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Path–kipas Ramsey numbers

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Abstract

For two given graphs F and H, the Ramsey number $R(F, H)$ is the smallest positive integer p such that for every graph G on p vertices the following holds: either G contains F as a subgraph or the complement of G contains H as a subgraph. In this paper, we study the Ramsey numbers $R(P_n, \hat{K}_m)$, where P_n is a path on n vertices and $\hat{K}_m$ is the graph obtained from the join of K_1 and P_m. We determine the exact values of $R(P_n, \hat{K}_m)$ for the following values of n and m: $1 \leqslant n \leqslant 5$ and $m \geqslant 3$; $n \geqslant 6$ and (m is odd, $3 \leqslant m \leqslant 2n-1$) or ($m$ is even, $4 \leqslant m \leqslant n+1$); $6 \leqslant n \leqslant 7$ and $m = 2n-2$ or $m \geqslant 2n$; $n \geqslant 8$ and $m = 2n-2$ or $m = 2n$ or $(q \cdot n - 2q + 1 \leqslant m \leqslant q \cdot n - q + 2$ with $3 \leqslant q \leqslant n-5)$ or $m \geqslant (n-3)^2$; odd $n \geqslant 9$ and $(q \cdot n - 3q + 1 \leqslant m \leqslant q \cdot n - 2q$ with $3 \leqslant q \leqslant (n-3)/2)$ or $(q \cdot n - q - n + 4 \leqslant m \leqslant q \cdot n - 2q$ with $(n-1)/2 \leqslant q \leqslant n-4)$. Moreover, we give lower bounds and upper bounds for $R(P_n, \hat{K}_m)$ for the other values of m and n.

MSC: 05C55; 05D10

Keywords: Kipas; Path; Ramsey number

1. Introduction

Throughout this paper, all graphs are finite and simple. Let G be such a graph. We write $V(G)$ or V for the vertex set of G and $E(G)$ or E for the edge set of G. The graph $\overline{G}$ is the *complement* of G, i.e., the graph obtained from the complete graph on $|V(G)|$ vertices by deleting the edges of G. The graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ (implying that the edges of H have all their end vertices in V').

If $e = \{u, v\} \in E$ (in short, $e = uv$), then u is called *adjacent* to v, and u and v are called *neighbors*. For $x \in V$, define $N(x) = \{y \in V | xy \in E\}$ and $N[x] = N(x) \cup \{x\}$. If $S \subset V(G)$, $S \neq V(G)$, then $G - S$ denotes the subgraph of G induced by $V(G) \backslash S$. If $e \in E(G)$, then $G - e = (V(G), E(G) \backslash \{e\})$.

We denote by P_n, C_n, and K_n the *path*, the *cycle* and the *complete graph* on n vertices, respectively. A *wheel* W_m with $m \geqslant 3$ is the graph on $m+1$ vertices obtained from a cycle on m vertices by adding a new vertex and edges joining it to all the vertices of the cycle (W_m is the join of K_1 and C_m). A *kipas* $\hat{K}_m$ with $m \geqslant 3$ is the graph on $m+1$ vertices obtained from the join of K_1 and P_m. A *fan* F_m with $m \geqslant 2$ is a graph on $2m+1$ vertices obtained from m disjoint

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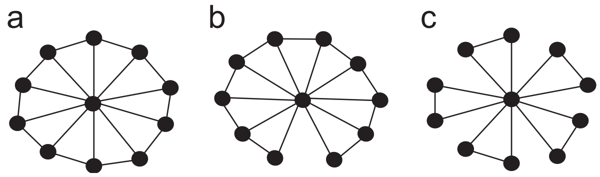


Fig. 1. (a) The wheel W_{10}. (b) The kipas $\hat{K}_{10}$. (c) The fan F_5.

triangles (K_3s) by identifying precisely one vertex of every triangle (F_m is the join of K_1 and mK_2). It is also known in the literature as 'dutch windmill'. Note that some authors use the term fan for graphs we defined as kipases. For illustration, consider W_{10} in Fig. 1(a), $\hat{K}_{10}$ in Fig. 1(b), and F_5 in Fig. 1(c). The vertex corresponding to K_1 in a wheel or in a kipas or in a fan is called the *hub* of the wheel or the *hub* of the kipas or the *hub* of the fan, respectively.

Given two graphs F and H, the *Ramsey number* $R(F, H)$ is defined as the smallest positive integer p such that every graph G on p vertices satisfies the following condition: G contains F as a subgraph or $\overline{G}$ contains H as a subgraph.

