

# Existence of $\Delta_\lambda$ -Cycles and $\Delta_\lambda$ -Paths

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## ABSTRACT

A cycle  $C$  of a graph  $G$  is called a  $D_\lambda$ -cycle if every component of  $G - V(C)$  has order less than  $\lambda$ . A  $D_\lambda$ -path is defined analogously.  $D_\lambda$ -cycles and  $D_\lambda$ -paths were introduced by Veldman. Here a cycle  $C$  of a graph  $G$  is called a  $\Delta_\lambda$ -cycle if all vertices of  $G$  are at distance less than  $\lambda$  from a vertex of  $C$ . A  $\Delta_\lambda$ -path is defined analogously. In particular, in a connected graph, a  $D_\lambda$ -cycle is a  $\Delta_\lambda$ -cycle and a  $D_\lambda$ -path is a  $\Delta_\lambda$ -path. Furthermore, a  $\Delta_1$ -cycle is a Hamilton cycle and a  $\Delta_1$ -path is a Hamilton path. Necessary conditions and sufficient conditions are derived for graphs to have a  $\Delta_\lambda$ -cycle or  $\Delta_\lambda$ -path. The results are analogues of theorems on  $D_\lambda$ -cycles and  $D_\lambda$ -paths. In particular, a result of Chvátal and Erdős on Hamilton cycles and Hamilton paths is generalized. A recent conjecture of Bondy and Fan is settled.

## 1. TERMINOLOGY

We use [2] for basic terminology and notation not introduced here, and consider simple graphs only. Let  $G$  be a graph. We will sometimes identify a trail in  $G$  with the subgraph induced by its edges. Hence a subgraph  $T$  of  $G$  is a trail if and only if  $T$  is connected and at most two vertices of  $T$  have odd degree in  $T$ . Let  $\lambda$  be an integer with  $\lambda \geq 1$ . If  $T$  is a trail in  $G$ , then  $b_\lambda(T)$  denotes the number of vertices of  $G$  that are at distance less than  $\lambda$  from a vertex of  $T$ . Following Veldman [6], a trail  $T$  of  $G$  is defined to be a  $D_\lambda$ -trail of  $G$  if all components of  $G - V(T)$  have order less than  $\lambda$ . Here we define  $T$  to be a  $\Delta_\lambda$ -trail of  $G$  if all vertices of  $G$  are at distance less than  $\lambda$  from a vertex of  $T$ . Note that, in a connected graph, a  $D_\lambda$ -trail is a  $\Delta_\lambda$ -trail, whereas the converse is only true in general for  $\lambda = 1$ . A *circuit* is a nontrivial closed trail. Graphs containing a  $\Delta_\lambda$ -cycle ( $D_\lambda$ -cycle) will be called  $\Delta_\lambda$ -cyclic ( $D_\lambda$ -cyclic); graphs containing a  $\Delta_\lambda$ -path ( $D_\lambda$ -path, Hamilton path) will be called  $\Delta_\lambda$ -traceable ( $D_\lambda$ -traceable, traceable). If  $C$  is a cycle of  $G$  with a fixed orientation and  $v \in V(C)$ , then  $v^-$

