



Variational derivation of improved KP-type of equations

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ABSTRACT

The Kadomtsev–Petviashvili equation describes nonlinear dispersive waves which travel mainly in one direction, generalizing the Korteweg–de Vries equation for purely uni-directional waves. In this Letter we derive an improved KP-equation that has exact dispersion in the main propagation direction and that is accurate in second order of the wave height. Moreover, different from the KP-equation, this new equation is also valid for waves on deep water. These properties are inherited from the AB-equation (E. van Groesen, Andonowati, 2007 [1]) which is the unidirectional improvement of the KdV equation. The derivation of the equation uses the variational formulation of surface water waves, and inherits the basic Hamiltonian structure.

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1. Introduction

The Kadomtsev–Petviashvili equation, or briefly the KP-equation, is derived in 1970 as a generalization of the Korteweg–de Vries (KdV)-equation for two spatial dimensions [5,6,10]. It is a well-known model for dispersive, weakly nonlinear and almost unidirectional waves. Although the equation is of relevance in many applications with various dispersion relations, we will concentrate in this Letter on the application to surface water waves on a – possibly infinitely deep – layer of fluid; for other applications the dispersion relation can be adapted in the following.

In 2007, the AB-equation, a new type of KdV-equation, is derived in [1]. This equation is exact up to and including quadratic nonlinear terms and has exact dispersive properties. In [2], it is shown that the AB-equation can accurately model waves in hydrodynamics laboratories that are generated by a flap motion. Moreover, unlike any other KdV-type of equation, this AB-equation is valid and accurate for waves in deep water. By using Hamiltonian theory, it was shown that AB-equation approximates accurately the highest 120° Stokes wave in deep water [2,4]. Different from all steady traveling wave profiles which are smooth, the highest Stokes wave has a peak at the crest position and travel periodically with a constant speed [2,4,11–13].

This Letter is a continuation, and actually a combination, of these previous works. Our objective is to obtain an AB2-equation that deals just as KP with mainly uni-directional waves in two

space dimensions, but shares the properties of the AB-equation of being accurate to second order in the wave height and having exact dispersion in the main direction of propagation. We will derive AB2 in a consistent way from the variational formulation of surface waves; in poorer approximations, AB2 will give various types of improved KP-type of equations.

Before dealing with the somewhat technical variational aspects, we will illustrate the basic result with a simple intuitive derivation of the classical KP and the new AB2-equation.

We start with the simplest second order wave equation:

$$\partial_t^2 \eta = c_0^2 (\partial_x^2 + \partial_y^2) \eta. \quad (1)$$

For constant $c_0 = \sqrt{gh}$ with h the constant depth of the layer, this is a description of very long waves in the linear approximation. The dispersion relation is given by $\omega^2 = c_0^2 (k_x^2 + k_y^2)$ which relates the frequency ω of harmonic waves to the wave numbers in the x and y -direction. Purely unidirectional waves in the positive x -direction would be described by $(\partial_t + c_0 \partial_x) \eta = 0$ which has the dispersion relation $\omega = c_0 k_x$. Multi-directional waves that *mainly* travel in the positive x -direction will have $|k_y| \ll k_x$. This makes it tempting to approximate the second order spatial operator $c_0^2 (\partial_x^2 + \partial_y^2)$ via the dispersion relation like

$$\begin{aligned} \omega &= c_0 k_x \sqrt{1 + (k_y/k_x)^2} \\ &\approx c_0 k_x \left(1 + \frac{1}{2} \frac{k_y^2}{k_x^2} \right) = c_0 \left(k_x + \frac{1}{2} \frac{k_y^2}{k_x} \right). \end{aligned}$$

The corresponding equation is therefore $\partial_t \eta = -c_0 (\partial_x \eta + \frac{1}{2} \partial_x^{-1} \partial_y^2 \eta)$, which can be rewritten in a more appealing way like

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$$\partial_x[\partial_t \eta + c_0 \partial_x \eta] + \frac{c_0}{2} \partial_y^2 \eta = 0. \quad (2)$$

This is the approximate equation for infinitesimal long waves traveling mainly in the positive x -direction. If we want to include approximate dispersion and nonlinearity in the x -direction only, the KdV-equation could be taken as approximation: $\partial_t \eta + c_0 \partial_x \eta + c_0 \frac{h^2}{6} \partial_x^3 \eta + \frac{3c_0}{2h} \eta \partial_x \eta = 0$. Changing the term in brackets by this expression leads to the standard form of the KP-equation:

$$\partial_x \left[\partial_t \eta + c_0 \left(\partial_x \eta + \frac{h^2}{6} \partial_x^3 \eta \right) + \frac{3c_0}{2h} \eta \partial_x \eta \right] + \frac{c_0}{2} \partial_y^2 \eta = 0. \quad (3)$$

A simple scaling can normalize all coefficients in the bracket in a frame moving with speed c_0 .