In 1967 Geréncser and Gyárfás [4] determined all Ramsey numbers for paths versus paths. After that, Ramsey numbers $R(P_n, H)$ for paths versus other graphs H have been investigated in several papers, for example: Parsons [6] when H is a complete graph; Faudree et al. [2] when H is a cycle; Parsons [7] when H is a star; Burr et al. [1] when H is a sparse graph; Häggkvist [5] when H is a complete bipartite graph; Faudree Schelp and Simonovits [3] when H is a tree; Salman and Broersma when H is a fan [9]; Surahmat and Baskoro [10], Salman and Broersma [8] when H is a wheel. We study Ramsey numbers for paths versus kipases.

Clearly, $R(P_n, \hat{K}_{2m}) \geqslant R(P_n, F_m)$, since F_m is a spanning subgraph of $\hat{K}_{2m}$. In this paper we show that $R(P_n, \hat{K}_{2m}) = R(P_n, F_m)$ for the Ramsey numbers $R(P_n, F_m)$ that are determined in [9]. Since $\hat{K}_m$ is a spanning subgraph of W_m, it is obvious that $R(P_n, \hat{K}_m) \leqslant R(P_n, W_m)$. In this paper we also show that $R(P_n, \hat{K}_m) = R(P_n, W_m)$ for the Ramsey numbers $R(P_n, W_m)$ that are determined in [9]. Moreover, we determine $R(P_n, \hat{K}_m)$ for some other values of m and n, namely for the following values of m and n: $n = 4$ or $n = 6$ and $m = 2n - 2$ or $m \geqslant 2n$; n is even, $n \geqslant 8$ and $m = 2n - 2$ or $m = 2n$ or $m \geqslant (n - 3)^2$ or $q \cdot n - 2q + 1 \leqslant m \leqslant q \cdot n - q + 2$ with $3 \leqslant q \leqslant n - 5$; $n = 7$ and $m = 15$; n is odd, $n \geqslant 9$ and $q \cdot n - 3q + 1 \leqslant m \leqslant q \cdot n - 2q$ with $3 \leqslant q \leqslant (n - 3)/2$ or $q \cdot n - q - n + 4 \leqslant m \leqslant q \cdot n - 2q$ with $(n - 1)/2 \leqslant q \leqslant n - 4)$.

2. Main results

In this paper we determine the Ramsey numbers $R(P_n, \hat{K}_m)$ for the following values of n and m: $1 \leqslant n \leqslant 5$ and $m \geqslant 3$; $n \geqslant 6$ and (m is odd, $3 \leqslant m \leqslant 2n - 1$) or ($m$ is even, $4 \leqslant m \leqslant n + 1$); $6 \leqslant n \leqslant 7$ and $m = 2n - 2$ or $m \geqslant 2n$; $n \geqslant 8$ and $m = 2n - 2$ or $m = 2n$ or ($q \cdot n - 2q + 1 \leqslant m \leqslant q \cdot n - q + 2$ with $3 \leqslant q \leqslant n - 5$) or $m \geqslant (n - 3)^2$; odd $n \geqslant 9$ and ($q \cdot n - 3q + 1 \leqslant m \leqslant q \cdot n - 2q$ with $3 \leqslant q \leqslant (n - 3)/2$) or ($q \cdot n - q - n + 4 \leqslant m \leqslant q \cdot n - 2q$ with $(n - 1)/2 \leqslant q \leqslant n - 4$). The Ramsey numbers for 'small' paths versus kipases or paths versus 'small' kipases will be given in Corollary 2. The Ramsey numbers for paths versus 'large' kipases will be given in Corollaries 5 and 7. Moreover, we also give nontrivial lower bounds and upper bounds for $R(P_n, \hat{K}_m)$ for (odd $n \geqslant 11$ and $q \cdot n - q + 3 \leqslant m \leqslant q \cdot n - 3q + n - 3$ with $2 \leqslant q \leqslant (n - 7)/2$) or (even $n \geqslant 8$ and $q \cdot n - q + 3 \leqslant m \leqslant q \cdot n - 2q + n - 2$ with $2 \leqslant q \leqslant n - 5$) or ($n \geqslant 6$ and m is even, $n + 2 \leqslant m \leqslant 2n - 4$) in Corollaries 8 and 9 and Theorem 10.

In [9] we have determined the Ramsey numbers for paths versus wheels for the values of m and n that are presented in Theorem 1. This theorem provides upper bounds that yield several exact Ramsey numbers for paths versus kipases.

