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Note

Edge-cuts leaving components of order at least three

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $X \subseteq V(G)$ let $G[X]$ be the subgraph induced by X, $\bar{X} = V(G) - X$, and $(X,\bar{X})$ the set of edges in G with one end in X and the other in $\bar{X}$. If G is a connected graph and $S \subset E(G)$ such that $G - S$ is disconnected, and each component of $G - S$ consists of at least three vertices, then we speak of an *order*-3 *edge-cut*. The minimum cardinality $|S|$ over all order-3 edge-cuts in G is called the *order*-3 *edge-connectivity*, denoted by $\lambda_3 = \lambda_3(G)$. A connected graph G is λ_3-*connected*, if $\lambda_3(G)$ exists. An order-3 edge-cut S in G is called a λ_3-*cut*, if $|S| = \lambda_3$. First of all, we characterize the class of graphs which are not λ_3-connected. Then we show for λ_3-connected graphs G that $\lambda_3(G) \leqslant \xi_3(G)$, where $\xi_3(G)$ is defined by

$$\xi_3(G) = \min\{|(X,\bar{X})|: X \subset V(G),\ |X| = 3,\ G[X] \text{ is connected}\}.$$

A λ_3-connected graph G is called λ_3-*optimal*, if $\lambda_3(G) = \xi_3(G)$. If $(X,\bar{X})$ is a λ_3-cut, then $X \subset V(G)$ is called a λ_3-*fragment*. Let

$$r_3(G) = \min\{|X|: X \text{ is a } \lambda_3\text{-fragment of } G\}.$$

We prove that a λ_3-connected graph G is λ_3-optimal if and only if $r_3(G) = 3$. Finally, we study the λ_3-optimality of some graph classes. In particular, we show that the complete bipartite graph $K_{r,s}$ with $r,s \geqslant 2$ and $r + s \geqslant 6$ is λ_3-optimal.

Keywords: Edge-connectivity; Order-3 edge-connectivity; Atoms

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1. Introduction and terminology

The classical edge-connectivity $\lambda(G)$ of a graph G is the minimum cardinality $|S|$ of a set of edges such that $G-S$ is disconnected or K_1. Note that in this definition, absolutely no restrictions are imposed on the components of $G-S$. Thus, it would seem natural to generalize the notion of connectivity by introducing some conditions on the components of $G-S$. Harvey Greenberg (see [3]) asked if anyone had studied the minimum $|S|$ such that each component of $G-S$ has at least p vertices. This is exactly the question, we shall discuss in the presented paper. There is extensive literature for the case $p=2$, see for example [1,2,4,6–8]. In view of this, it is somewhat surprising that the closely-related question for $p\geqslant 3$ has, as yet, received no attention. In this paper we develop the first contributions to this problem, and we shall see that, in particular, the case $p=3$ leads to interesting results.

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. For $X\subseteq V(G)$ let $G[X]$ be the subgraph induced by X, $\bar{X}=V(G)-X$, and $(X,\bar{X})=(X,\bar{X})_G$ the set of edges in G with one end in X and the other in $\bar{X}$. By $m(A,B)=m_G(A,B)=|(A,B)|$, we denote the number of edges between two disjoint vertex sets A and B. For $A=\{x\}$, we write $m(x,B)$. If x is a vertex of a graph G, then $N(x)=N_G(x)$ denotes the set of vertices adjacent to x and $N[x]=N_G[x]=N(x)\cup\{x\}$. Furthermore, $N(X)=N_G(X)=\bigcup_{x\in X}N(x)$ and $N[X]=N_G[X]=N(X)\cup X$ for a subset X of $V(G)$. The vertex v is an end vertex if $d_G(v)=1$, and an isolated vertex if $d_G(v)=0$, where $d(x)=d_G(x)=|N(x)|$ is the degree of $x\in V(G)$. We denote by $\delta=\delta(G)$ the minimum degree, by $\Delta=\Delta(G)$ the maximum degree, and by $n=n(G)=|V(G)|$ the order of G. For two vertices x and y the distance $d(x,y)=d_G(x,y)$ between them is defined as the length of a shortest path from x to y. For a given vertex v, we call $e(v)=\max_{x\in V(G)}d(x,v)$ the excentricity of v. We write C_n for a cycle of length n, P_n for a path of order n, K_n for the complete graph of order n, and $K_{n,m}$ for the complete bipartite graph. A star is a complete bipartite graph $K_{1,m}$ with $m\geqslant 2$, and the unique vertex of degree m is its center.