Improving on this heuristic derivation, we will show in this Letter that the more accurate AB2-equation can be derived in a consistent way. A simplified form of AB2 could be obtained by replacing the term in square brackets in (3) by the AB-equation, and at the same time adding dispersive effects related to the group velocity in the term with the second order transversal derivatives that are consistently related to the dispersion within the square bracket. The most appealing equation is found that matches second order accuracy in wave height with second order transversal accuracy, to be called AB22, given by

$$\partial_x \left[\partial_t \eta + \sqrt{g} A \left(\eta - \frac{1}{4} (A\eta)^2 + \frac{1}{2} A(\eta A\eta) + \frac{1}{4} (B\eta)^2 + \frac{1}{2} B(\eta B\eta) \right) \right] + \frac{1}{2} V \partial_y^2 \eta = 0, \quad (4)$$

where the term in square brackets is the AB-equation of [1].

This Letter will follow the consistent derivation for waves in one direction as given in [1]. The major difference is to account for small deviations in the unidirectionalization procedure; this requires some careful dealing with skew symmetric square-root operators.

The content of the Letter is as follows. In the next section we describe the variational structure and the approximation of the action principle that will be used in the following sections. Section 3 discusses the linear 2-dimensional wave equation and the unidirectional constraint to obtain an approximate linear 2D-equation that has exact dispersion in the main propagation direction. This will be the linearized AB2-equation which will be derived in Section 4, and shown to be accurate in second order of the wave height. In Section 5 some approximate cases are considered. Some conclusions and remarks are given in Section 6.

2. Variational structure

We consider surface waves on a layer of irrotational, inviscid and incompressible fluid that propagate in the $\mathbf{x} = (x, y)$ direction over a finite depth h_0 or over infinite depth. We denote the wave elevation by $\eta(\mathbf{x}, t)$ and the fluid potential by $\Phi(\mathbf{x}, z, t)$ with $\phi(\mathbf{x}, z = \eta, t)$ the fluid potential at the surface.

Based on previous work of [3,7–9], the dynamic equations can be derived from variations of the action principle, i.e. $\delta A(\eta, \phi) = 0$, where

$$A(\eta, \phi) = \int \left[\int \phi \cdot \partial_t \eta \, d\mathbf{x} - H(\eta, \phi) \right] dt. \quad (5)$$

Variations with respect to ϕ and η lead to the following coupled equations:

$$\partial_t \eta = \delta_\phi H(\eta, \phi), \quad (6)$$

$$\partial_t \phi = -\delta_\eta H(\eta, \phi). \quad (7)$$

The Hamiltonian $H(\eta, \phi)$ is the total energy, the sum of the potential and the kinetic energy, expressed in the variables η, ϕ . The potential energy is calculated with respect to the undisturbed water level, leading to

$$H(\eta, \phi) = \int \frac{1}{2} g \eta^2 \, d\mathbf{x} + K(\phi, \eta). \quad (8)$$

The value of the kinetic energy is given by:

$$K(\phi, \eta) = \iint \frac{1}{2} |\nabla \Phi|^2 \, dz \, d\mathbf{x}$$

where the fluid potential Φ satisfies the Laplace equation $\Delta \Phi = 0$ in the fluid interior, the surface condition $\Phi = \phi$ at $z = \eta(\mathbf{x})$ and the impermeable bottom boundary condition. This kinetic energy cannot be expressed explicitly as a functional in terms of ϕ and η , and therefore approximations are needed. To that end, we will use the notation of the fluid potential and the vertical velocity at the still water as

$$\phi_0 = \Phi(\mathbf{x}, z = 0, t), \quad W_0 = \partial_z \Phi(\mathbf{x}, z = 0, t)$$

respectively. To approximate the kinetic energy, we split the kinetic energy in an expression till the still water level and an additional term to account for the actual surface elevation:

$$K = K_0 + K_s = \int \int_{z=-h}^{z=0} \frac{1}{2} |\nabla \Phi|^2 \, dz \, d\mathbf{x} + \int \int_{z=0}^{z=\eta} \frac{1}{2} |\nabla \Phi|^2 \, dz \, d\mathbf{x}.$$