Theorem 1.

$$R(P_n, W_m) = \begin{cases} 1 & \textit{for } n = 1 \textit{ and } m \geqslant 3, \\ m + 1 & \textit{for either } (n = 2 \textit{ and } m \geqslant 3) \textit{ or } (n = 3 \textit{ and even } m \geqslant 4), \\ m + 2 & \textit{for } (n = 3 \textit{ and odd } m \geqslant 5), \\ 3n - 2 & \textit{for either } (n = 3 \textit{ and } m = 3) \textit{ or } (n \geqslant 4 \textit{ and } m \textit{ is odd}, 3 \leqslant m \leqslant 2n - 1), \\ 2n - 1 & \textit{for } n \geqslant 4 \textit{ and } m \textit{ is even}, \ 4 \leqslant m \leqslant n + 1. \end{cases}$$

Corollary 2.

$$R(P_n,\hat{K}_m)=\begin{cases}1 & \textit{for } n=1 \textit{ and } m\geqslant 3,\\ m+1 & \textit{for either } (n=2 \textit{ and } m\geqslant 3) \textit{ or } (n=3 \textit{ and even } m\geqslant 4),\\ m+2 & \textit{for } (n=3 \textit{ and odd } m\geqslant 5),\\ 3n-2 & \textit{for either } (n=3 \textit{ and } m=3) \textit{ or } (n\geqslant 4 \textit{ and } m \textit{ is odd}, 3\leqslant m\leqslant 2n-1),\\ 2n-1 & \textit{for } n\geqslant 4 \textit{ and } m \textit{ is even}, \ 4\leqslant m\leqslant n+1.\end{cases}$$

Proof. The graphs

$$\begin{cases}P_1 & \text{for } n=1 \text{ and } m\geqslant 3,\\ mP_1 & \text{for } n=2 \text{ and } m\geqslant 3,\\ \left\lfloor\frac{m+1}{2}\right\rfloor K_2 & \text{for } n=3 \text{ and } m\geqslant 4,\\ 3K_{n-1} & \text{for } (n=3 \text{ and } m=3) \text{ or } (n\geqslant 4 \text{ and } m \text{ is odd}, \ 3\leqslant m\leqslant 2n-1),\\ 2K_{n-1} & \text{for } n\geqslant 4 \text{ and } m \text{ is even}, \ 4\leqslant m\leqslant n+1\end{cases}$$

give lower bounds for $R(P_n,\hat{K}_m)$ for the values of m and n in Corollary 2. Since $\hat{K}_m$ is a subgraph of W_m, Theorem 1 completes the proof. □

The next lemma plays a key role in our proofs of Lemmas 4 and 6. The proof of this lemma has been given in [8].

Lemma 3. *Let $n\geqslant 3$ and G be a graph on at least n vertices containing no P_n. Let the paths $P^1, P^2, \ldots, P^k$ in G be chosen in the following way: $\bigcup_{j=1}^{k} V(P^j)=V(G)$, P^1 is a longest path in G, and, if $k>1$, P^{i+1} is a longest path in $G-\bigcup_{j=1}^{i} V(P^j)$ for $1\leqslant i\leqslant k-1$. Denote by ℓ_j the number of vertices on the path P^j. Let z be an end vertex of P^k. Then*:

(i) $\ell_1\geqslant \ell_2\geqslant \cdots \geqslant \ell_k$;
(ii) *If $\ell_k\geqslant \lfloor n/2\rfloor$, then $N(z)\subset V(P^k)$*;
(iii) *If $\ell_k<\lfloor n/2\rfloor$, then $|N(z)|\leqslant \lfloor n/2\rfloor-1$.*

The following lemma provides upper bounds that yield several exact Ramsey numbers in the sequel.

Lemma 4. *If $n\geqslant 4$ and $m\geqslant 2n-2$, then*

$$R(P_n,\hat{K}_m)\leqslant\begin{cases}m+n-1 & \textit{for } m=1 \bmod (n-1),\\ m+n-2 & \textit{for other values of } m.\end{cases}$$

Proof. Let G be a graph that contains no P_n and has order

$$|V(G)|=\begin{cases}m+n-1 & \text{for } m=1 \ mod(n-1),\\ m+n-2 & \text{for other values of } m.\end{cases} \tag{1}$$