and  $v^+$  denote the immediate predecessor and immediate successor of  $v$  on  $C$ , respectively. If  $H$  is an oriented path or cycle of  $G$  and  $u$  and  $v$  are vertices of  $H$ , then  $u\overrightarrow{H}v$  and  $v\overleftarrow{H}u$  denote, respectively, the segment of  $H$  from  $u$  to  $v$  and the reverse segment from  $v$  to  $u$ . Two vertices  $u$  and  $v$  of  $G$  are  $\lambda$ -neighbors if  $d(u, v) < \lambda$ .  $N_\lambda(v)$  denotes the set of  $\lambda$ -neighbors of a vertex  $v$  of  $G$ . Note that  $N_1(v) = \{v\}$  and  $N_2(v) = N(v) \cup \{v\}$ . We let  $R_\lambda(v) = N_\lambda(v) - \{v\}$ . Two vertices  $u$  and  $v$  of  $G$  are  $\lambda$ -distant if  $d(u, v) \geq \lambda$ , i.e., if they are not  $\lambda$ -neighbors. In [6] Veldman introduced  $\omega_\lambda(G)$ , the number of components of  $G$  of order at least  $\lambda$ , and  $\alpha_\lambda(G)$ , the maximum number of mutually disjoint connected subgraphs of order  $\lambda$  of  $G$  such that no edge of  $G$  joins two vertices of different subgraphs. If  $S$  is a subset of  $V(G)$ , then  $\hat{\omega}_\lambda(G, S)$  denotes the number of components of  $G - S$  that contain a vertex that is  $\lambda$ -distant from all vertices of  $S$ . Note that, in a connected graph  $G$ ,  $\hat{\omega}_\lambda(G, S) \leq \omega_\lambda(G - S)$  and  $\hat{\omega}_1(G, S) = \omega_1(G - S) = \omega(G - S)$ . By  $\hat{\alpha}_\lambda(G)$  we denote the maximum cardinality of a set of mutually  $\lambda$ -distant vertices in  $G$ . Note that, in a connected graph  $G$ ,  $\hat{\alpha}_{2\lambda}(G) \leq \alpha_\lambda(G)$  and  $\hat{\alpha}_2(G) = \alpha_1(G) = \alpha(G)$ . If  $P$  is a nontrivial path in  $G$  with origin  $v_1$  and terminus  $v_2$ , then  $P$  is called a  $\lambda$ -bridge if all edges of  $P$  are cut edges of  $G$  and, for  $i = 1, 2$ , the component of  $G - E(P)$  containing  $v_i$  also contains a vertex  $u_i$  satisfying  $d(u_i, v_{3-i}) \geq \lambda$ . Note that a cut edge is a 1-bridge, and a cut edge incident with two vertices of degree at least 2 is a 2-bridge.  $G$  is  $\lambda$ -bridgeless if  $G$  contains no  $\lambda$ -bridge.

## 2. INTRODUCTION

Our concern will be the existence of  $\Delta_\lambda$ -cycles and  $\Delta_\lambda$ -paths in graphs. Recognizing  $\Delta_\lambda$ -cyclic graphs is an NP-complete problem. This is easily seen, using the NP-completeness of the Hamilton cycle problem. In Sections 3 and 4 necessary conditions and sufficient conditions are derived for the existence of  $\Delta_\lambda$ -cycles and  $\Delta_\lambda$ -paths. There is a nice analogy with known results and proof techniques concerning the existence of  $D_\lambda$ -cycles and  $D_\lambda$ -paths.  $D_\lambda$ -cycles and  $D_\lambda$ -paths were studied in [4] and [6];  $D_2$ -cycles and  $D_2$ -paths in [5]. The conditions for  $\Delta_\lambda$ -cyclicity ( $\Delta_\lambda$ -traceability) are weaker than the corresponding ones for  $D_\lambda$ -cyclicity ( $D_\lambda$ -traceability), in accordance with the fact that every  $D_\lambda$ -cyclic ( $D_\lambda$ -traceable) connected graph is  $\Delta_\lambda$ -cyclic ( $\Delta_\lambda$ -traceable), whereas the converse is not true in general.

## 3. NECESSARY CONDITIONS

The following statement is obvious:

**Proposition 1.** If a graph  $G$  contains a  $\Delta_\lambda$ -circuit, then  $G$  is  $\lambda$ -bridgeless.

For  $\lambda = 1$ , Proposition 1 coincides with the statement that a graph containing a spanning circuit is 2-edge-connected.

Theorem 4.2 of [2] states that if a graph  $G$  is hamiltonian, then  $\omega(G - S) \leq |S|$  for every nonempty proper subset  $S$  of  $V(G)$ .

Veldman [6] showed that if a graph  $G$  is  $D_\lambda$ -cyclic, then  $\omega_\lambda(G - S) \leq |S|$  for every nonempty proper subset of  $S$  of  $V(G)$ .

Here we give a similar condition on  $\Delta_\lambda$ -cyclic graphs.

**Theorem 2.** If a graph  $G$  is  $\Delta_\lambda$ -cyclic, then, for every nonempty proper subset  $S$  of  $V(G)$ ,  $\hat{\omega}_\lambda(G, S) \leq |S|$ .