The bulk kinetic energy K_0 can be rewritten using $\Delta \Phi = 0$ and the divergence theorem with the permeability bottom condition as follows:

$$K_0 = \int \int_{z=-h}^{z=0} \frac{1}{2} \nabla \cdot (\Phi \nabla \Phi) \, dz \, d\mathbf{x} = \int \frac{1}{2} \phi_0 W_0 \, d\mathbf{x}.$$

Taking in K_s the approximation of lowest order in the wave height, we obtain:

$$\begin{aligned} K_s &\approx \int \int_{z=0}^{z=\eta} \left[\frac{1}{2} |\nabla \Phi|^2 \right]_{z=0} \, dz \, d\mathbf{x} \\ &= \frac{1}{2} \int \eta [(\partial_x \phi_0)^2 + (\partial_y \phi_0)^2 + W_0^2] \, d\mathbf{x}. \end{aligned}$$

Taken together, we obtained the following approximation for the Hamiltonian:

$$\begin{aligned} H(\eta, \phi_0) &\approx \frac{1}{2} \int \left\{ g \eta^2 + \phi_0 W_0 + \eta [(\partial_x \phi_0)^2 + (\partial_y \phi_0)^2 + W_0^2] \right\} \, d\mathbf{x}. \quad (9) \end{aligned}$$

Observe that this expression is in the variables η and ϕ_0 (and W_0 which will be easily expressed in terms of ϕ_0 in the next section). These are not the canonical variables, and hence $H(\eta, \phi_0)$ cannot be used in the Hamilton equations, except in the linear approximation of the next section. To include nonlinear effects, we will need to translate the pair η, ϕ_0 to the canonical pair η, ϕ in Section 4.

3. Mainly unidirectional linear waves

We start this section on linear wave theory with some notation to reduce possible confusion. For irrotational waves in one dimension, the dispersion relation between frequency ω and wavenumber k is given by

$$\omega^2 = gk \tanh(kh).$$

By defining

$$\Omega(k) = \text{sign}(k)\sqrt{gk \tanh(kh)},$$

the corresponding pseudo-differential operator $i\Omega(-i\partial_x)$ is skew-symmetric.

As usual we define the phase and group velocity respectively as

$$C = \frac{\Omega}{k}, \quad \text{and} \quad V = \frac{d\Omega}{dk},$$

we will use the same notation for the corresponding symmetric operators in the following.

Then the second order dispersive equation $\partial_t^2 \eta = -\Omega^2(-i\partial_x)\eta = \partial_x^2 C^2 \eta$ can be written like

$$(\partial_t^2 + \Omega^2)\eta = (\partial_t - \partial_x C)(\partial_t + \partial_x C)\eta = 0,$$

showing that each solution is a combination of a wave running to the right, i.e. satisfying $(\partial_t + \partial_x C)\eta = 0$, and a wave running to the left $(\partial_t - \partial_x C)\eta = 0$.

For multi-directional waves, the dispersion relation is given with the wave vector $\mathbf{k} = (k_x, k_y)$ by

$$\omega^2 = g|\mathbf{k}| \tanh(|\mathbf{k}|h).$$

To deal with waves traveling mainly in the positive x -direction, we define

$$\Omega_2(\mathbf{k}) = \text{sign}(k_x)\sqrt{g|\mathbf{k}| \tanh(|\mathbf{k}|h)},$$

then, with $\nabla = (\partial_x, \partial_y)$, the operator $i\Omega_2(-i\nabla)$ is skew-symmetric. Note that $\Omega_2(\mathbf{k})$ is discontinuous for $k_x = 0$ if $k_y \neq 0$, which will not happen when the assumption of mainly unidirectionality holds, since then $|k_y| \ll k_x$.