Choose the paths $P^1,\ldots,P^k$ and the vertex z in G as in Lemma 3. Because of (1), not all P^i can have $n-1$ vertices, so $\ell_k\leqslant n-2$. If $\ell_k<\lfloor n/2\rfloor$ then by Lemma 3(iii) we obtain $|N(z)|\leqslant\lfloor n/2\rfloor-1\leqslant n-3$. If $\lfloor n/2\rfloor\leqslant \ell_k\leqslant n-2$ then by Lemma 3(ii) we obtain $|N(z)|\leqslant \ell_k-1\leqslant n-3$. Hence, $|N[z]|\leqslant n-2$. We will use the following result that has been proved in [2]: $R(P_t,C_s)=s+\lfloor t/2\rfloor-1$ for $s\geqslant\lfloor(3t+1)/2\rfloor$. We distinguish the following cases.

Case 1: $|N(z)|\leqslant\lfloor n/2\rfloor-2$ or n is odd and $|N(z)|=\lfloor n/2\rfloor-1$. Since $|V(G)\backslash N[z]|\geqslant m+\lfloor n/2\rfloor-1$, we find that $\overline{G-N[z]}$ contains a C_m. So, there is a $\hat{K}_m$ in $\overline{G}$ with z as a hub.

Case 2: n is even and $|N(z)|=n/2-1$. Since $|V(G)\backslash N[z]|\geqslant(m+n-2)-n/2=m+n/2-2$, we find that $\overline{G-N[z]}$ contains a C_{m-1}; denote its vertices by $v_1,v_2,v_3,\ldots,v_{m-1}$ in the order of appearance on the cycle with a fixed orientation. There are $n/2-1$ vertices in $U=V(G)\backslash(V(C_{m-1})\cup N[z])$, say $u_1,u_2,\ldots,u_{n/2-1}$. If some vertex v_i $(i=1,\ldots,m-1)$ is no neighbor of some vertex u_j $(j=1,\ldots,n/2-1)$, w.l.o.g. assume $v_{m-1}u_1\notin E(G)$. Then $\overline{G}$

contains a $\hat{K}_m$ with z as a hub and its other vertices $v_1, v_2, v_3, \ldots, v_{m-2}, v_{m-1}, u_1$. Now let us assume each of the v_i is adjacent to all u_j in G. For every choice of a subset of $n/2$ vertices from $V(C_{m-1})$, there is a path on $n-1$ vertices in G alternating between the vertices of this subset and the vertices of U, starting and terminating in two arbitrary vertices from the subset. Since G contains no P_n, there are no edges $v_iv_j \in E(G)$ $(i, j \in \{1, \ldots, m-1\})$. This implies that $V(C_{m-1}) \cup \{z\}$ induces a K_m in $\overline{G}$. Since G contains no P_n, no v_i is adjacent to a vertex of $N(z)$. This implies that $\overline{G}$ contains a $K_{m+1} - zw$ for any vertex $w \in N(z)$, and hence $\overline{G}$ contains a $\hat{K}_m$ with one of the v_i as a hub.

Case 3: Suppose that there is no choice for P^k and z such that one of the former cases applies. Then $|N(w)| \geqslant \lfloor n/2 \rfloor$ for any end vertex w of a path on ℓ_k vertices in $G - \bigcup_{j=1}^{k-1} V(P^j)$. This implies all neighbors of such w are in $V(P^k)$ and $\ell_k \geqslant \lfloor n/2 \rfloor + 1$. So for the two end vertices z_1 and z_2 of P^k we have that $|N(z_i) \cap V(P^k)| \geqslant \lfloor n/2 \rfloor \geqslant \ell_k/2$. By standard arguments in Hamiltonian graph theory, we can find an index $i \in \{2, \ldots, \ell_k - 1\}$ such that z_1v_{i+1} and z_2v_i are edges of G. It is clear that we can find a cycle on ℓ_k vertices in G. This implies that any vertex of $V(P^k)$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^k)$ and the other vertices. This also implies that all vertices of P^k have degree at least m in $\overline{G}$.

We now turn to P^{k-1} and consider one of its end vertices w. Since $\ell_{k-1} \geqslant \ell_k \geqslant \lfloor n/2 \rfloor + 1$, similar arguments as in the proof of Lemma 3 show that all neighbors of w are on P^{k-1}. If $|N(w)| < \lfloor n/2 \rfloor$, we get a $\hat{K}_m$ in $\overline{G}$ as in Case 1 or Case 2. So we may assume $|N(w_i) \cap V(P^{k-1})| \geqslant \lfloor n/2 \rfloor \geqslant \ell_{k-1}/2$ for both end vertices w_1 and w_2 of P^{k-1}. By similar arguments as before we obtain a cycle on ℓ_{k-1} vertices in G. This implies that any vertex of $V(P^{k-1})$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^{k-1})$ and the other vertices. This also implies that all vertices of P^{k-1} have degree at least $m-1$ in $\overline{G}$. (Note that P^{k-1} can have $n-1$ vertices, whereas $\ell_k \leqslant n-2$.)

