A Note on Bounds for Target Tracking with $p_d < 1$

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Recently several new results for Cramer-Rao lower bounds (CRLBs) in dynamical systems have been developed. Several different approaches and approximations have been presented. For the general case of target tracking with a detection probability smaller than one and possibly in the presence of false measurements, two main approaches have been presented. The first approach is the information reduction factor (IRF) approach. The second approach is the enumeration (ENUM) approach, also referred to as the conditioning approach. It has been found that the ENUM approach leads to a strictly larger covariance matrix than the IRF approach, however, still providing a lower bound on the attainable error covariance. Thus, the ENUM approach provides a strictly tighter bound on the attainable performance. It has been conjectured that these bounds converge to one another in the limit or equivalently after an initial transition stage. We demonstrate, using some recent results from the modified Riccati equation (MRE) and by means of counter examples, that this conjecture does not hold true in general. We also demonstrate that the conjecture does hold true in the special case of deterministic target motion, or equivalently in the absence of process noise. Furthermore, we show that the detection probability has an influence on the limiting behaviors of the bounds. Moreover, we show that the MRE approximation provides a very good and computationally efficient approximation of the ENUM bound. The various results are illustrated by means of representative examples.

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I. INTRODUCTION

The Cramer-Rao lower bound (CRLB) is a lower bound on the estimation error variance, see [1], for an estimation problem. Thus, the CRLB associated with a parameter estimation problem provides an estimation error covariance that cannot be improved upon. An estimator attaining this bound is therefore optimal in the minimum variance sense. Originally the CRLB was developed for bounds on estimation error variances for static parameter estimation problems. Recently a number of results on CRLBs for state estimation problems in dynamical systems has been derived, starting with the seminal paper [2]. These bounds admit e.g. system performance analysis even before the system is actually implemented, and especially in the dynamical case they can be used for optimizing sensor scheduling and deployment strategies, see [3] and references therein.

In [2] recursive equations for the calculation of CRLBs for general nonlinear dynamical systems have been derived. In the case of dynamical systems the CRLB is a bound on the performance that a filter may optimally achieve. The CRLB is stated in terms of the covariance of the filter, like the covariance of the classical CRLB in parameter estimation, see [1]. As the CRLB is a bound on the performance of a dynamical system and because it can be attained by an estimator in the linear Gaussian case, it is not surprising that in the linear Gaussian case the recursive equations for the CRLB derived in [2] coincide with the recursive equations for the Kalman filter, see [4]. This is almost trivially true when recalling that the Kalman filter is the minimum variance solution to a linear Gaussian filtering problem; see [4] and [5].

Following the seminal result from [2], a number of extensions to this result were derived. In [6], a CRLB for filtering with a probability of detection of less than one was proposed; this bound is generally referred to as the enumeration (ENUM) bound. In [3] this bound was shown to be tighter than an alternative lower bound, referred to as the information reduction factor (IRF) bound. In [7], this alternative bound was presented and analyzed for the more general case of target tracking in clutter and in [8] the ENUM bound for target tracking in clutter was presented. In [3] it was proven and shown by example that the information reduction bound can be quite far off the ENUM bound, especially in the initial or transition phase.

The ENUM bound, in the case of possible miss detections, requires an exhaustive enumeration over all miss-detect sequences and thus, its computational load grows exponentially over time. Recently an effective approximation for the ENUM bound was derived. This approximation was provided by the so-called modified Riccati equation (MRE); see [9] and [10]. As already mentioned in [3] it was shown that the IRF bound is always smaller than the ENUM bound. Also in [3] as well as in the more recent paper [8] it was conjectured, but not proven, that in the limit the IRF bound and the ENUM bound approach one another. If this conjecture were to hold true, this would mean that in the steady state case either the IRF or the ENUM bound could be used.

In [11] the authors consider CRLB-like bounds for systems with missing observations. However, they assume that the miss-detect sequences are correlated over time through a Markov process. We emphasize that this leads to a fundamentally different system model as well as different solutions from what is presented here and in the previously mentioned works, where one of the key assumptions is that miss-detect events are not correlated over time.

In the remainder of this note we concentrate amongst others on this conjecture. The contributions of this note are as follows.

1) We demonstrate that in the case of zero process noise or equivalently deterministic target motion the conjecture from [3] holds true.
2) We show by means of a counter example that in the general case, i.e., the case of non-zero process noise, this conjecture does not always hold true.
3) We show that in the case of non-zero process noise the difference between the steady state values of the two bounds can grow with a decreasing probability of detection.