The phase velocity vector is given by

$$C_2(\mathbf{k}) = \Omega_2(\mathbf{k}) \frac{\mathbf{k}}{|\mathbf{k}|^2} = C(|\mathbf{k}|) \frac{\mathbf{k}}{|\mathbf{k}|},$$

hence we have

$$\Omega_2(\mathbf{k}) = -i\alpha(\mathbf{k})C(|\mathbf{k}|), \quad \text{with} \quad \alpha(\mathbf{k}) = i \cdot \text{sign}(k_x)|\mathbf{k}|.$$

To make the analogy with the 1D case complete, we defined here the skew symmetric square root α operator of $-\nabla^2$; indeed, $\alpha = \partial_x$ if $\partial_y = 0$, and $\alpha^2 = (\partial_x^2 + \partial_y^2)$ and $\alpha\alpha^* = -(\partial_x^2 + \partial_y^2)$.

For later reference, we note that Ω_2 has for each $k_x \neq 0$ the following expression as second order Taylor expansion in k_y at 0:

$$\Omega_2(\mathbf{k}) \approx \Omega(k_x) + \frac{1}{2} \frac{k_y^2}{k_x} V(k_x). \quad (10)$$

Using the above notation, we recall that the solution of the Laplace problem for the fluid potential implies the following expression for the vertical velocity at the surface

$$W_0 = -\frac{1}{g}\Omega_2^2\phi_0 = -\frac{1}{g}C^2\alpha^2\phi_0.$$

This leads to the following expression for the quadratic part of the Hamiltonian:

$$\begin{aligned} H_0(\eta, \phi_0) &= \frac{1}{2} \int \{g\eta^2 + \phi_0 W_0\} d\mathbf{x} \\ &= \frac{1}{2} \int \left\{g\eta^2 - \frac{1}{g}\phi_0\alpha^2 C^2\phi_0\right\} d\mathbf{x}, \end{aligned}$$

and the corresponding Hamilton equations are given by

$$\partial_t \eta = \delta_\phi H_0(\eta, \phi) = -\frac{1}{g}\alpha^2 C^2 \phi_0, \quad (11)$$

$$\partial_t \phi_0 = -\delta_\eta H_0(\eta, \phi) = -g\eta, \quad (12)$$

which are equivalent with the second order equation:

$$\partial_t^2 \eta = C^2 \alpha^2 \eta. \quad (13)$$

We notice that when C is a constant, this equation is exactly the hyperbolic equation that describes two-dimensional non-dispersive traveling waves as described in the introduction. This linear equation can be rewritten as the application of two first order linear equation:

$$(\partial_t - C\alpha)(\partial_t + C\alpha)\eta = 0. \quad (14)$$

Hence we find the dispersive equation for mainly uni-directional waves in the positive x -direction as

$$(\partial_t + C\alpha)\eta = 0.$$

As in [1], we will execute this 'unidirectionalization' procedure in another way via the variational principle. To illustrate that this leads to the correct result above, we note that the dynamic equation for ϕ_0 leads for mainly unidirectional waves to a constraint between ϕ_0 and η given by $\partial_t \phi_0 = -C\alpha\phi_0 = -g\eta$, i.e. $\phi_0 = g\alpha^{-1}C^{-1}\eta$.

Restricting the action functional to this constraint, leads to:

$$\overline{\mathcal{A}}_0(\eta) = \int \left[\int g\alpha^{-1}C^{-1}\eta \cdot \partial_t \eta d\mathbf{x} - \overline{H}_0(\eta) \right] dt,$$

with the restricted Hamiltonian

$$\overline{H}_0(\eta) = \frac{g}{2} \int \eta^2 - (\partial_x^2 + \partial_y^2)^{-1} \alpha \eta \cdot \alpha \eta d\mathbf{x} = g \int \eta^2 d\mathbf{x}.$$

Notice that the kinetic energy and the potential energy functionals have the same value, i.e. equipartition of energy as is known to hold for linear wave evolutions.

The dynamic equation of η then follows from the action principle, i.e. $\delta \overline{\mathcal{A}}_0(\eta) = 0$, leading to:

$$\partial_t \eta = -C\alpha \eta. \quad (15)$$

This is the linear equation for waves mainly running in the positive x -direction. Expansion of the operators to second order in k_y using (10), it can be written like:

$$\partial_x [\partial_t \eta + \partial_x C(-i\partial_x)\eta] + \frac{1}{2} V(-i\partial_x)\partial_y^2 \eta = 0. \quad (16)$$

This is recognized as an improvement of the linear KP-equation that is of the same second order in the transversal direction. Observe, in particular, the appearance of the group-velocity operator in the last term, which is consistently linked to the (possibly approximated) phase velocity operator in the square brackets. When we use the approximation for rather long waves, as in the KP-equation, we have $C = c_0(1 + \frac{h^2}{6}\partial_x^2)$ and then consistently $V = c_0(1 + \frac{h^2}{2}\partial_x^2)$; this last operator is missing in the original KP-equation.