Repeating the above arguments for $P^{k-2}, \ldots, P^1$ we eventually conclude that all vertices of G have degree at least $m-1$ in $\overline{G}$. Now let $H = \overline{G} - V(P^k)$. Then all vertices in $V(H)$ have degree at least $m - 1 - \ell_k \geqslant m/2 + (n-1) - 1 - \ell_k \geqslant \frac{1}{2}(m + 2n - 4 - \ell_k - (n-2)) = \frac{1}{2}(m + n - 2 - \ell_k) = \frac{1}{2}(|V(H)| - 1)$. Hence, there exists a Hamilton path in H. Since $|V(H)| \geqslant m$ and z is a neighbor of all vertices in H (in $\overline{G}$), it is clear that $\overline{G}$contains a $\hat{K}_m$ with z as a hub. This completes the proof of Lemma 4. □

Corollary 5. *If* $(4 \leqslant n \leqslant 6$ *and* $m = 2n-2$ *or* $m \geqslant 2n)$ *or* $(n \geqslant 7$ *and* $m = 2n-2$ *or* $m = 2n$ *or* $m \geqslant (n-3)^2)$ *or* $(n \geqslant 8$ *and* $q \cdot n - 2q + 1 \leqslant m \leqslant q \cdot n - q + 2$ *with* $3 \leqslant q \leqslant n-5)$, *then*

$$R(P_n, \hat{K}_m) = \begin{cases} m + n - 1 & \text{for } m = 1 \bmod (n-1), \\ m + n - 2 & \text{for other values of } m. \end{cases}$$

Proof. Let r denote the remainder of m divided by $n-1$, so $m = p(n-1) + r$ for some $0 \leqslant r \leqslant n-2$. Then for $(4 \leqslant n \leqslant 6$ and $m = 2n-2$ or $m \geqslant 2n)$ or $(n \geqslant 7$ and $m = 2n-2$ or $m = 2n$ or $m \geqslant (n-3)^2)$ or $(n \geqslant 8$ and $q \cdot n - 2q + 1 \leqslant m \leqslant q \cdot n - q + 2$ with $3 \leqslant q \leqslant n-5)$, the graphs

$$\begin{cases} (p-1)K_{n-1} \cup 2K_{n-2} & \text{for } r = 0, \\ (p+1)K_{n-1} & \text{for } r = 1 \text{ or } 2, \\ (p + r + 1 - n)K_{n-1} \cup (n + 1 - r)K_{n-2} & \text{for other values of } r \end{cases}$$

show that

$$R(P_n, \hat{K}_m) > \begin{cases} m + n - 2 & \text{for } m = 1 \bmod (n-1), \\ m + n - 3 & \text{for other values of } m. \end{cases}$$

Lemma 4 completes the proof. □

Lemma 6. *If n is odd*, $n \geqslant 7$ *and* $q \cdot n - q + 3 \leqslant m \leqslant q \cdot n - 2q + n - 2$ *with* $2 \leqslant q \leqslant n - 5$, *then* $R(P_n, \hat{K}_m) \leqslant m + n - 3$.

Proof. The proof is modelled along the lines of the proof of Lemma 4. Let G be a graph on $m + n - 3$ vertices, and assume G contains no P_n. We will show that $\overline{G}$ contains a $\hat{K}_m$. Choose the paths $P^1, \ldots, P^k$ and the vertex z in G as in

Lemma 3. Since $|V(G)|=m+n-3$ with $n\geqslant 7$ and $q\cdot n-q+3\leqslant m\leqslant q\cdot n-2q+n-2$ with $2\leqslant q\leqslant n-5$, $k\geqslant q+2$, and therefore not all P^i can have more than $n-3$ vertices. So $\ell_k\leqslant n-3$. By similar arguments as in the proof of Lemma 4, this implies $|N(z)|\leqslant n-4$. We will use the following result that has been proved in [2]: $R(P_t, C_s)=s+\lfloor t/2\rfloor-1$ for $s\geqslant\lfloor(3t+1)/2\rfloor$. We distinguish the following cases.