If G is a connected graph and $S\subset E(G)$ such that $G-S$ is disconnected, and each component of $G-S$ consists of at least p vertices, then we speak of an *order-p edge-cut*. The minimum cardinality $|S|$ over all order-p edge-cuts in G is called the *order-p edge-connectivity*, denoted by $\lambda_p=\lambda_p(G)$. A connected graph G is *λ_p-connected*, if $\lambda_p(G)$ exists. Clearly, if G is a λ_p-connected graph for $p\geqslant 2$, then G is also λ_{p-1}-connected and $\lambda_{p-1}(G)\leqslant\lambda_p(G)$. Furthermore, $\lambda_1(G)=\lambda(G)$ and $\lambda_2(G)$ is the so-called *restricted edge-connectivity*, often denoted by $\lambda'(G)$. An order-p edge-cut S in G is called a *λ_p-cut*, if $|S|=\lambda_p$. Obviously, for any λ_p-cut S, the graph $G-S$ consists of exactly two components. If $(X,\bar{X})$ is a λ_p-cut, then $X\subset V(G)$ is called a *λ_p-fragment*. Clearly, if X is a λ_p-fragment, then $\bar{X}$ is also a λ_p-fragment. Let

$$r_p=r_p(G)=\min\{|X|:\ X \text{ is a } \lambda_p\text{-fragment of } G\}.$$

A λ_p-fragment X of G is called a *λ_p-atom* of G, if $|X|=r_p(G)$. Obviously, $p\leqslant r_p(G)\leqslant\frac{1}{2}|V(G)|$ and $G[X]$ as well as $G[\bar{X}]$ are connected, when X is a λ_p-atom. In addition, let

$$\xi_p = \xi_p(G) = \min\{|(X,\bar{X})|:\ X \subset V(G),\ |X| = p,\ G[X] \text{ is connected}\}.$$

In 1988, Esfahanian and Hakimi [2] have shown that for every connected graph G of order $n \geqslant 4$, except a star, $\lambda_2(G)$ exists and satisfies the inequality $\lambda_2(G) \leqslant \xi_2(G)$. Recently, Xu and Xu [8] have proved that a λ_2-connected graph fulfills $\lambda_2(G) = \xi_2(G)$ if and only if $r_2(G) = 2$. In the presented paper, we shall prove some analogous results for λ_3-connected graphs.

First of all, we characterize the family of λ_3-connected graphs, and then we prove for these graphs G the inequality $\lambda_3(G) \leqslant \xi_3(G)$. Examples will show that this inequality is no longer true for $p \geqslant 4$. Because of $\lambda_3(G) \leqslant \xi_3(G)$, the following definition is rich in meaning. A λ_3-connected graph G is called λ_3-*optimal*, if $\lambda_3(G) = \xi_3(G)$. Inspired by the above-mentioned result of Xu, we shall prove that a λ_3-connected graph G is λ_3-optimal if and only if $r_3(G) = 3$. Finally, we study the λ_3-optimality of some graph classes. In particular, we show that the complete bipartite graph $K_{r,s}$ with $r, s \geqslant 2$ and $r + s \geqslant 6$ is λ_3-optimal.