*Proof.* Let  $S$  be a nonempty proper subset of  $V(G)$  and  $C$  a  $\Delta_\lambda$ -cycle of  $G$ . Then every vertex of  $G - S$  is at distance at most  $\lambda - 1$  from a vertex of  $C$  in  $G$ . If  $S \cap V(C) = \emptyset$ ,  $\hat{\omega}_\lambda(G, S) \leq 1 \leq |S|$ . Otherwise,  $\hat{\omega}_\lambda(G, S) \leq |S \cap V(C)| \leq |S|$ . ■

Analogously, one proves a cut set theorem for  $\Delta_\lambda$ -traceable graphs.

**Theorem 3.** If a graph  $G$  is  $\Delta_\lambda$ -traceable, then, for every nonempty proper subset  $S$  of  $V(G)$ ,  $\hat{\omega}_\lambda(G, S) \leq |S| + 1$ .

#### 4. SUFFICIENT CONDITIONS

Chvátal and Erdős [3] showed that a graph  $G$  with independence number  $\alpha(G)$  and connectivity  $\kappa(G)$  is hamiltonian if  $\alpha(G) \leq \kappa(G)$ , while  $\alpha(G) \leq \kappa(G) + 1$  implies that  $G$  is traceable. Veldman [6] proved the following generalization on  $D_\lambda$ -cyclic graphs:

**Theorem 4.** (Veldman [6]). Let  $k$  and  $\lambda$  be positive integers such that either  $k \geq 2$  or  $k = 1$  and  $\lambda \leq 2$ . If  $G$  is a  $k$ -connected graph, other than a tree (in case  $k = 1$ ), with  $\alpha_\lambda \leq k$ , then  $G$  is  $D_\lambda$ -cyclic.

Here we prove an analogue of Theorem 4. For convenience, we deal with graphs of connectivity 1 separately.

**Theorem 5.** Let  $G$  be a  $k$ -connected graph ( $k \geq 2$ ). If  $\hat{\alpha}_{2\lambda} \leq k$ , then  $G$  is  $\Delta_\lambda$ -cyclic.

*Proof.* Let  $G$  be a non- $\Delta_\lambda$ -cyclic  $k$ -connected graph ( $k \geq 2$ ). We will exhibit  $k + 1$  mutually  $2\lambda$ -distant vertices. Let  $C$  be a cycle of  $G$  such that

- (1)  $b_\lambda(C)$  is maximum.

Fix an orientation on  $C$ . Since  $G$  is connected and  $C$  is not a  $\Delta_\lambda$ -cycle of  $G$ , there exists a vertex  $u \in A = V(G) - V(C)$  such that

- (2)  $N_\lambda(u) \cap V(C) = \emptyset$ .

Suppose  $|V(C)| < k$  and let  $xy \in E(C)$ . By Menger's theorem there exists a path  $Q_x$  from  $u$  to  $x$  and a path  $Q_y$  from  $u$  to  $y$  such that  $Q_x$  and  $Q_y$  meet only at

$u$  and no other vertex of  $C$  lies on  $Q_x$  or  $Q_y$ . The cycle  $C'$  with  $E(C') = E(C) \cup E(Q_x) \cup E(Q_y) - \{xy\}$  satisfies  $b_\lambda(C') > b_\lambda(C)$ , contradicting (1). Hence  $V(C) \geq k$ . Since  $G$  is  $k$ -connected and  $|V(C)| \geq k$ , a variation on Menger's theorem asserts that  $u$  is connected to at least  $k$  distinct vertices of  $C$  by internally-disjoint paths. Let  $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$  be a collection of paths with the following properties:

- (3)  $P_i$  has origin  $u$  and terminus  $v_i$  on  $C$  ( $i = 1, 2, \dots, k$ ).
- (4) Two distinct paths of  $\mathcal{P}$  have only  $u$  in common.
- (5) No internal vertex of  $P_i$  is on  $C$  ( $i = 1, 2, \dots, k$ ).

Furthermore, assume that  $C$  and  $\mathcal{P}$  are chosen such that, subject to conditions (1)–(5),

- (6)  $|\bigcup_{i=1}^k V(P_i)|$  is minimum.

