cdots, r_l$ and $q_l = \Sigma_{i=1}^{l} r_i$. We have $h = (h^1, \cdots, h^l, \bar{h}^l)^T$ and the parametrized dynamics.

$$\dot{x} = f^l(x, \bar{U}^{l-1}) + g^l(x, \bar{U}^{l-1})\begin{pmatrix} U^l \\ \bar{u}^l \end{pmatrix} \tag{3.15}$$

where $U^l = (u^1, \cdots, u^l)^T$. Similarly, to the second step we examine now the dependency of the outputs $\bar{y}^l = \bar{h}^l(x)$ on the remaining inputs $\bar{u}^l$. So we differentiate these outputs with respect to (3.15) until $\bar{u}^l$ appears. Let for $i > q_l$ the integer ρ_i^{l+1} denote the smallest number such that the ρ_i^{l+1}th time derivative of y_i explicitly depends on $\bar{u}^l$. Similarly to (3.8) we obtain

$$\begin{pmatrix} (y_{q_l+1})^{(\rho^{l+1}_{q_l+1})} \\ \vdots \\ (y_p)^{(\rho^{l+1}_p)} \end{pmatrix} = E^{l+1}(x, \bar{U}^l) + D^{l+1}(x, \bar{U}^l)\bar{u}^l \tag{3.16}$$

for a $(p - q_l, 1)$-vector $E^{l+1}(x, \bar{U}^l)$ and a $(p - q_1, m - q_1)$-matrix $D^{l+1}(x, \bar{U}^l)$, where $\bar{U}^l$ consists of $\bar{U}^{l-1}$ and (time derivatives of) u^l which occur in the above expression. Note that as in Step 2 $\mu_l = \dim \bar{U}^l$ satisfies $\mu_l \le (n - 1)q_l$; see [34]. Let $r_{l+1}(x, \bar{U}^l) = \operatorname{rank} D^{l+1}(x, \bar{U}^l)$, then $r_{l+1}(\cdot, \cdot)$ is constant on an open and dense submanifold M_{l+1} of $M'_{l+1} = M \times \mathbb{R}^{\mu_l}$. Reorder the output functions $\bar{h}^l$ such that the first r_{l+1} rows of D^{l+1} are linearly independent and partition $\bar{h}^l = (h^{l+1}, \bar{h}^{l+1})^T$ accordingly. Let

$$q_{l+1} = \sum_{i=1}^{l+1} r_i. \tag{3.17}$$

In the same way partition $\bar{y}^l = (y^{l+1}, \bar{y}^{l+1})^T$ and $(\rho^{l+1}, \bar{\rho}^{l+1})$ as the vector of characteristic numbers. Choose an $(m - q_l, 1)$-vector $\alpha_{l+1}(x, \bar{U}^l)$ and an invertible $(m - q_l)$ matrix $\beta_{l+1}(x, \bar{U}^l)$ on a neighborhood in M_{l+1} such that after applying the feedback

$$\bar{u}^l = \alpha_{l+1}(x, \bar{U}^l) + \beta_{l+1}(x, \bar{U}^l) \begin{pmatrix} u^{l+1} \\ \bar{u}^{l+1} \end{pmatrix} \tag{3.19}$$

we arrive at

$$\begin{pmatrix} (y^{l+1})^{(\rho^{l+1})} \\ (\bar{y}^{l+1})^{(\bar{\rho}^{l+1})} \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda^{l+1}(x, \bar{U}^l) \end{pmatrix} + \begin{pmatrix} I_{r_{l+1}} & 0 \\ \mu^{l+1}(x, \bar{U}^l) & 0 \end{pmatrix} \begin{pmatrix} u^l \\ \bar{u}^{l+1} \end{pmatrix} \tag{3.19}$$