For the formal proof of the first item, we employ some recent results on the MRE in relation to CRLBs; see [9] and [10]. We illustrate this result by means of simulation examples. Furthermore, the second point and the third point are shown by means of the examples.

The implications of the above points are that in general for the (almost) steady state case the ENUM bound cannot simply be replaced with the IRF bound. Furthermore, it is shown that there are two contributing factors for the difference between the bounds, namely the process noise level and the probability of detection. It is also shown that the modified Riccati does provide a good and computationally very efficient approximation for the ENUM bound.

The organization of the reminder of this note is as follows: First, we provide a little bit of a recapitulation of the MRE and its relevance to the problem at hand. Thereafter, we present the main results of this note. Lastly, we provide extensive simulations illustrating the main result and also showing its limitations. We end with a thorough discussion on the interpretation and implications of the findings in this note and of course the conclusions.

II. THE MODIFIED RICCATI EQUATION REVISITED

In this section, we briefly revisit some properties and recent results on the MRE. The majority of the results can be found in more detail in [9] and [10].

Consider the following discrete time linear Gaussian state space model

$$s_{k+1} = Fs_k + w_k, \qquad k \in \mathbb{N} \tag{1}$$

$$z_k = Hs_k + v_k, \qquad k \in \mathbb{N} \tag{2}$$

where $s_k \in \mathbb{R}^n$ is the state of the system and $z_k \in \mathbb{R}^p$ are the measurements. F and H are matrices of appropriate dimensions. It is assumed that the process noise is distributed according to $w_k \sim \mathcal{N}(0,Q)$ and the measurement noise according to $v_k \sim \mathcal{N}(0,R)$. This model can be used in a target tracking context to describe a linear target motion and measurement model. Let us assume that we receive a measurement with a probability of detection p_d, where $0 \leq p_d \leq 1$. Let us now examine the Kalman filter. The one step ahead prediction for the performance of a Kalman filter for a system with a $0 \leq p_d \leq 1$ is given by

$$P_{k|k} = (I - p_d K_k H)P_{k|k-1} \tag{3}$$

where K_k is the Kalman gain. We can now formulate Kalman-like recursions for a general system with $0 \leq p_d \leq 1$. These recursions are given by

$$P_{k+1|k} = FP_{k|k}F^T + Q \tag{4}$$

$$K_k = P_{k|k-1}H^T(HP_{k|k-1}H^T + R)^{-1} \tag{5}$$

$$P_{k|k} = (I - p_d K_k H)P_{k|k-1}. \tag{6}$$

Note that it is obvious that for $p_d = 1$ the standard Kalman filter equations are obtained; see e.g. [4] and [5]. If we apply the recursions (4), (5), and (6) and perform the proper substitutions, we arrive at

$$P_{k+1|k} = FP_{k|k-1}F^T - p_d FP_{k|k-1}H^T(HP_{k|k-1}H^T + R)^{-1} \times HP_{k|k-1}F^T + Q \tag{7}$$

with $P_{0|-1} \equiv P_0 = P_0^T > 0$.

For $p_d = 1$, (7) reduces to the standard Riccati difference equation associated with the Kalman filter. However, for $p_d < 1$ it may be verified that the recursion (7) is not a Riccati equation, that is, it cannot be transformed by smart manipulations in an equation that is of the Riccati form. Equation (7) is generally referred to as the modified Riccati difference equation. As in the case of the standard difference Riccati equation, the modified Riccati difference equation also has a steady state algebraic equivalent, obtained for $k \to \infty$ and given by

$$P = FPF^T - p_d FPH^T(HPH^T + R)^{-1}HPF^T + Q. \tag{8}$$

Again, for $p_d = 1$, (8) reduces to the standard discrete time algebraic Riccati equation; see also [5]. Several theoretical and practical results regarding these modified Riccati equations and their meanings and uses have been recently derived and proven; see [9] and [10]. One result is of special use to us and has been provided in [9] and is restated below.