Assume that  $v_1, v_2, \dots, v_k$  occur on  $C$  in the order of their indices. From the maximality of  $b_\lambda(C)$  it follows that  $v_i v_{i+1} \notin E(C)$  ( $i = 1, 2, \dots, k$ , indices mod  $k$ ); otherwise the cycle  $C'$  with  $E(C') = E(C) \cup E(P_i) \cup E(P_{i+1}) - \{v_i v_{i+1}\}$  contradicts the choice of  $C$ . Define a vertex  $u_i$  on  $C$  by the following requirements:

- (7) for all  $x$  in  $X_i = V(v_i \vec{C} u_i) \cup \{v \in A \mid v \in N_\lambda(w) \text{ for some } w \in V(v_i \vec{C} u_i)\}$  there exists a vertex  $y \in V(u_i \vec{C} v_i)$  with  $d(x, y) < \lambda$ , and
- (8)  $|V(v_i \vec{C} u_i)|$  is maximum ( $i = 1, 2, \dots, k$ ).

Since (7) is satisfied for  $u_i = v_i^+$ ,  $u_i$  exists ( $i = 1, 2, \dots, k$ ). Furthermore, (1) implies that  $u_i \in V(v_i \vec{C} v_{i+1}^-)$  ( $i = 1, 2, \dots, k$ , indices mod  $k$ ); otherwise (7) is satisfied for  $u_i = v_{i+1}$ , and the cycle  $C'$  with  $E(C') = E(C) \cup E(P_i) \cup E(P_{i+1}) - E(v_i \vec{C} v_{i+1})$  satisfies  $b_\lambda(C') > b_\lambda(C)$ , since all vertices of  $v_i \vec{C} v_{i+1}$  and their  $\lambda$ -neighbors have  $\lambda$ -neighbors on  $C'$  and  $u$  is on  $C'$ . This contradicts (1). Now  $X_i$  contains at least one vertex  $x_i$  such that all  $\lambda$ -neighbors of  $x_i$  on  $C$  are in  $X_i$  ( $i = 1, 2, \dots, k$ ); otherwise (7) is satisfied with  $u_i$  replaced by  $u_i^+$ , in contradiction to (8). Thus

- (9)  $N_\lambda(x_i) \cap V(C) \subset X_i$  ( $i = 1, 2, \dots, k$ ).

Let  $H_i$  denote the component of  $G[X_i]$  containing  $x_i$ . We make two more observations. Here  $i, j \in \{1, 2, \dots, k\}$ .

- (10) There exists no path from a vertex of  $H_i$  to a vertex of  $V(P_j) - \{v_j\}$  that is internally disjoint from  $C$  and the paths of  $\mathcal{P}$ .

This is a consequence of (1) and (6). Suppose to the contrary that there is such a path. Without loss of generality, assume there exists a path  $Q$  from a vertex  $x \in V(H_i) \cap V(C)$  to a vertex  $y \in V(P_j) - \{v_j\}$  that is internally disjoint from  $C$  and the paths of  $\mathcal{P}$ . If  $i \neq j$ , consider the cycle  $C'$  with  $E(C') = E(C) \cup E(Q) \cup E(P_i) \cup E(u \vec{P}_j y) - E(v_i \vec{C} x)$  (possibly  $y = u$ ); if  $i = j$ , consider the cycle  $C'$  with  $E(C') = E(C) \cup E(Q) \cup E(y \vec{P}_i v_i) - E(v_i \vec{C} x)$ . Now  $b_\lambda(C') \geq b_\lambda(C)$ , since all vertices of  $v_i \vec{C} x$  and their  $\lambda$ -neighbors have  $\lambda$ -neighbors on  $C'$ . If  $b_\lambda(C') = b_\lambda(C)$ , then  $i = j$ , and there exist paths  $P'_1, P'_2, \dots, P'_k$  with prop-

erties (3), (4), and (5) with respect to  $u$  and  $C'$ , and  $|\bigcup_{i=1}^k V(P'_i)| < |\bigcup_{i=1}^k V(P_i)|$ . Hence  $C'$  contradicts the choice of  $C$ .

- (11) For  $i \neq j$ , there exists no path from a vertex of  $H_i$  to a vertex of  $H_j$  that is internally disjoint from  $C$ .