Lemma 1.1. *Let G be a λ_p-connected graph. If A is a subset of $V(G)$ such that $G[A]$ as well as $G[\bar{A}]$ contain a component with at least p vertices, then $|(A,\bar{A})| \geqslant \lambda_p(G)$.*

Proof. Firstly, assume that $G[A]$ is connected. If H is a component of $G[\bar{A}]$ with at least p vertices, then let $B = V(H)$. Since G is connected, we see that $G - V(H) = G[\bar{B}]$ is also connected with $A \subseteq \bar{B}$. Hence $(B,\bar{B})$ is an order-p edge-cut of G, and we conclude that $\lambda_p(G) \leqslant |(B,\bar{B})| \leqslant |(A,\bar{A})|$.

If $G[A]$ is not connected, then let $A' \subset A$ be a maximal subset such that $G[A']$ is connected. Since there is no edge from A' to $A - A'$, it follows from the first case that $|(A,\bar{A})| \geqslant |(A',\bar{A}')| \geqslant \lambda_p(G)$. □

2. Characterization of λ_3-connected graphs

Theorem 2.1. *Let G be a connected graph and $p \in \mathbf{N}$. The graph G is λ_p-connected if and only if there exist two disjoint sets $X = \{x_1, x_2, \ldots, x_p\} \subset V(G)$ and $Y = \{y_1, y_2, \ldots, y_p\} \subset V(G)$ such that $G[X]$ and $G[Y]$ are connected.*

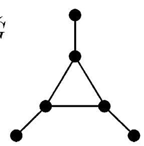
Proof. Firstly, let G be λ_p-connected and S a λ_p-cut of G. Then $G - S$ consists of two components with at least p vertices each. So the sets X and Y exist.

Now assume that G is a graph and $X, Y \subset V(G)$ are disjoint sets of cardinality p with $G[X]$ and $G[Y]$ connected. Let q be the number of edge-disjoint paths starting in X and ending in Y such that the inner vertices are contained in $\bar{X} \cap \bar{Y}$. If we remove a minimal set S of edges from these q paths such that $G - S$ is disconnected, then $G - S$ consists of two components. Clearly, one of them contains X and the other one Y. This implies that S is an order-p edge-cut and G is λ_p-connected. □

Theorem 2.2. *A connected graph of order $n \geqslant 6$ is not λ_3-connected if and only if either*

(a) 1. *G contains no cycles of length greater than* 3 *and*
2. *there exists exactly one $v_0 \in V(G)$ with degree greater than* 2, *and this v_0 has excentricity equal or less than* 2 *or*

(b) *G is isomorphic to the net $\tilde{G}$ pictured below.*



Proof. Clearly, $\tilde{G}$ is not λ_3-connected. If G is a graph satisfying (a), then every vertex set $X = \{x_1, x_2, x_3\}$ with $G[X]$ connected contains v_0. Thus, according to Theorem 2.1, G is not λ_3-connected.

Now let G be a connected graph of order $n \geqslant 6$ that is not isomorphic to $\tilde{G}$ and does not fulfill (a). We will apply Theorem 2.1 to conclude that G is λ_3-connected. We have to show that corresponding vertex sets X and Y exist.

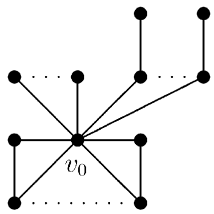
In the case that G contains a cycle C of length at least 6, this is trivial. If the cycle C consists of 4 or 5 vertices, X and Y can be found by choosing the vertices of C and one or two of their neighbors (or neighbors of their neighbors).