Define the new parametrized vector fields

$$f^{l+1}(x, \bar{U}^l) = f^l(x, \bar{U}^{l-1}) + g^l(x, \bar{U}^{l-1}) \begin{pmatrix} 0 \\ \alpha_{l+1}(x, \bar{U}^l) \end{pmatrix},$$

$$g^{l+1}(x, \bar{U}^l) = g^l(x, \bar{U}^{l-1}) \begin{pmatrix} I_{q_l} & 0 \\ 0 & \beta_{l+1}(x, \bar{U}^l) \end{pmatrix} \tag{3.20}$$

and consider the dynamics

$$\dot{x} = f^{l+1}(x, \bar{U}^l) + g^{l+1}(x, \bar{U}^l) \begin{pmatrix} U^{l+1} \\ \bar{u}^{l+1} \end{pmatrix} \tag{3.21}$$

where $U^{l+1} = (u^1, \cdots, u^{l+1})$. Observe that the sequence of q_i's is increasing and bounded by the number min (p, m), and therefore the algorithm will terminate after finite steps with a maximal number q^*, i.e., $q^* = q_l$ for l sufficiently large, say $l > k$, for a certain k. This integer is well defined on an open and dense submanifold $M_* = \tilde{M}_1 \cap \cdots \cap \tilde{M}_l$ of M where $\tilde{M}_i$ is the projection of M_i onto M. In the next section we will show that q^* is exactly the number of dynamically decouplable input-output channels.

IV. Main Theorem

In this section we will state and prove our main result on dynamic decoupling.

Theorem 4.1: Suppose the analytic system (2.1) is given. Then the system is locally input-output decouplable by precompensation and feedback if and only if $q^* = p$.

Remark 4.2: As it can be seen in the proof, if $q^* < p$ then q^* is maximal number of decouplable input-output channels.

Remark 4.3: The concept of reproducibility in the necessity part of the proof is obviously related to the notion of right-invertibility, and therefore there are links with the approach of Fliess [5], [6]; see also [33].

Proof: "$\leftarrow$" Suppose $q^* = p$, on an open and dense submanifold M_* of M with $q^* = q_{k+1}$. For every $1 \le i \le q_k$ let ν_i denote the highest time derivative of u_i present in $\bar{U}^k$. Introduce the precompensator

$$\dot{z}_{ij} = z_{ij+1}, \quad 1 \le j < \nu_i, \quad \dot{z}_{i\nu_i} = w_i \tag{4.1}$$

for all $1 \le i \le q_k$ satisfying $\nu_i > 1$. Define r_j-dimensional vectors $\tilde{z}^j$, $j = 1, \cdots, k$, by

$$\tilde{z}^j = (\tilde{z}_{q_{j-1}+1}, \cdots, \tilde{z}_{q_j}) \tag{4.2a}$$

where

$$\tilde{z}_i = z_{i1} \text{ if } \nu_i > 0 \text{ and } \tilde{z}_i = w_i \text{ if } \nu_i = 0. \tag{4.2b}$$

Now define inductively the map linking the precompensator (4.1) with the system (2.1) as follows. Make the composition of (3.4) with (4.1), (4.2) via

$$u^1 = \tilde{z}^1 \tag{4.3a}$$

$$\bar{u}^1 = \alpha_2(x, \bar{U}^1) + \beta_2(x, \bar{U}^1) \begin{pmatrix} u^2 \\ \bar{u}^2 \end{pmatrix} \tag{4.3b}$$

where α_2 and β_2 are given by (3.11). At the lth step make the composition of the so far obtained system together with

$$u^l = \tilde{z}^l \tag{4.4a}$$

$$\bar{u}^l = \alpha_{l+1}(x, \bar{U}^1) + \beta_{l+1}(x, \bar{U}^l) \begin{pmatrix} u^{l+1} \\ \bar{u}^{l+1} \end{pmatrix} \tag{4.4b}$$

with α_{l+1} and β_{l+1} given by (3.19). This is done for all $l = 1, \cdots, k$. Observe that $u_i^{(j)}$ present in $\bar{U}^l$ can be expressed as z_{ij}. Moreover, according to Lemma 2.3, all above defined maps are affine with respect to the inputs w_i. These two facts imply that the resulting composition is of the desired form (2.7b). In order to describe the input-output behavior of the extended system, let

$$w^l = (w_{q_{l-1}+1}, \cdots, w_{q_l}), \; \nu^l = (\nu_{q_l}), \qquad l = 1, \cdots, k \tag{4.5}$$

and put

$$\sigma^l = \rho^l + \nu^l, \qquad l = 1, \cdots, k. \tag{4.6}$$

Then from (3.5), (3.12), (3.19), (4.3a), (4.3b), (4.4a), (4.4b), and (4.5), (4.6) we get

$$\begin{pmatrix} (y^1)^{(\sigma^1)} = w^1 \\ \vdots \\ (y^k)^{(\sigma^k)} = w^k \\ (y^k)^{(\rho^{k+1})} = u^{k+1}. \end{pmatrix} \tag{4.7}$$