This is a consequence of (1) and (10). Suppose to the contrary that there is such a path. Then, by (10), this path is disjoint from the paths of  $\mathcal{P}$ . Without loss of generality, assume there exists a path  $Q$  from a vertex  $x \in V(H_i) \cap V(C)$  to a vertex  $y \in V(H_j) \cap V(C)$  that is internally disjoint from  $C$  and the paths of  $\mathcal{P}$  such that  $|V(v_i\vec{C}x)|$  is minimum. The choice of  $x$  implies that all vertices in  $X_i \cup X_j$  have a  $\lambda$ -neighbor in  $V(x\vec{C}v_j) \cup V(y\vec{C}v_i)$ . Now the cycle  $C'$  with  $E(C') = E(C) \cup E(Q) \cup E(P_i) \cup E(P_j) - (E(v_i\vec{C}x) \cup E(v_j\vec{C}y))$  contradicts the choice of  $C$ .

We complete the proof by showing that  $\{u, x_1, \dots, x_k\}$  is a set of mutually  $2\lambda$ -distant vertices. If  $1 \leq i < j \leq k$ , then, by (9) and (11),  $d(x_i, x_j) \geq (\lambda - 1) + (\lambda - 1) + 2 = 2\lambda$ . For arbitrary  $i \in \{1, 2, \dots, k\}$ , consider a shortest path  $P$  from  $u$  to  $x_i$ . By (10), at least one of the internal vertices of  $P$  is on  $C$ . Let  $x$  be the first vertex on  $P$  that is on  $C$ . By (10),  $x \notin X_i$ . By (2),  $d(u, x) \geq \lambda$ , and by (9),  $d(x_i, x) \geq (\lambda - 1) + 1$ . Hence  $d(u, x_i) \geq 2\lambda$ . ■

Note that  $\hat{\alpha}_{2\lambda} = 1$  for a graph  $G$  if and only if the diameter of  $G$  is at most  $2\lambda - 1$ . Hence the following result can be viewed as the case  $k = 1$  of Theorem 5:

**Theorem 6.** Let  $G$  be a connected  $\lambda$ -bridgeless graph other than a tree. If the diameter of  $G$  is at most  $2\lambda - 1$ , then  $G$  is  $\Delta_\lambda$ -cyclic.

*Proof.* Let  $G$  be a connected non- $\Delta_\lambda$ -cyclic  $\lambda$ -bridgeless graph other than a tree. Let  $C$  be a cycle of  $G$  such that  $b_\lambda(C)$  is maximum. Fix an orientation on  $C$ . Since  $C$  is not a  $\Delta_\lambda$ -cycle of  $G$  and since  $G$  is connected, there exists a vertex  $u \in V(G) - V(C)$  such that  $N_\lambda(u) \cap V(C) = \emptyset$  and  $d(u, v_1) = \lambda$  for some  $v_1 \in V(C)$ . Define  $u_1$  and  $x_1$  as in the proof of Theorem 5. If  $u_1 \neq v_1$ , then, like in the proof of Theorem 5, one can show that  $d(u, x_1) \geq 2\lambda$ . Now suppose  $u_1 = v_1$ . Then all vertices of  $C$  and their  $\lambda$ -neighbors are  $\lambda$ -neighbors of  $v_1$ . Let  $P$  be a shortest  $(u, v_1)$ -path and let  $z_1$  denote the immediate predecessor of  $v_1$  on  $P$ . Now  $v_1z_1$  is a cut edge of  $G$ ; otherwise there is a cycle  $C'$  of  $G$  containing  $v_1$  and  $z_1$ . Since  $d(u, z_1) = \lambda - 1$ ,  $b_\lambda(C') > b_\lambda(C)$ , a contradiction.