Now we consider a connected graph that is either a tree or has only cycles of length 3. Assume that there exists no $v_0 \in V(G)$ with degree greater than 2. This implies that G is isomorphic to a path P_n, $n \geqslant 6$. So the two disjoint vertex sets X and Y can easily be found. Assume now that there exist two vertices x and y in G with degrees greater than 2. Let their neighborhoods be $N_G(x) = \{x_1, x_2, \ldots, x_s\}$ and $N_G(y) = \{y_1, y_2, \ldots, y_t\}$, $s, t \geqslant 3$. The vertices x and y have at most one common neighbor, because else there would exist a cycle of length greater than 3 in G. If $N_G(x) \cap N_G(y) = \emptyset$, then the sets X and Y can be chosen to be $\{x, x_1, x_2\}$ and $\{y, y_1, y_2\}$, where $x \notin \{y_1, y_2\}$ and $y \notin \{x_1, x_2\}$. If they have one common neighbor, say $z = x_1 = y_1$, let us first consider the case that x and y are not adjacent. Then we can choose $X := \{x, x_2, x_3\}$ and $Y := \{y, y_2, y_3\}$. Assume now that x and y are adjacent, and let $x = y_2$ and $y = x_2$. Then $H := G[\{x, y, z, x_3, y_3\}]$ is isomorphic to the bull. Since $n \geqslant 6$ and G is not isomorphic to the net, it is a simple matter to obtain the desired sets X and Y.

Let us now consider a connected graph G with exactly one $v_0 \in V(G)$ that is of degree greater than 2. Suppose that its excentricity $r := e(v_0)$ is at least 3.

Let $x \in V(G)$ with $d_G(x, v_0) = e(v_0)$ and $P = v_0 v_1 \ldots v_{r-1} x$ be a shortest path from v_0 to x in G. With $X := \{x, v_{r-2}, v_{r-1}\}$ and $Y := \{v_0, w, y\}$, where w and y are neighbors of v_0 that are not in $V(P)$, we can apply Theorem 2.1 again to conclude that G is λ_3-connected. Since we have discussed all possible cases, the proof is complete. □

Remark 2.3. The graphs satisfying (a) in Theorem 2.2 consist of a vertex v_0 that incides with several paths of lengths 1 and/or 2 and a number of triangles (cf. the figure below).



3. An upper bound for the order-3 edge connectivity

Theorem 3.1. *If G is λ_3-connected, then*

- *either $\lambda_3(G) < \xi_3(G)$,*
- *or $\lambda_3(G) = \xi_3(G)$ and $r_3(G) = 3$.*

Proof. Consider a vertex set $X = \{x, y, z\}$ of G such that $G[X]$ is connected and $|(X, \bar{X})| = \xi_3(G)$.

If $G[\bar{X}]$ is connected, then $(X, \bar{X})$ is an order-3 edge cut, and hence $\lambda_3(G) \leqslant \xi_3(G)$. Therefore, we have either $\lambda_3(G) = \xi_3(G)$ and thus X is a λ_3-atom, which yields $r_3(G) = 3$. Or else if $\lambda_3(G) < \xi_3(G)$, then there exists a λ_3-atom U with $|(U, \bar{U})| < |(X, \bar{X})| = \xi_3(G)$. This implies $r_3(G) \geqslant |U| \geqslant 4$.

Now consider the case that $G[\bar{X}]$ is disconnected. If $\bar{X}$ contains a component $C \subset \bar{X}$ with at least 3 vertices, then $|(C, \bar{C})| < |(X, \bar{X})| = \xi_3(G)$ and $G[\bar{C}]$ is connected. Thus, we have again $\lambda_3(G) < \xi_3(G)$ and $r_3(G) \geqslant |C| \geqslant 4$.

Finally, consider the case that $\bar{X}$ consists only of K_1- and K_2-components. The k components ($k \geqslant 2$) of $\bar{X}$ are identified by their vertex sets $C_1, C_2, \ldots, C_k$. Let $(S, \bar{S})$ be a λ_3-cut of G. It is clear that neither $\{x, y, z\} \not\subseteq S$ nor $\{x, y, z\} \not\subseteq \bar{S}$. So assume, without loss of generality, that $\{x, y\} \subset S$ and $\{z\} \subset \bar{S}$.

Now define for each component C_i the number of edges this component adds to the cut $(X, \bar{X})$ by

$$x(C_i) = |(C_i, X)|.$$

Note that $\sum_{i=1}^{k} x(C_i) = |(X, \bar{X})|$. Define for each component C_i the number of edges this component adds to the cut $(S, \bar{S})$:

$$s(C_i) = |\{uv \in (S, \bar{S}):\ u \in C_i \vee v \in C_i\}|.$$

Every edge in $(S, \bar{S})$ is counted exactly once except for the possible edges xz and yz, so $\sum_{i=1}^{k} s(C_i) + 2 \geqslant |(S, \bar{S})|$.