Since $q_{k+1} = q_* = p$ it follows from (4.7) that we have obtained input-output decoupling. Moreover, the open and dense submanifold M_k of $M \times \mathbb{R}^{\mu_k}$ gives the set of pairs of initial states (x_0, z_0) for which the input-output decoupling as described in (4.7) holds.

"$\leftarrow$" Assume that there exist a precompensator of the form

$$\dot{z} = \phi(x, z) + \psi(x, z)w, \; z \in \mathbb{R}^\nu, \tag{4.8}$$

initial state $z_0 \in \mathbb{R}^\nu$, and a feedback law

$$u = \alpha(x, z) + \beta(x, z)w \tag{4.9}$$

such that the precompensated system denoted by Σ_e is decoupled in a neighborhood of (x_0, z_0). This implies that locally the behavior of the system is described by

$$\frac{d^{\sigma_i}}{dt^{\sigma_i}} y_i = w_i, \qquad i = 1, \cdots, p \tag{4.10}$$

for suitable σ_i's and w_i, $p < i \le m$ do not influence the output vector $y = (y_1, \cdots, y_p)^T$. Observe that this implies the following local reproducibility property. Given a set of any analytic functions $\phi_i = \phi_i(t)$, $i = 1, \cdots, p$ one is able to find controls $v_i(t)$, $i = 1, \cdots, m$ such that Σ_e feeded by $v(t) = (v_1(t), \cdots, v_m(t))^T$ products on a small time interval the output

$y(t) = (y_1(t), \cdots, y_p(t))^T$ such that

$$\frac{d^{\nu_i}}{dt^{\nu_i}} y_i = \phi_i(t), \qquad i=1, \cdots, p \tag{4.11}$$

for any fixed $\nu_i \geq \sigma_i$. Therefore, the original system (3.1), denoted by Σ, has the same property. To see this one should apply to Σ the control $u(t)$ given by

$$u(t)=\alpha(x(t), z(t))+\beta(x(t), z(t))v(t) \tag{4.12}$$

which obviously produces output $y(t)$ satisfying (4.11). To prove that $q_* = p$ is necessary for decoupling, we show that if $q_* < p$, then Σ does not possess the above reproducibility property. To see this take the decoupling procedure, based on the algorithm, up to the q_*th step. We get

$$\frac{d^{\sigma_i}}{dt^{\sigma_i}} y_i = w_i, \qquad 1 \le i \le q_* \tag{4.13}$$

for suitable σ_i's, and y_i, $i > q_*$ do not depend on u_i, $i > q_*$ (compare the proof of the implication). We show that given $\phi_i = \phi_i(t)$, $i = 1, \cdots, p$ it is not possible to find a control $v(t) = (v_1(t), \cdots, v_m(t))^T$ such that (4.11) is satisfied. Observe that (4.13) (locally) gives $v_i(t)$, $1 \le i \le q_*$ in a unique way, however, this choice also uniquely determines y_i, $i > q_*$. Therefore, the derivatives of y_i, $i > q_*$ are specified by those of y_i, $i \le q_*$ and this contradicts the desired reproducibility.

V. Final Remarks

In this section we will elucidate our results and compare them to other approaches existing in the literature.