Since  $v_1z_1$  is not a  $\lambda$ -bridge of  $G$ ,  $N_\lambda(v_1) \subset N_\lambda(z_1)$ . Let  $z$  be the vertex on  $P$  such that all edges of  $z\vec{P}v_1$  are cut edges of  $G$  and  $|V(z\vec{P}v_1)|$  is maximum. Since  $G$  is  $\lambda$ -bridgeless,  $z \neq u$ , implying the existence of a cycle  $C'$  containing  $z$ . Furthermore, since  $G$  is  $\lambda$ -bridgeless,  $N_\lambda(v_1) \subset N_\lambda(z)$ . This implies that  $b_\lambda(C') > b_\lambda(C)$ , a contradiction. ■

Theorem 5 and Theorem 6 generalize the mentioned result of Chvátal and Erdős, and are best possible in the sense that, for any positive integers  $k$  and  $\lambda$ , there exist infinitely many  $k$ -connected non- $\Delta_\lambda$ -cyclic graphs with

$\hat{\alpha}_{2\lambda} = k + 1$  (which are  $\lambda$ -bridgeless and not trees). Consider, e.g., the graphs  $G(k, n_{1,1}, \dots, n_{1,k+1}, n_{2,1}, \dots, n_{2,k+1}, \dots, n_{\lambda,1}, \dots, n_{\lambda,k+1})$  that are sketched in Figure 1.

They consist of the following mutually disjoint subgraphs: a subgraph  $H \cong K_k$ ,  $\lambda(k + 1)$  subgraphs  $H_{i,j} \cong K_{n_{i,j}}$ , where  $1 \leq i \leq \lambda$  and  $1 \leq j \leq k + 1$ , and the following additional edges:

$$\{xy \mid x \in V(H); y \in \bigcup_{j=1}^{k+1} V(H_{1,j})\} \\ \cup \{xy \mid x \in V(H_{i,j}); y \in V(H_{i+1,j}); 1 \leq i < \lambda; 1 \leq j \leq k + 1\}.$$

If  $n_{i,j} \geq k$ , for  $1 \leq i < \lambda$  and  $1 \leq j \leq k + 1$ , and  $n_{\lambda,j} \geq 1$ , for  $1 \leq j \leq k + 1$ , then  $G(k, n_{1,1}, \dots, n_{\lambda,k+1})$  obviously is  $k$ -connected. It is not  $\Delta_\lambda$ -cyclic by Theorem 2 (with  $S = V(H)$ ). Obviously,  $d(u, v) \geq 2\lambda$  if and only if there exist integers  $i$  and  $j$  with  $1 \leq i < j \leq k + 1$  such that  $u \in V(H_{\lambda,i})$  and  $v \in V(H_{\lambda,j})$ . Hence  $\hat{\alpha}_{2\lambda} = k + 1$  (for  $k = 1$  we take  $n_{1,1} \geq n_{1,2} \geq 2$  to obtain  $\lambda$ -bridgeless graphs other than trees).

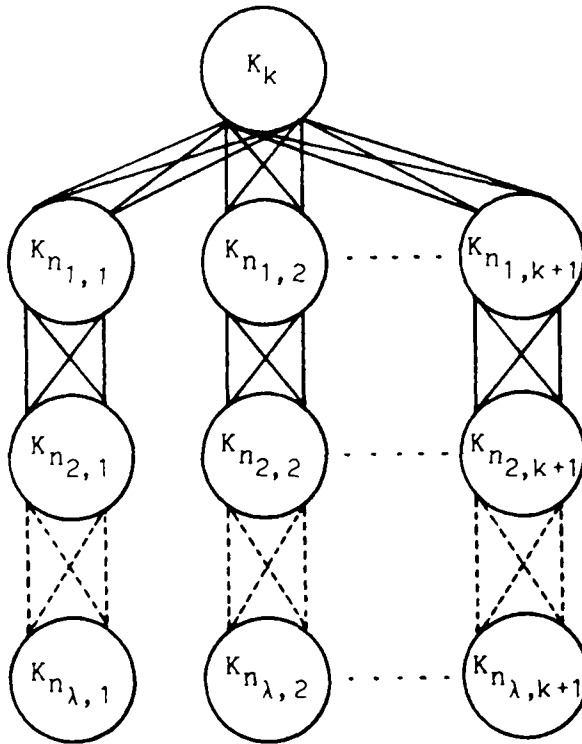


FIGURE 1

Theorem 5 and Theorem 6 have some interesting corollaries. The following one, in terms of the connectivity and independence number, is another generalization of the mentioned result of Chvátal and Erdős.