THEOREM 1 *Consider the system given by* (1) *and* (2), *with initial condition* $P_{0|-1} \equiv P_0 = P_0^T > 0$. *Assume that* $(F, Q^{1/2})$ *is controllable and that* (F, H) *is observable.*[1] *Let the* $P_{k|k}^{\mathrm{MR}}$ *be calculated using the MRE recursions, see* (4), (5), *and* (6) *and the covariances* $P_{k|k}^{\mathrm{IRF}}$ *and* $P_{k|k}^{\mathrm{ENUM}}$ *according to the methods in* [3]. *The following relations hold:*

$$P_{k|k}^{\mathrm{IRF}} \le P_{k|k}^{\mathrm{ENUM}} \le P_{k|k}^{\mathrm{MR}}. \tag{9}$$

Theorem 1 is also helpful in proving the result in the next section.

III. RESULTS—THEORY

In this section we show that the conjecture from [3] is true for the case of deterministic target motion or equivalently zero process noise. The conjecture that was formulated in the text of the discussion section, i.e., [3, sect. IVC], can be formally formulated for linear systems. We formulate the conjecture as a theorem for the special case of zero process noise and will of course prove it.

THEOREM 2 *Consider the linear time-invariant system defined by* (1) *and* (2). *Let* $0 < p_d \le 1$. *Let* (F, H) *be observable. Let the covariance of the process noise satisfy:* $Q = 0$. *Then it holds that:*

$$\lim_{k\to\infty} P_{k|k}^{\mathrm{IRF}} = \lim_{k\to\infty} P_{k|k}^{\mathrm{ENUM}} = 0. \tag{10}$$

PROOF The IRF bound for the case of zero process noise is defined by the Riccati difference equation

$$P_{k+1|k}^{\mathrm{IRF}} = FP_{k|k-1}^{\mathrm{IRF}}F^T - FP_{k|k-1}^{\mathrm{IRF}}H^T \times \left(HP_{k|k-1}^{\mathrm{IRF}}H^T + \frac{1}{p_d}R\right)^{-1} HP_{k|k-1}^{\mathrm{IRF}}F^T \tag{11}$$

with $P_{0|-1}^{\mathrm{IRF}} \equiv P_0 = P_0^T > 0$.

The modified Riccati equation with zero process noise reads as

$$P_{k+1|k}^{\mathrm{MR}} = FP_{k|k-1}^{\mathrm{MR}}F^T - p_d FP_{k|k-1}^{\mathrm{MR}}H^T(HP_{k|k-1}^{\mathrm{MR}}H^T + R)^{-1} \times HP_{k|k-1}^{\mathrm{MR}}F^T \tag{12}$$

with $P_{0|-1}^{\mathrm{MR}} \equiv P_0 = P_0^T > 0$.

Note that a Riccati difference equation under mild assumptions satisfies only one unique positive limiting solution, see e.g. [5]. Equation (11) constitutes a standard Riccati equation and the solution $P_{\infty|\infty}^{\mathrm{IRF}} = 0$ satisfies the steady version of (11). Thus, the zero solution necessarily must be the unique positive limiting solution.

Due to the results from [9], in particular theorem I therein, the same holds for the limiting solution of the MRE (12).

Now using inequality (9) from Theorem 1, i.e., using the fact that the ENUM bound is squeezed in between the IRF and the MR solution, completes the proof.

REMARK 1 The proof of the theorem could also have been obtained without resorting to the MR argument, by using the argument that the IRF bound is exact in the limit, see [9], tending to zero, see (11), and combining this with the observation that the ENUM bound is exact, see [3].

Thus, the above theorem confirms the conjecture from [3], at least for the zero process noise case.

In the next section it is shown that the conjecture from [3] cannot be generalized to the case of non-zero process noise, or, especially in a target tracking example, nondeterministic target motion.

IV. RESULTS—SIMULATIONS

In this section, we illustrate the theorem from Section III. We also provide a counter example for the general conjecture, as it was formulated in [3]. Lastly we show that for nondeterministic target motion the difference between the steady state values of the IRF and ENUM bounds also grows as the detection probability decreases. Before continuing, we explain how the different bounds and approximations, used in this section have been calculated.

1) The IRF bound is calculated using the recursion in (11).

2) The ENUM bound is calculated by evaluating the Kalman filter prediction covariances for all possible miss-detect sequences and taking the normalized sum over these covariances.

3) The MRE approximation is calculated using the recursion in (7).

Consider the following linear system, which represents one-dimensional target motion. $s_k = [x_k \ \ \dot{x}_k]^T \in \mathbb{R}^2$,

$$F = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 \\ \frac{1}{2}T^3 & T^2 \end{pmatrix} \sigma_w^2$$

$$R = \sigma_v^2 \qquad \text{and} \qquad H = [1 \ \ 0].$$

For this system, the value of the measurement noise parameter is set to $\sigma_v = 5$.