We notice also that Eq. (16) is for purely unidirectional waves, $\partial_y = 0$, the same as the linear part of AB-equation as derived in [1].

4. Nonlinear AB2-equation

In the previous section, we used the quadratic part of the Hamiltonian and the unidirectionalization procedure in the action principle to obtain the linear AB2-equation. In this section, we will include the cubic terms of η that appear in the approximation of the Hamiltonian (9). In the following, we use the same unidirectionalization constraint as describe above, i.e. taking $\phi_0 = gC^{-1}\alpha^{-1}\eta$. For simplification, we introduce the following operators

$$A_2 = \frac{C(-i\nabla)\alpha}{\sqrt{g}}, \quad B_2 = \partial_x A_2^{-1} \quad \text{and} \quad \gamma_2 = \partial_y A_2^{-1},$$

and rewrite the Hamiltonian (9) as

$$H(\eta, \phi_0) = g \int \left\{ \eta^2 + \frac{1}{2} \eta [(A_2 \eta)^2 + (B_2 \eta)^2 + (\gamma_2 \eta)^2] \right\} d\mathbf{x} \quad (17)$$

$$= [H_2 + H_3](\eta), \quad (18)$$

with H_2 and H_3 the quadratic and the cubic terms of the Hamiltonian respectively.

For the nonlinear case considered here, we have to use the action principle with $\int \int \phi \partial_t \eta d\mathbf{x} dt$ which contains the potential ϕ at the surface $z = \eta$. The strategy is to relate ϕ to ϕ_0, η by a direct expansion $\phi = \phi_0 + \eta W_0$. Then using the unidirectional constraint $\phi_0 = \sqrt{g} A_2^{-1} \eta$ and $W_0 = -\sqrt{g} A_2 \eta$, we get

$$\int \int \phi \partial_t \eta d\mathbf{x} dt = \int \int \sqrt{g} [A_2^{-1} \eta - \eta A_2 \eta] \partial_t \eta d\mathbf{x} dt.$$

The action functional becomes

$$\mathcal{A}(\eta) = \int \left\{ \int \sqrt{g} [A_2^{-1} \eta - \eta A_2 \eta] \partial_t \eta d\mathbf{x} - H(\eta) \right\} dt, \quad (19)$$

and vanishing of the variational derivative leads to the evolution equation

$$\sqrt{g} [-2A_2^{-1} \partial_t \eta + A_2 (\eta \partial_t \eta) + \eta A_2 \partial_t \eta] = \delta_\eta \bar{H}(\eta)$$

with

$$\delta_\eta \bar{H}(\eta) = g \left[2\eta + \frac{1}{2} (A_2 \eta)^2 - A_2 (\eta A_2 \eta) \right. \\ \left. + \frac{1}{2} (B_2 \eta)^2 + B_2 (\eta B_2 \eta) + \frac{1}{2} (\gamma_2 \eta)^2 + \gamma_2 (\eta \gamma_2 \eta) \right].$$

Although this formulation is correct, the expression involving $\partial_t \eta$ is rather complicated. We will simplify it in the same way as in [1]. To that end, we note that

$$-2\sqrt{g} A_2^{-1} \partial_t \eta = 2g\eta + O(\eta^2), \quad \text{i.e. } \partial_t \eta = -\sqrt{g} A_2 \eta + O(\eta^2),$$

and substitute this approximation in the action to obtain

$$\int \sqrt{g} [A_2^{-1} \eta - \eta A_2 \eta] \partial_t \eta d\mathbf{x} = \int [\sqrt{g} A_2^{-1} \eta \partial_t \eta + g\eta (A_2 \eta)^2] d\mathbf{x}.$$

In this way, the total action functional is approximated correctly up to and including cubic terms in wave height. We write the result as a modified action principle, reading explicitly

$$A_{mod}(\eta) = \int \left[\int \sqrt{g} A_2^{-1} \eta \partial_t \eta d\mathbf{x} - \bar{H}_{mod}(\eta) \right] dt \quad (20)$$

where the modified Hamiltonian \bar{H}_{mod} contains a term from the original action and is given by

$$\bar{H}_{mod}(\eta) = \bar{H} - g \int \eta (A_2 \eta)^2 d\mathbf{x} \quad (21)$$

$$= g \int \eta^2 + \frac{1}{2} \eta \{ -(A_2 \eta)^2 + (B_2 \eta)^2 + (\gamma_2 \eta)^2 \} d\mathbf{x}. \quad (22)$$