In [11] Hirschorn proposed an algorithm, a nonlinear version generalization of Silverman structure algorithm [32], for studying the (left-) invertibility of affine control systems. A modified version of this algorithm was proposed by Singh [26]. As can be seen from Example 2.1 there are some connections between left invertibility and our algorithm. In [25] Singh has shown that for those nonlinear systems that are left-invertible under the condition of Hirschorn [11] one can achieve decoupling via precompensation and feedback. It is interesting to observe that Hirschorn's algorithm [11] (and its modification by Singh [26]) allows for state-dependent transformations of the output (postcompensation), whereas we allow for state-dependent input transformations (precompensation); so this is, in fact, a dual approach of our method.

Descusse and Moog [13] (see also [33]) proposed an algorithm for solving the nonlinear decoupling problem and showed its converge under the nonverifiable assumption of the left-invertibility. However, in the very recent paper [35] it is shown that the assumption given in [33] may in fact be verifiable. The algorithm we propose is based on the same idea of precompensation of those inputs which appear too "early" when differentiating the outputs. However, both approaches differ substantially in the feedback laws they suggest. In [3] one changes at every step only the controlled vector fields g_i's (β_l is applied) while we change at every step both the drift term f and the controlled vector fields g_i's, i.e., both α_l and β_l are applied. In a sense our algorithm can be viewed as a dynamic feedback generalization of Krener's algorithm for computing the maximum local controlled invariant distribution contained in ker dh; see [15]. In the first step both algorithms yield the same but in the next steps they differ, since Krener uses only state feedback, whereas we use feedback which also depends on the controls that are treated as parameters or state variables of the extended system.

The pair (α_l, β_l) is chosen in our algorithm in such a way that at step l we transform a part of the studied system into q_l independent ρ_i^l-fold integrators. This allows us to obtain a verifiable necessary and sufficient condition for dynamic decoupling: the problem is solvable if and only if $q^* = p$. This result can also be expressed in the following form: a nonlinear system is decouplable by means of general dynamic feedback (2.7a), (2.7b) if and only if it can be decoupled by our method. Observe that the latter implies in particular that if a nonlinear system is not decouplable by preintegration, then it cannot be by any precompensator of the general form (2.7a), (2.7b) either. This generalizes an analogous result shown for linear systems by Wonham and Morse [31]. Finally, observe, what follows easily from the proof, that if $q^* < p$, then the decoupling problem is not solvable and q^* gives the maximal number of decouplable I/O channels.

Very recently Fliess introduced in nonlinear system theory some very interesting new ideas based on differential algebraic techniques [5]–[7]. Using this frame he also solves the dynamic input-output decoupling problem in the following way. The dynamic decoupling problem is solvable if and only if the differential algebraic transcendence degree of the system equals p, the number of output channels. We refer the reader to [5]–[7] for a precise statement of the problem, the needed concepts, and the proof. For those who are not familiar with differential algebra we emphasize that the above statement is equivalent to the fact that there is no differential equation involving components of the outputs and their time derivatives. We refer to [35] for a comparison of the analytic approach and that of Fliess.

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Design for Noninteracting Decomposition of Nonlinear Systems

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Abstract—**This note tackles the general input-output noninteracting decomposition problem of nonlinear systems. Under less regularity assumptions we give an alternative proof of the same necessary and sufficient conditions as in [4]. Our result gives an algorithm which constructs the feedback law α and β explicitly. Finally, we prove that the decomposed form is a canonical form.**

I. Introduction

Consider an affine nonlinear system

$$\dot{x}=f(x)+\sum_{i=1}^{m} g_i(x)u_i \tag{1.1.a}$$

$$y=h(x) \tag{1.1.b}$$

where $x \in M; f(x), g_i(x) \in V(M); h: M \to N$ is a C^∞ mapping: M and N are C^∞ manifolds with dimensions n and r, respectively. The input-output noninteracting decomposition problem (NDP) may be stated as follows. Given a partition of the outputs y, whether there exist a feedback control

$$u=\alpha(x)+\beta(x)v \tag{1.2}$$

and a partition of the controls v, such that each block of v completely controls the corresponding block of y, and does not affect the other blocks of the outputs.

The NDP has been studied extensively and from various points of view. The discussion for linear systems is founded in [1], [2], etc.

Recently, the NDP of nonlinear systems has been studied in [3] and [4]. Reference [3] gives precise formulation and solves NDP for the single-input and block-output case by "controllability distribution" approach. Reference [4] proves the same results for the block-input and block-output case under more regularity assumptions by using the concept of zeros at infinity.