In Table I "almost" steady state values for the position parts of the different accuracies, that are the squared roots of position parts of the covariances $P_{k|k}^{\mathrm{IRF}}$ and $P_{k|k}^{\mathrm{MR}}$, are shown for different values of the process noise. The probability of detection

[1]See e.g. [5] for the definitions of controllability and observability.

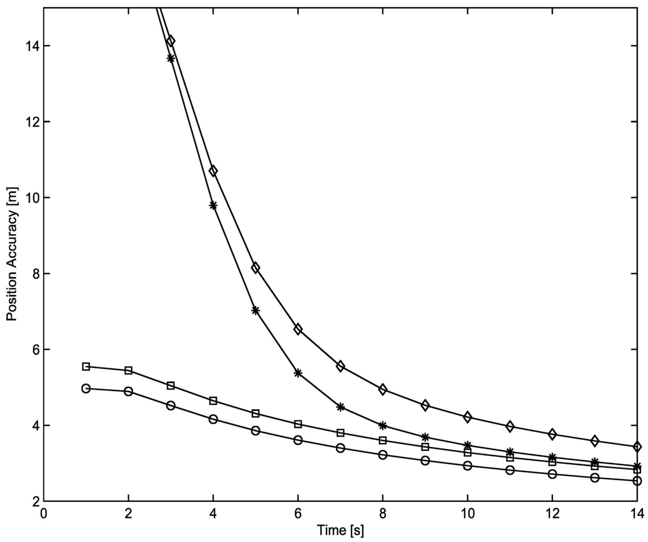

Fig. 1. Different bounds for process noise level $\sigma_w = 0$ and $p_d = 0.8$. $*$ ENUM CRLB, $\diamond$ MR upper bound, $\square$ IR lower bound, $\circ$ Kalman filter with $p_d = 1$.

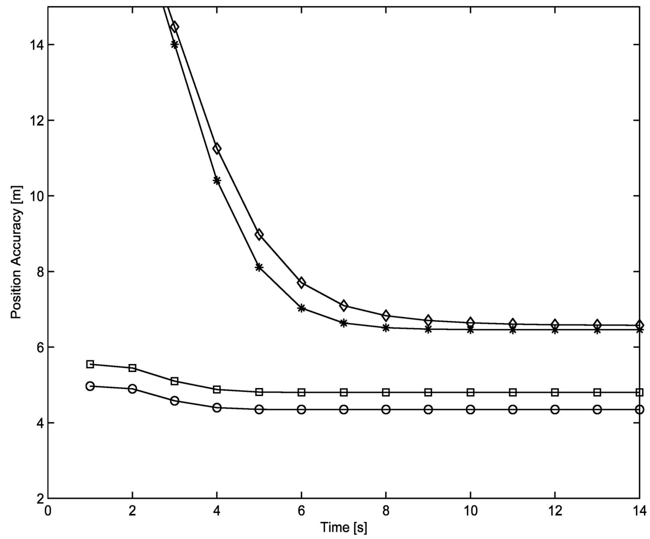

Fig. 2. Different bounds for process noise level $\sigma_w = 5$ and $p_d = 0.8$. $*$ ENUM CRLB, $\diamond$ MR upper bound, $\square$ IR lower bound, $\circ$ Kalman filter with $p_d = 1$.

TABLE I
Almost Steady State Values
$k = 10000/k = 50000$

	MR	IRF
$\sigma_w = 5$	6.5746	4.8003
$\sigma_w = 1$	4.0891	3.7507
$\sigma_w = 0$	0.1119/0.0500	0.1118/0.0500