Next we will show that $x(C_i) \geqslant s(C_i) + 1$ for every component C_i. If $C_i \subset S$, then

$$x(C_i) = |(C_i, X)| = |(C_i, \{z\})| + |(C_i, \{x, y\})| = s(C_i) + |(C_i, \{x, y\})|.$$

Since S is connected, it follows that $x(C_i) \geqslant s(C_i) + 1$. The case $C_i \subset \bar{S}$ is similar. It remains the case that $C_i = \{u, v\}$ with $u \in S$ and $v \in \bar{S}$. Now because of $s(C_i) = 1 + |(\{u\}, \{z\})| + |(\{v\}, \{x, y\})|$, we conclude that

$$x(C_i) = |(C_i, X)| = s(C_i) - 1 + |(\{u\}, \{x, y\})| + |(\{v\}, \{z\})|,$$

and since S and $\bar{S}$ are connected, we obtain $x(C_i) \geqslant s(C_i) - 1 + 2 = s(C_i) + 1$.

Altogether, we have

$$|(S, \bar{S})| \leqslant 2 + \sum_{i=1}^{k} s(C_i) \leqslant 2 + \sum_{i=1}^{k} x(C_i) - k = |(X, \bar{X})| + 2 - k. \quad (*)$$

If $k \geqslant 3$, then it follows from $(*)$ that $|(S, \bar{S})| < |(X, \bar{X})|$ and hence $\lambda_3(G) < \xi_3(G)$.

If $k = 2$, then $|V(G)| \leqslant |X| + |C_1| + |C_2| \leqslant 7$, so $|S| = 3$ or $|\bar{S}| = 3$ and $r_3(G) = 3$. Furthermore, $|S| = 3$ or $|\bar{S}| = 3$ and $(*)$ yield

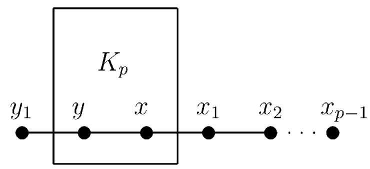
$$\xi_3(G) = |(X, \bar{X})| \leqslant |(S, \bar{S})| = \lambda_3(G) \leqslant |(X, \bar{X})| = \xi_3(G),$$

and therefore $\lambda_3(G) = \xi_3(G)$. □

Corollary 3.2. *If G is a λ_3-connected graph, then*

$$\lambda_3(G) \leqslant \xi_3(G).$$

Remark 3.3. For $p \geqslant 4$, the inequality $\lambda_p(G) \leqslant \xi_p(G)$ is no longer true in general. Let G be the disjoint union of a complete graph K_p and the vertices $y_1, x_1, x_2, \ldots, x_{p-1}$ together with the edges $yy_1, xx_1, x_i x_{i+1}$, $1 \leqslant i \leqslant p-2$, where $x, y \in V(K_p)$ (cf. the figure below).



Then $\xi_p(G) = (V(K_p), \overline{V(K_p)})_G = 2$ and $\lambda_p(G) = p - 1 > \xi_p(G)$.

Corollary 3.4. *A λ_3-connected graph G is λ_3-optimal if and only if $r_3(G) = 3$.*

Proof. Let G be λ_3-optimal. Then, by definition, $\lambda_3(G) = \xi_3(G)$. Thus, according to Theorem 3.1, $r_3(G) = 3$.