Case 1: $|N(z)|\leqslant\lfloor n/2\rfloor-2$. Since $|V(G)\backslash N[z]|\geqslant m+\lfloor n/2\rfloor-1$, we find that $\overline{G-N[z]}$ contains a C_m. So, there is a $\hat{K}_m$ in $\overline{G}$ with z as a hub.

Case 2: $|N(z)|=\lfloor n/2\rfloor-1$. Since $|V(G)\backslash N[z]|=(m+n-3)-\lfloor n/2\rfloor=m+\lfloor n/2\rfloor-2$, we find that $\overline{G-N[z]}$ contains a C_{m-1}; denote its vertices by $v_1, v_2, v_3, \ldots, v_{m-1}$ in the order of appearance on the cycle with a fixed orientation. There are $\lfloor n/2\rfloor-1$ vertices in $U=V(G)\backslash(V(C_{m-1})\cup N[z])$, say $u_1, u_2, \ldots, u_{\lfloor n/2\rfloor-1}$. If some vertex v_i $(i=1,\ldots,m-1)$ is no neighbor of some vertex u_j $(j=1,\ldots,\lfloor n/2\rfloor-1)$, w.l.o.g. assume $v_{m-1}u_1\notin E(G)$. Then $\overline{G}$ contains a $\hat{K}_m$ with z as a hub and its other vertices $v_1, v_2, v_3, \ldots, v_{m-2}, v_{m-1}, u_1$. Now let us assume each of the v_i is adjacent to all u_j in G. For every choice of a subset of $\lfloor n/2\rfloor$ vertices from $V(C_{m-1})$, there is a path on $n-2$ vertices in G alternating between the vertices of this subset and the vertices of U, starting and terminating in two arbitrary vertices from the subset. Let $z_1\in N(z)$. Since G contains no P_n, there are no edges $v_iz\in E(G)$ and $v_iz_1\in E(G)$ $(i\in\{1,\ldots,m-1\})$ and there is at most one edge $v_iv_j\in E(G)$ (for some $i, j\in\{1,\ldots,m-1\}$). Assume (at most) $v_1v_2\in E(G)$. This implies $\overline{G}$ contains a $\hat{K}_m$ with hub v_{m-1} and its other vertices $v_1, z, v_2, z_1, v_3, \ldots, v_{m-4}, v_{m-3}, v_{m-2}$.

Case 3: Suppose that there is no choice for P^k and z such that one of the former cases applies. Then $|N(w)|\geqslant\lfloor n/2\rfloor$ for any end vertex w of a path on ℓ_k vertices in $G-\bigcup_{j=1}^{k-1}V(P^j)$. This implies all neighbors of such w are in $V(P^k)$ and $\ell_k\geqslant\lfloor n/2\rfloor+1$. So for the two end vertices z_1 and z_2 of P^k we have that $|N(z_i)\cap V(P^k)|\geqslant\lfloor n/2\rfloor\geqslant\ell_k/2$. By similar arguments as in the proof of Lemma 4 we obtain a cycle on ℓ_k vertices in G. This implies that any vertex of $V(P^k)$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^k)$ and the other vertices. This also implies that all vertices of P^k have degree at least m in $\overline{G}$.

We now turn to P^{k-1} and consider one of its end vertices w. Since $\ell_{k-1}\geqslant\ell_k\geqslant\lfloor n/2\rfloor+1$, similar arguments as in the proof of Lemma 3 show that all neighbors of w are on P^{k-1}. If $|N(w)|<\lfloor n/2\rfloor$, we get a $\hat{K}_m$ in $\overline{G}$ as in Case 1 or Case 2. So we may assume $|N(w_i)\cap V(P^{k-1})|\geqslant\lfloor n/2\rfloor\geqslant\ell_{k-1}/2$ for both end vertices w_1 and w_2 of P^{k-1}. By similar arguments as before we obtain a cycle on ℓ_{k-1} vertices in G. This implies that any vertex of $V(P^{k-1})$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^{k-1})$ and the other vertices. This also implies that all vertices of P^{k-1} have degree at least $m-2$ in $\overline{G}$. (Note that P^{k-1} can have $n-1$ vertices, whereas $\ell_k\leqslant n-3$.)