The main goal of this note is to give an alternative proof of the same result of [4] under less regularity assumptions as required in [3]. Our proof is constructive, thus it yields an algorithm. Using it, an input-output decomposed form has been obtained. Finally, we prove that the decomposed form obtained is a canonical form.

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For investigating the decoupling problem of linear systems, the geometric concepts of (A, B)-invariant subspaces and controllability subspaces play a very important role. In the geometric approach to nonlinear systems, the concept of (A, B)-invariant subspaces has been extended to that of (f, g)-invariant distributions [5], [6], and the concept of controllability subspaces has also been extended to that of controllability distributions [7], [8].

Since our discussion depends particularly upon the concept of (f, g)-invariance, we state the following definition which is slightly different from the original one given in [6].

For the sake of compactness, let $C_m^\infty(U)$ be the set of $m \times 1$ vectors with the entries in $C^\infty(U)$, and $Gl(m, C^\infty(U))$ be the set of $m \times m$ nonsingular matrices with the entries in $C^\infty(U)$ too, where U is an open subset of M.

Definition 1.1: A distribution Δ is said to be weakly (f, g)-invariant at $p \in M$ if there exists a neighborhood U of p, such that on U

$$[f, \Delta] \subset \Delta+G, \tag{1.3.a}$$

$$[g_i, \Delta] \subset \Delta+G, \qquad i=1, \cdots, m \tag{1.3.b}$$

where $G = Sp\{g_1, \cdots, g_m\}$. Δ is said to be strongly (f, g)-invariant at $p \in M$ if there exist a neighborhood U of p, $\alpha \in C_m^\infty(U)$ and $\beta \in Gl(m, C^\infty(U))$, such that on U

$$[f+g\alpha, \Delta] \subset \Delta, \tag{1.4.a}$$

$$[(g\beta)_i, \Delta] \subset \Delta, \qquad i=1, \cdots, m. \tag{1.4.b}$$

The local equivalence of these two kinds of (f, g)-invariances is proved in [6] and [12] independently.

II. Compatible (f, g)-Invariance

To study decoupling problems of nonlinear systems, we have to consider several (f, g)-invariant distributions simultaneously. Thus, we introduce the concept of compatible (f, g)-invariance.

Definition 2.1: Let $\Delta_1, \cdots, \Delta_k$ be k weakly (f, g)-invariant distributions at p. $\Delta_1, \cdots, \Delta_k$ are said to be compatible (f, g)-invariant at p, if there exist a neighborhood U of p, $\alpha \in C_m^\infty(U)$ and $\beta \in Gl(m, C^\infty(U))$, such that on U

$$[f+g\alpha, \Delta_i] \subset \Delta_i \tag{2.1.a}$$

$$[(g\beta)_j, \Delta_i] \subset \Delta_i, \qquad j=1, \cdots, m; \; i=1, \cdots, k. \tag{2.1.b}$$

Let Δ be an involutive distribution with constant dimension. According to Frobenius' theorem, there exists a local coordinate chart $(U, (x, y))$, $x = (x_1, \cdots, x_p)$, and $y = (y_1, \cdots, y_{n-p})$, such that

$$\Delta = Sp\left\{\frac{\partial}{\partial x_i}: \qquad i=1, \cdots, p\right\}$$

This coordinate chart is called a flat chart [10]. Let $(W, (x', y'))$ be another flat coordinate chart and $W \cap U \neq \phi$. Then on $W \cap U$

$$y'=y'(y). \tag{2.2}$$

Assume a vector field X is expressed in a flat chart (x, y) as

$$X=(a_1, \cdots, a_n)^T \in T(U).$$

Then the canonical projection $\pi(X)$ of X on TM/Δ is defined as

$$\pi(X)=(a_{p+1}, \cdots, a_n)^T \tag{2.3}$$

and denoted as X/Δ. Using (2.2), it is easy to prove that X/Δ is independent of the choice of the flat frame.

Likewise, for a distribution G we may define the canonical projection

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