Conversely, let G be λ_3-connected such that $r_3(G) = 3$. Then there exists a λ_3-atom X of G with $|X| = r_3(G) = 3$ and $|(X, \bar{X})| = \lambda_3(G)$. Since $G[X]$ is connected, Corollary 3.2 implies $\xi_3(G) \leqslant |(X, \bar{X})| = \lambda_3(G) \leqslant \xi_3(G)$ and thus, G is λ_3-optimal. □

4. A study of the λ_3-optimality of some graph classes

Theorem 4.1. *Let G be a λ_3-connected graph. If G is not λ_3-optimal, then*

$$r_3(G) \geqslant \max\{4, \delta(G) - 1\}.$$

Proof. Let X be a λ_3-atom of G. In view of the hypothesis that G is not λ_3-optimal, it follows from Corollary 3.4, that $r_3 = r_3(G) = |X| \geqslant 4$. Therefore, it remains to show that $r_3 \geqslant \delta(G) - 1$.

Let $u \in X$ such that $s = d_G(u) = \min_{x\in X} d_G(x)$. Clearly, $G[X]$ is connected, and hence there exist two vertices $v, w \in X$ such that $H := G[\{u, v, w\}]$ is connected. Because of

$$d_G(v) + d_G(w) \geqslant \xi_3(G) - s + \begin{cases} 4 & \text{if } H \cong P_3 \\ 6 & \text{if } H \cong C_3 \end{cases}$$

and since G is not λ_3-optimal, we obtain

$$\begin{aligned} \xi_3(G) > \lambda_3(G) = |(X, \bar{X})| &\geqslant \sum_{x\in X} d_G(x) - r_3(r_3 - 1) + \begin{cases} 2 & \text{if } H \cong P_3 \\ 0 & \text{if } H \cong C_3 \end{cases} \\ &\geqslant d_G(v) + d_G(w) + (r_3 - 2)d_G(u) - r_3(r_3 - 1) + \begin{cases} 2 & \text{if } H \cong P_3 \\ 0 & \text{if } H \cong C_3 \end{cases} \\ &\geqslant \xi_3(G) - s + 6 + (r_3 - 2)s - r_3(r_3 - 1). \end{aligned}$$

This inequality implies

$$(r_3 - (s - 2))(r_3 - 3) > 0$$

and consequently, because of $r_3 > 3$, we deduce that $r_3 \geqslant s - 1 = d_G(u) - 1 \geqslant \delta(G) - 1$. □

Example 4.2. Let $n \geqslant 4$ be an integer and let K_n^i be a complete graph with vertex set $V_i = \{v_1^i, v_2^i, \ldots, v_n^i\}$ for $i = 1, 2, 3$. Define G as the disjoint union of K_n^1, K_n^2 and K_n^3 together with the edges $v_k^i v_k^j$ for $k = 1, 2, \ldots, n$ and $1 \leqslant i < j \leqslant 3$. Then $\xi_3(G) = 3n - 3$, $\lambda_3(G) = 2n$, $\delta(G) = n + 1$, and $r_3(G) = n = \delta(G) - 1$. Therefore, G is not λ_3-optimal. This example shows that Theorem 4.1 is best possible.

Using Turán's [5] bound $|E(G)| \leqslant \frac{1}{4}|V(G)|^2$ for triangle-free graphs G, one can prove similarly the next result.

Theorem 4.3. *Let G be a λ_3-connected and triangle-free graph. If G is not λ_3-optimal, then*

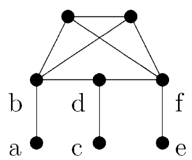
$$r_3(G) \geqslant \max\{4, 2\delta(G) - 2\}.$$

Example 4.4. Let $n \geqslant 3$ be an integer and let $K^i_{n,n}$ be a complete bipartite graph with vertex set $V_i = \{v^i_1, v^i_2, \ldots, v^i_{2n}\}$ for $i = 1, 2$. Define G as the disjoint union of $K^1_{n,n}$ and $K^2_{n,n}$ together with the edges $v^1_j v^2_j$ for $j = 1, 2, \ldots, 2n$. Then $\xi_3(G) = 3n - 1$, $\delta(G) = n+1$, $\lambda_3(G) = r_3(G) = 2n$. Therefore, G is not λ_3-optimal and $r_3(G) = 2\delta(G) - 2$. This example shows that Theorem 4.3 is best possible.