Repeating the above arguments for $P^{k-2}, \ldots, P^1$ we eventually conclude that all vertices of G have degree at least $m-2$ in $\overline{G}$. Now let $H=\overline{G}-V(P^k)$. Then all vertices in $V(H)$ have degree at least $m-2-\ell_k\geqslant m/2+n-2-\ell_k\geqslant\frac{1}{2}(m+2n-4-\ell_k-(n-3))=\frac{1}{2}(m+n-1-\ell_k)=\frac{1}{2}(|V(H)|+2)$. This implies there exists a Hamilton cycle in H. Since $|V(H)|\geqslant m$ and z is a neighbor of all vertices in H (in $\overline{G}$), it is clear that $\overline{G}$contains a $\hat{K}_m$ with z as a hub. This completes the proof of Lemma 6. □

Corollary 7. *If* $(n=7$ *and* $m=15)$ *or* $(n$ *is odd*, $n\geqslant 9$ *and* $(q\cdot n-3q+1\leqslant m\leqslant q\cdot n-2q$ *with* $3\leqslant q\leqslant(n-3)/2)$ *or* $(q\cdot n-q-n+4\leqslant m\leqslant q\cdot n-2q$ *with* $(n-1)/2\leqslant q\leqslant n-4))$, *then* $R(P_n, \hat{K}_m)=m+n-3$.

Proof. For $n=7$ and $m=15$, the graph $3K_6$ and for odd $n\geqslant 9$ and $m=q\cdot n-2q-j$ with either $(3\leqslant q\leqslant(n-3)/2$ and $0\leqslant j\leqslant q-1)$ or $((n-1)/2\leqslant q\leqslant n-5$ and $0\leqslant j\leqslant n-q-4)$, the graph $(q-j-1)K_{n-2}\cup(j+2)K_{n-3}$ shows that $R(P_n, \hat{K}_m)>m+n-4$. Lemma 6 completes the proof. □

Corollary 8. *If n is odd*, $n\geqslant 11$ *and* $q\cdot n-q+3\leqslant m\leqslant q\cdot n-3q+n-3$ *with* $2\leqslant q\leqslant(n-7)/2$, *then*

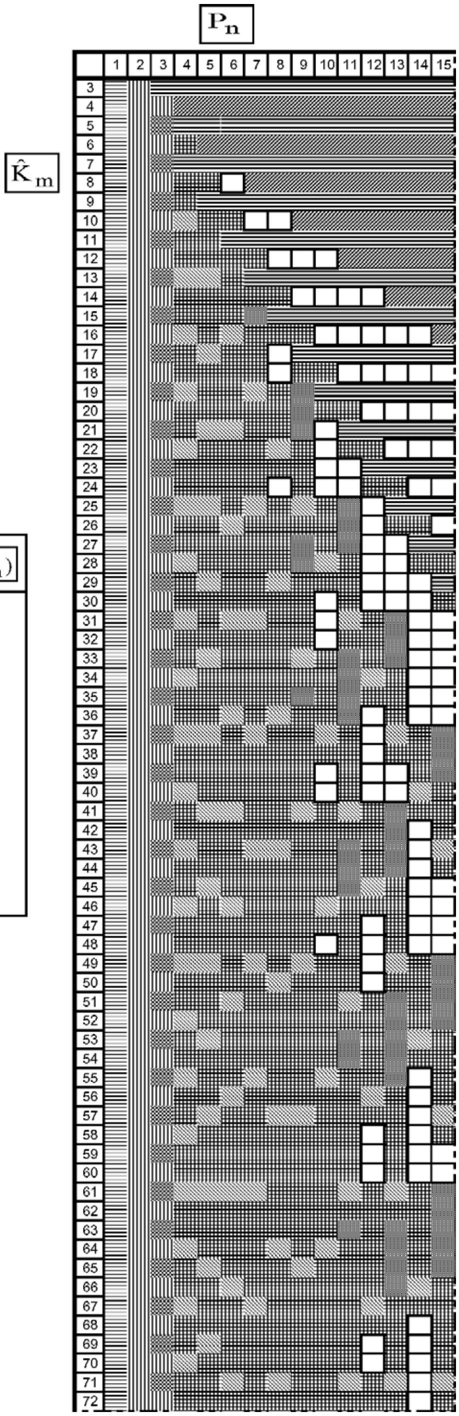
$$m+n-3\geqslant R(P_n, \hat{K}_m)\geqslant\max\left\{\left\lfloor\frac{m}{n-1}\right\rfloor(n-1)+n,\ m+\left\lfloor\frac{m-1}{\lceil m/(n-1)\rceil}\right\rfloor\right\}.$$

Proof. Let $t=\lceil m/(n-1)\rceil$ and s denote the remainder of $m-1$ divided by t. Then for m and n satisfying $\lfloor m/(n-1)\rfloor(n-1)+n\geqslant m+\lfloor(m-1)/t\rfloor$, the graph tK_{n-1} shows that $R(P_n, \hat{K}_m)>\lfloor m/(n-1)\rfloor(n-1)+n-1$.