The resulting equation $\delta A_{mod}(\eta) = 0$:

$$-2\sqrt{g} A_2^{-1} \partial_t \eta = \delta \bar{H}_{mod}(\eta) \quad (23)$$

will be called the AB2-equation and it is explicitly given by:

$$\partial_t \eta = -\sqrt{g} A_2 \begin{bmatrix} \eta - \frac{1}{4} (A_2 \eta)^2 + \frac{1}{2} A_2 (\eta A_2 \eta) \\ + \frac{1}{4} (B_2 \eta)^2 + \frac{1}{2} B_2 (\eta B_2 \eta) \\ + \frac{1}{4} (\gamma_2 \eta)^2 + \frac{1}{2} \gamma_2 (\eta \gamma_2 \eta) \end{bmatrix}. \quad (24)$$

From the derivation above, we notice that AB2-equation is exact up to and including the second order in the wave height, with (possible approximations of the) dispersion relation reflected in the operators A_2, B_2 and γ_2 .

5. Approximations of the AB2-equation

In this section, we consider some special limiting cases of the AB2-equation.

If there is no dependence on the y -direction, i.e. waves travel uniformly in the positive x -direction, we can ignore the last two terms of Eq. (24) since $\gamma_2 = 0$, and the operators α, A_2 and B_2 can be rewritten as follows:

$$\alpha = \partial_x, \quad A_2 = A \equiv \frac{C(-i\partial_x)}{\sqrt{g}} \partial_x \quad \text{and} \quad B_2 = B \equiv \partial_x A^{-1}.$$

Substituting these operators in Eq. (24), we obtain the original AB-equation for the unidirectional wave as derived in [1]:

$$\partial_t \eta = -\sqrt{g} A \left(\eta - \frac{1}{4} (A\eta)^2 + \frac{1}{2} A(\eta A\eta) + \frac{1}{4} (B\eta)^2 + \frac{1}{2} B(\eta B\eta) \right).$$

This equation is exact in the dispersion relation and is second order accurate in the wave height. It is an improvement of the KdV-equation for waves above finite depth. For infinite depth, this is a new equation in which all four quadratic terms are of the same order.

In Section 3 we investigated the approximation of the exact linear equation to second order in k_y , which leads to the improvement of the linear KP-equation but retains its characteristic form. Now we will make a similar approximation for the nonlinear AB2-equation. To keep the basic variational structure, any approximation can be obtained by taking approximations of the operators A_2 and the consistent approximations for $B_2 = \partial_x A_2^{-1}$ and $\gamma_2 = \partial_y A_2^{-1}$.

Here we will consider only one special case, namely an approximation for which we neglect transversal effects in the nonlinear terms. Stated differently, we approximate Eq. (24) such that second order nonlinear and transversal effects are considered to be of the same order. Using the operators A, B as above, we obtain the AB22-equation (4) mentioned in the introduction.

It should be remarked that in all the results above we did not restrict the wavelength in the x -direction; all results are valid also for short waves in the main propagation direction, the dispersion properties in the x -direction are exact. Also, note that, just like KdV, the KP-equation has no sensible limit for infinite depth, but that, just like the AB-equation, the AB2- and AB22-equation, are also valid for deep water waves.

6. Conclusion and remarks

The KP-equation as a model for mainly unidirectional surface water waves has been improved in this Letter to the AB2-equation that has exact dispersion and is exact up to and including second order in the wave height. As was found for the unidirectional version of this equation, the AB-equation, we expect that with AB2 very accurate simulations can be performed. In a subsequent paper we will show results of comparisons with measurements in a hydrodynamic laboratory. The most difficult aspect to predict beforehand is the practical validity in the transversal direction, i.e. what the maximal deviation from the main propagation direction can be so that the waves are still accurately modeled. We will also

present results that show the effect of the additional group velocity operator in the term with second order transversal derivatives, which will be one comparison of the accuracy between the AB2-equation and the KP-equation. A further extension of this work could be to derive an equation for surface wave that takes vorticity into account [14].

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