**Corollary 7.** Let  $G$  be a  $k$ -connected  $\lambda$ -bridgeless graph other than a tree ( $k \geq 1$ ). If  $\lambda$  is odd and  $\alpha \leq \frac{1}{2}(\lambda k + \lambda + k - 1)$ , or  $\lambda$  is even and  $\alpha \leq \frac{1}{2}(\lambda k + \lambda)$ , then  $G$  is  $\Delta_\lambda$ -cyclic.

*Proof.* Let  $G$  be a non- $\Delta_\lambda$ -cyclic  $k$ -connected  $\lambda$ -bridgeless graph other than a tree ( $k \geq 1$ ). By Theorem 5 and Theorem 6 there exist  $k + 1$  vertices  $v_0, v_1, \dots, v_k$  that are mutually  $2\lambda$ -distant. Now, for  $0 \leq i \leq k$ , let  $x_{i,j}$  be a vertex at distance  $2j$  of  $v_i$ , with  $0 \leq j \leq \lambda/2$ . Since, for  $0 \leq i < j \leq k$ ,  $d(v_i, v_j) \geq 2\lambda$ , the following observations are obvious:

- (I) If  $\lambda$  is odd, then  $\{x_{i,j} \mid 0 \leq i \leq k; 0 \leq j \leq (\lambda - 1)/2\}$  is a set of mutually independent vertices.
- (II) If  $\lambda$  is even, then  $\{x_{i,j} \mid 0 \leq i \leq k; 0 \leq j \leq (\lambda - 2)/2\} \cup \{x_{0,\lambda/2}\}$  is a set of mutually independent vertices.

Hence, if  $\lambda$  is odd,  $\alpha \geq ((\lambda + 1)/2)(k + 1) = \frac{1}{2}(\lambda k + \lambda + k + 1)$ , and, if  $\lambda$  is even,  $\alpha \geq (\lambda/2)(k + 1) + 1 = \frac{1}{2}(\lambda k + \lambda) + 1$ . ■

The graphs showing that Theorem 5 and Theorem 6 are best possible also show that Corollary 7 is best possible. Before we state two other corollaries of Theorem 5 and Theorem 6, we prove the following lemma:

**Lemma 8.** Let  $G$  be a  $k$ -connected graph ( $k \geq 1$ ) and let  $\lambda \geq 2$ .

If  $\{v_0, v_1, \dots, v_k\}$  is a set of mutually  $2\lambda$ -distant vertices, then

$$\sum_{i=0}^k d(v_i) < \nu - 2k - (\lambda - 2)k(k + 1) \quad \text{and} \quad \sum_{i=0}^k |R_\lambda(v_i)| < \nu - 2k.$$

*Proof.* Let  $G$  be a  $k$ -connected graph ( $k \geq 1$ ) and let  $\lambda \geq 2$ .

Let  $\{v_0, v_1, \dots, v_k\}$  be a set of mutually  $2\lambda$ -distant vertices.

Define  $V_{i,t} = \{v \in V(G) \mid d(v_i, v) = t\}$  ( $0 \leq i \leq k; 1 \leq t \leq \lambda$ ). Since  $G$  is  $k$ -connected,  $|V_{i,t}| \geq k$  ( $0 \leq i \leq k; 1 \leq t \leq \lambda$ ). Since, for  $0 \leq i < j \leq k$ ,  $d(v_i, v_j) \geq 2\lambda$ , the sets in the collection  $\mathcal{R} = \{V_{i,t} \mid 0 \leq i \leq k; 2 \leq t \leq \lambda - 1\} \cup \{V_{0,\lambda}\}$  are mutually disjoint.

Furthermore, no vertex in the union of the sets in  $\mathcal{R}$  is adjacent to any of the vertices  $v_0, v_1, \dots, v_k$ . Hence

$$\begin{aligned} \sum_{i=0}^k d(v_i) &\leq \nu - (k + 1) - \\ &(k + 1)(\lambda - 2)k - k = \nu - 2k - (\lambda - 2)k(k + 1) - 1. \end{aligned}$$

Finally, since  $R_\lambda(v_i) = \bigcup_{t=1}^{\lambda-1} V_{i,t}$  and  $R_\lambda(v_i) \cap R_\lambda(v_j) = \emptyset$ , we get  $\sum_{i=0}^k |R_\lambda(v_i)| \leq \nu - 2k - 1$ . ■

The next two corollaries are easily obtained by combining Lemma 8 with Theorem 5 and Theorem 6.