For other values of m and n, the graph $sK_{\lceil(m-1)/t\rceil}\cup(t-s+1)K_{\lfloor(m-1)/t\rfloor}$ shows that $R(P_n, \hat{K}_m)>m-1+\lfloor(m-1)/\lceil m/(n-1)\rceil\rfloor$.

The upper bound comes from Lemma 6. □

Table 1
The Ramsey numbers for paths versus kipases



From	$R(P_n, \hat{K}_m)$
Corollary 2	1
Corollary 2	m + 1
Corollary 2	m + 2
Corollary 2	3n - 2
Corollary 2	2n - 1
Corollary 5	m + n - 2
Corollary 5	m + n - 1
Corollary 7	m + n - 3

Corollary 9. *If n is even,* $n \geqslant 8$ *and* $q \cdot n - q + 3 \leqslant m \leqslant q \cdot n - 2q + n - 2$ *with* $2 \leqslant q \leqslant n - 5$, *then*

$$m + n - 2 \geqslant R(P_n, \hat{K}_m) \geqslant \max\left\{\left\lfloor \frac{m}{n-1} \right\rfloor (n-1) + n,\ m + \left\lfloor \frac{m-1}{\lceil m/(n-1) \rceil} \right\rfloor\right\}.$$

Proof. Let $t = \lceil m/(n-1) \rceil$ and s denote the remainder of $m-1$ divided by t. Then for m and n satisfying $\lfloor m/(n-1) \rfloor (n-1) + n \geqslant m + \lfloor (m-1)/t \rfloor$, the graph tK_{n-1} shows that $R(P_n, \hat{K}_m) > \lfloor m/(n-1) \rfloor (n-1) + n - 1$.

For other values of m and n, the graph $sK_{\lceil (m-1)/t \rceil} \cup (t-s+1)K_{\lfloor (m-1)/t \rfloor}$ shows that $R(P_n, \hat{K}_m) > m - 1 + \lfloor (m-1)/\lceil m/(n-1) \rceil \rfloor$.

The upper bound comes from Lemma 4. □

Theorem 10. *If $n \geqslant 6$ and m is even with $n+2 \leqslant m \leqslant 2n-4$, then*

$$m + \left\lfloor \frac{3n}{2} \right\rfloor - 2 \geqslant R(P_n, \hat{K}_m) \geqslant \begin{cases} 2n-1 & \text{for } n+2 \leqslant m \leqslant n + \lfloor n/3 \rfloor, \\ \dfrac{3m}{2} - 1 & \text{for } n + \lfloor n/3 \rfloor < m \leqslant 2n-4. \end{cases}$$

Proof. For $n \geqslant 6$ and m is even with $n+2 \leqslant m \leqslant n + \lfloor n/3 \rfloor$, the graph $2K_{n-1}$ shows that $R(P_n, \hat{K}_m) > 2n-2$. For $n \geqslant 6$ and m is even, $n + \lfloor n/3 \rfloor < m \leqslant 2n-4$, the graph $K_{m/2} \cup 2K_{m/2-1}$ shows that $R(P_n, \hat{K}_m) > 3m/2 - 2$.

Let G be a graph on $m + \lfloor 3n/2 \rfloor - 2$ vertices, and assume G contains no P_n. Choose the paths $P^1, \ldots, P^k$ and the vertex z in G as in Lemma 3. By Lemma 3, $|N(z)| \leqslant n-2$. Hence, $|V(G) \backslash N[z]| \geqslant m + \lfloor n/2 \rfloor - 1$. We can apply the result from [2] that $R(P_n, C_m) = m + \lfloor n/2 \rfloor - 1$ for m is even and $2 \leqslant n \leqslant m$. This implies that $\overline{G - N[z]}$ contains a C_m. So, there is a $\hat{K}_m$ in $\overline{G}$ with z as a hub (there is even a wheel on $m+1$ vertices). □

3. Conclusion

In this paper we determined the Ramsey numbers for paths versus kipases of varying orders. The numbers are indicated in Table 1. We used different shadings to distinguish the results in the previous section that led to these numbers. The white elements indicate open cases. For these cases we established lower bounds and upper bounds for $R(P_n, \hat{K}_m)$.

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