Remark 4.5. In [8], Xu and Xu proved that the λ_2-atoms of a non-λ_2-optimal graph are pairwise disjoint. This is no longer true for non-λ_3-optimal graphs, as the following examples show.

Consider a non-λ_3-optimal graph G with $\lambda_3(G) = \xi_3(G) - 1$. Let there exist two vertex sets $X, X' \subset V(G)$ of cardinality 3 with $X \cap X' = \{x_1\}$ and $\xi_3(G) = |(X, \bar{X})| = |(X', \overline{X'})|$. If x_1 has a neighbor x_0 of degree 1 in G, then $X \cup \{x_0\}$ and $X' \cup \{x_0\}$ are two λ_3-atoms of G which are not disjoint.

In the next figure we represent a special case of these examples.



For this graph, we have $\lambda_3(G) = 3 < 4 = \xi_3(G)$. The vertex sets $\{a, b, c, d\}$ and $\{c, d, e, f\}$ are λ_3-atoms of G that are not pairwise disjoint.

Theorem 4.6. *If $K_{r,s}$ is a complete bipartite graph with $r, s \geqslant 2$ and $r + s \geqslant 6$, then it is λ_3-optimal.*

Proof. Let $A = \{x_1, \ldots, x_r\}$ and $B = \{y_1, \ldots, y_s\}$ be the partitions of $V(K_{r,s})$ with $r \leqslant s$. Then we have $\xi_3(K_{r,s}) = s + 2r - 4$. Assume that $K_{r,s}$ is not λ_3-optimal. Let, without loss of generality, $X = \{x_1, x_2, \ldots, x_k, y_1, y_2 \ldots, y_l\}$ be a λ_3-atom of $K_{r,s}$. Then $\lambda_3(K_{r,s}) = ks + lr - 2kl$ with $1 \leqslant k \leqslant r - 1$, $1 \leqslant l \leqslant s - 1$ and $4 \leqslant k + l \leqslant \frac{r+s}{2}$. Furthermore, $l \leqslant \frac{s}{2}$, because else we can construct a λ_3-atom X' with $|X'| = |X|$ and $|(X', \bar{X}')| < |(X, \bar{X})|$ by removing one vertex from $X \cap B$ and adding one from $A \backslash X$. We will show by induction that $ks + lr - 2kl \geqslant s + 2r - 4$ for all k and l chosen as above. For $k = 1$, l is at least 3, and thus

$$ks + lr - 2kl = s + lr - 2l > s + 2(r - 2).$$

If $k \leqslant r - 2$ fulfills the inequality, we obtain for $k + 1$

$$(k + 1)s + lr - 2(k + 1)l \geqslant s + 2r - 4 + s - 2l \geqslant s + 2r - 4.$$

From this contradiction we conclude that $K_{r,s}$ is λ_3-optimal. $\square$

References

[1] A.-H. Esfahanian, Generalized measures of fault tolerance with application to n-cube networks, IEEE Trans. Comput. 38 (1989) 1586–1591.
[2] A.-H. Esfahanian, S.L. Hakimi, On computing a conditional edge-connectivity of a graph, J. Inform. Process. Lett. 27 (1988) 195–199.
[3] F. Harary, Conditional connectivity, Networks 13 (1983) 347–357.
[4] Q.L. Li, Q. Li, Reliability analysis of circulant graphs, Networks 31 (1998) 61–65.
[5] P. Turán, An extremal problem in graph theory, Mat.-Fiz. Lapok 48 (1941) 436–452 (in Hungarian).
[6] N. Ueffing, L. Volkmann, Restricted edge-connectivity and minimum edge-degree, Ars Combin., to appear.
[7] J.M. Xu, Restricted edge-connectivity of vertex-transitive graphs, Chinese Ann. Math. Ser. A 21 (5) (2000) 605–608 (in Chinese).
[8] J.M. Xu, K.-L. Xu, On restricted edge-connectivity of graphs, Discrete Math. 243 (2002) 291–298.