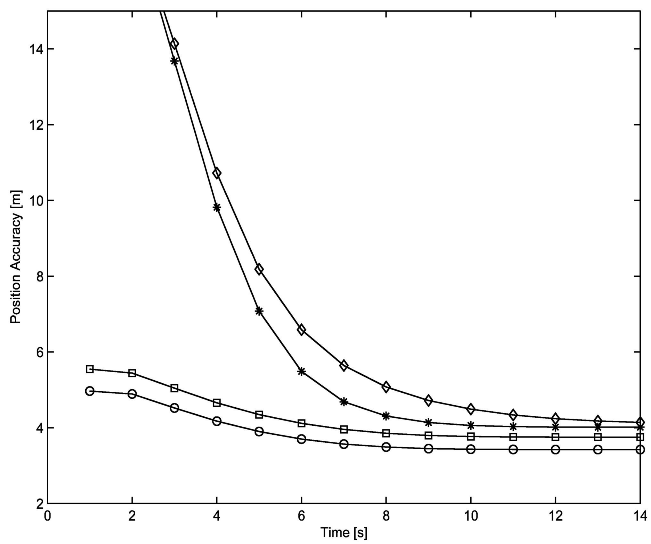

Fig. 3. Different bounds for process noise level $\sigma_w = 1$ and $p_d = 0.8$. $*$ ENUM CRLB, $\diamond$ MR upper bound, $\square$ IR lower bound, $\circ$ Kalman filter with $p_d = 1$.

used is 0.8. The steady state values are shown only for the IRF and MRE solution, as it is not feasible to calculate the ENUM bound for values of $k > 20$. The computational costs to calculate the covariance of the ENUM bound are $k \times 2^k$ a Kalman filter step and thus grow exponentially over time; see also [9]. From Table I it can be seen that the covariances of the MR solution and the IRF solution approach zero from above for the zero process noise case. This is consistent with Theorem 2. In Fig. 1, the different bounds are shown up to time step $k = 14$. It is clear that at this time step the different bounds have not reached their steady state value. Furthermore, this figure nicely illustrates the inequality relationships from Theorem 1. Note that those relationships hold regardless of the value of the process noise.

For the case of non-zero process noise, the results are shown in Table I and in the Figs. 2 and 3. It can be seen that for these non-zero values of process noise steady state is reached much faster. This is a well-known property of the standard Kalman filter as well, see [5]. Even more important is that it is quite clear, especially from Fig. 2, that for non-zero process noise the steady state values for the IRF bound and the ENUM bound are definitely not the same. In fact the steady state values for the ENUM bound are strictly greater than the steady state bound for the IRF and there is a considerable distance between them. This distance grows when the process noise levels would be further increased. Furthermore, for a constant value of process noise, but a decreasing value of the detection probability, the difference between the steady state values of the IRF and the ENUM bounds grows. This can be seen by comparing Figs. 2 and 4. Moreover, it shows that the increase in the difference is really considerable even for a relatively small decrease of the value of the probability of detection.

For the nonlinear example on a reentering ballistic object (see [3]) the different bounds are also shown; see Fig. 5. This example has deterministic target

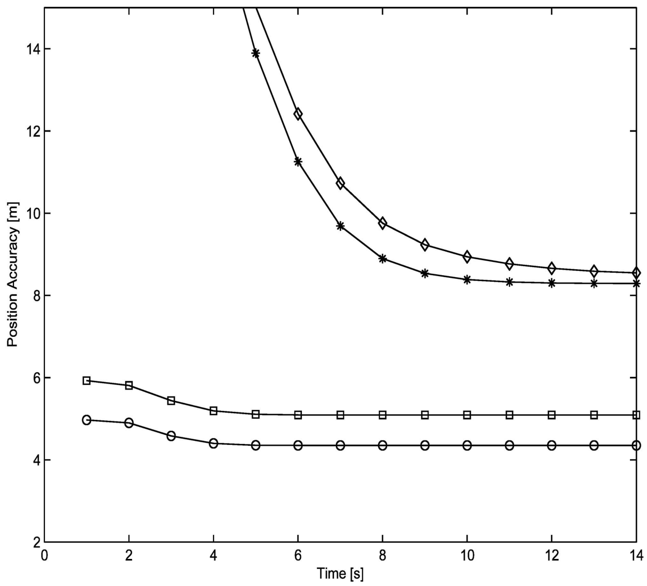

Fig. 4. Different bounds for process noise level $\sigma_w = 5$ and $p_d = 0.7$. $*$ ENUM CRLB, $\diamond$ MR upper bound, $\square$ IR lower bound, $\circ$ Kalman filter with $p_d = 1$.