**Corollary 9.** Let  $G$  be a  $k$ -connected  $\lambda$ -bridgeless graph other than a tree ( $k \geq 1$ ;  $\lambda \geq 2$ ). If the degree-sum of any  $k + 1$  mutually  $(2\lambda - 1)$ -distant vertices is at least  $\nu - 2k - (\lambda - 2)k(k + 1)$ , then  $G$  is  $\Delta_\lambda$ -cyclic.

**Corollary 10.** Let  $G$  be a  $k$ -connected  $\lambda$ -bridgeless graph other than a tree ( $k \geq 1$ ;  $\lambda \geq 2$ ). If any  $k + 1$  mutually  $(2\lambda - 1)$ -distant vertices  $v_0, v_1, \dots, v_k$  satisfy the inequality  $\sum_{i=0}^k |R_\lambda(v_i)| \geq \nu - 2k$ , then  $G$  is  $\Delta_\lambda$ -cyclic.

The graphs  $G(k, n_{1,1}, \dots, n_{\lambda, k+1})$ , with  $n_{i,j} = k$ , for  $1 \leq i < \lambda$  and  $1 \leq j \leq k + 1$ , and  $n_{\lambda, j} \geq 1$ , for  $1 \leq j \leq k + 1$ , show that Corollaries 9 and 10 are best possible for  $k \geq 2$ . Corollaries 9 and 10 are more general than the following result of Fraisse [4]:

**Corollary 11.** (Fraisse [4]). Let  $G$  be a  $k$ -connected graph ( $k \geq 2$ ). If  $\delta(G) > (\nu - 2k - 1)/(k + 1)$ , then  $G$  is  $\Delta_2$ -cyclic.

The case  $k \geq 2$  of Corollary 10 was recently conjectured by Bondy and Fan [1] and proved for  $\lambda = 2$ .

Without proof we mention the following analogue of Theorems 5 and 6 on  $\Delta_\lambda$ -traceable graphs:

**Theorem 12.** Let  $G$  be a  $k$ -connected graph ( $k \geq 1$ ). If  $\hat{\alpha}_{2\lambda} \leq k + 1$ , then  $G$  is  $\Delta_\lambda$ -traceable.

As analogues of Corollaries 7, 9, and 10 we find, respectively, the following:

**Corollary 13.** Let  $G$  be a  $k$ -connected graph ( $k \geq 1$ ). If  $\lambda$  is odd and  $\alpha \leq \frac{1}{2}(\lambda k + 2\lambda + k)$ , or  $\lambda$  is even and  $\alpha \leq \frac{1}{2}(\lambda k + 2\lambda)$ , then  $G$  is  $\Delta_\lambda$ -traceable.

**Corollary 14.** Let  $G$  be a  $k$ -connected graph ( $k \geq 1$ ) and let  $\lambda \geq 2$ . If the degree-sum of any  $k + 2$  mutually  $(2\lambda - 1)$ -distant vertices is at least  $\nu - 2k - 1 - (\lambda - 2)k(k + 2)$ , then  $G$  is  $\Delta_\lambda$ -traceable.

**Corollary 15.** Let  $G$  be a  $k$ -connected graph ( $k \geq 1$ ) and let  $\lambda \geq 2$ . If any  $k + 2$  mutually  $(2\lambda - 1)$ -distant vertices  $v_0, v_1, \dots, v_{k+1}$  satisfy the inequality  $\sum_{i=0}^{k+1} |R_\lambda(v_i)| \geq \nu - 2k - 1$ , then  $G$  is  $\Delta_\lambda$ -traceable.

The proofs of the above results on  $\Delta_\lambda$ -traceable graphs are similar to the proofs of the corresponding results on  $\Delta_\lambda$ -cyclic graphs. Both Theorem 12 and Corollaries 13–15 can be shown to be best possible by considering the graphs sketched in Figure 1 and replacing “ $k + 1$ ” by “ $k + 2$ .”



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