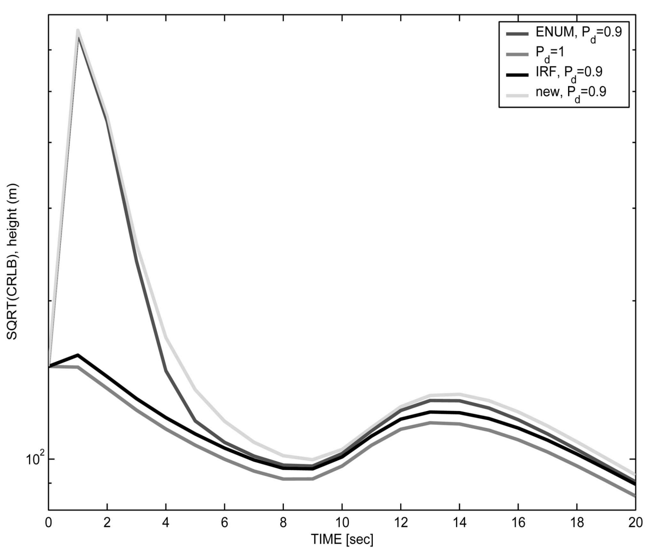

Fig. 5. Different bounds for ballistic example from [3]. From top down: MR upper bound, ENUM CRLB, IR lower bound, extended Kalman filter with $p_d = 1$.

motion, but is nonlinear. It can be seen from Fig. 5 that all the bounds seem to approach one another. It can also be seen that the inequality relationships from Theorem 1 also hold for this nonlinear example. For this example, the MRE solution is simply calculated by applying the MR recursions to the linearized system.

Moreover, although we have focussed on the steady state behavior of the different bounds, it is also important to mention that in the transient phase the MR solution is a much more realistic approximation of the CRLB, i.e., the ENUM bound, than the IRF bound. This observation has already been made in [9] and can be seen in the above figures as well.

V. DISCUSSION

The results of this note were presented in previous sections. Although we have also presented a nonlinear example, we have only proven results for a particular class of linear systems. In the face of the nonlinear example, it might be worthwhile to research for which possible class of nonlinear systems the results could be extended. Additionally, the results for non-zero process noise indicate that even at steady state, or at least for a relatively large time horizon, the IRF and the ENUM bounds do not approach one another and are far apart. This implies that one cannot simply use the IRF bound to approximate the CRLB for analysis or scheduling purposes Moreover, it was shown that in the case of non-zero process noise the IRF is grossly overoptimistic even for a larger time horizon. Ergo, for steady state analysis or scheduling many steps ahead, the IRF bound is not useful, once more, if the process noise is non-zero. We recall that the complexity of calculating the IRF bound does not grow over time and equals the complexity of a Kalman filter, in fact k times a Kalman filter step to predict the performance for a horizon k. However, as argued before, the IRF bound, although efficient, is inappropriate for systems with considerable process noise. The alternative ENUM bound is accurate and in the linear Gaussian case it is exact. However, the problem with this bound is that its computational complexity grows exponentially over time. In fact the cost is $k \times 2^k$ the cost of a Kalman filter step. For larger time horizons calculation of the ENUM bound is not feasible anymore, without making additional approximations; see e.g. [6]. This fact becomes even more important in on-line applications, such as sensor scheduling.

It has already been shown in [9], that the MR approximation is a very good alternative for the IRF and the ENUM bound. Moreover, the complexity of the calculation of this bound is equal to that of the IRF bound. The numerical value of the bound is very close and always greater than the ENUM bound, which is exact for the linear case. Also, numerical results show that the MRE approximation is very close to the ENUM bound for a wide range of process noise and detection probability values. Thus, the MR approximation is efficient to calculate, often close to the ENUM bound and also very important, never too optimistic. The results in this note, especially for larger time horizons provide additional motivation for the use of the MR approximation as a good alternative for the IRF and ENUM bound.

An interesting open question is whether it is possible to formulate and prove theoretical results on the distance between the ENUM bound and the MR approximation.

VI. CONCLUSIONS

In this note, we have proven a conjecture that was originally formulated in [3]. This conjecture stated that two approximations, i.e., the ENUM bound and the IRF bound for the CRLB for dynamical systems with probability of detection smaller than one approach one another in the limit, despite the fact that their transitional behavior is very different.

We have shown the conjecture to be true for the special case of zero process noise or equivalently deterministic target motion. We have also shown, by means of numerical counter examples that the conjecture does not hold for the general case of nondeterministic target motion. We have used recent results on the so-called MRE, that constitutes a numerically tractable approximation for the ENUM bound.

Finally, we have illustrated the value of the upper bound by means of the MR approach. We have shown for the case of a time-invariant linear system with possibly missing measurements and no false alarms, that the MRE approximation can be quite close to the computationally expensive ENUM bound.

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