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INFINITELY DIVISIBLE RENEWAL DISTRIBUTIONS

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1. Introduction. It is well-known that many waiting-time processes yield infinitely divisible (inf div) waiting-time distributions, independent of the divisibility properties of the arrival and service time distributions. One might expect that renewal processes also would often yield inf div renewal distributions. It turns out however that even when the original distribution (with characteristic function (cf) $\phi(t)$) is inf div then the renewal distribution (with cf $\{\phi(t) - 1\}(i\mu t)^{-1}$) need not be inf div On the other hand a distribution, which is not inf div may give rise to an inf div. renewal distribution. In this paper conditions, in terms of the original distribution function, are derived for the renewal distribution to be inf div. In relation with this the infinite divisibility of some waiting-time distributions is considered. It is shown that not all waiting-time distributions are inf div.

2. Preliminaries. We will be concerned with distributions having a probability density function (pdf) of the form

$$g(x) = \mu^{-1}(1 - F(x)), \tag{1}$$

where $F(x)$ is the distribution function of a non-negative random variable with mean $\mu > 0$. If F is a distribution function, then by F^{*k} we denote the kth convolution of F with itself. The derivative of F^{*k} will be denoted by f^{*k}. The Laplace-Stieltjes transform (LT) of distribution functions F, G, $\cdots$ are denoted by $\tilde{F}(\tau)$, $\tilde{G}(\tau)$, $\cdots$, their Fourier-Stieltjes transforms (cf's) by $\phi(t)$, $\gamma(t)$, $\cdots$. In the waiting-time examples we use Wishart's notation. We will have to consider the function $L(x)$ defined by

$$L(x) = \sum_{k=1}^{\infty} k^{-1}F^{*k}(x), \tag{2}$$

which has been studied by Smith [3].

LEMMA 1. *For all* $\tau > 0$

$$\sum_{k=1}^{\infty} k^{-1} \int_0^{\infty} e^{-\tau x}x \, dF^{*k}(x) = \int_0^{\infty} e^{-\tau x}x \, dL(x).$$

PROOF. By Fubini's theorem (partial integration) we have

$$\int_0^{\infty} e^{-\tau x}x \, dF^{*k}(x) = \int_0^{\infty} e^{-\tau x}(\tau x - 1)F^{*k}(x) \, dx. \tag{3}$$

Summation of (3) and the use of Fubini's theorem to invert the order of summation and integration yields

$$\sum_{k=1}^{\infty} k^{-1} \int_0^{\infty} e^{-\tau x}x \, dF^{*k}(x) = \int_0^{\infty} e^{-\tau x}(\tau x - 1)L(x) \, dx.$$

Using partial integration once more we obtain the required result.

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3. Renewal distributions. Examples of renewal-type cf's, i.e. cf's of the form

$$\{\phi(t) - 1\}(i\mu t)^{-1}, \tag{4}$$

which are inf div, are provided by mixtures of exponential distributions as studied in [5]. If

$$\phi(t) = \int_0^\infty \lambda(\lambda - it)^{-1}\, dG(\lambda)$$

with $G(+0) = 0$ and $\phi'(0) = i\mu$, then

$$(\phi - 1)(i\mu t)^{-1} = \int_0^\infty \lambda(\lambda - it)^{-1}\lambda^{-1}\mu^{-1}\, dG(\lambda)$$

is again the cf of a mixture of exponential distributions ($\int_0^\infty \lambda^{-1}\mu^{-1}\, dG(\lambda) = 1$) and therefore is inf div (cf. [5]).

Generally if

$$\psi(t) = \{1 - \phi(t)\}(-i\mu t)^{-1}$$

then we can write

$$\psi(t) = \lim_{\lambda\downarrow 0} \psi_o(t) = \lim_{\lambda\downarrow 0} \phi_{1\gamma}(t)/\phi_{2\lambda}(t),$$

where

$$\phi_{1\lambda}(t) = \lambda(\lambda - it\mu)^{-1}$$

and

$$\phi_{2\lambda}(t) = \lambda(\lambda + 1 - \phi)^{-1}.$$

Both $\phi_{1\lambda}$ and $\phi_{2\lambda}$ are inf div (for $\phi_{2\lambda}$ see [2] p. 203). It follows that $\psi_\lambda(t)$ has a Lévy-Khinchine representation determined by (see [2]) $\Theta_\lambda(x) = \Theta_{1\lambda}(x) - \Theta_{2\lambda}(x)$, where $\Theta_{1\lambda}$ and $\Theta_{2\lambda}$ correspond to $\Theta_{1\lambda}$ and $\Theta_{2\lambda}$, and $\Theta_{2\lambda}$ can be found in [6]. It is possible to obtain $\Theta(x) = \lim_{\lambda\downarrow 0} \Theta_\lambda(x)$ explicitly for a large class of distribution functions F. To avoid difficulties of convergence, however, it is easier to use Laplace transforms. We prove the following theorem.

THEOREM 1. *The Laplace transform of* (1)

$$\tilde{G}(\tau) = (\mu\tau)^{-1}(1 - \tilde{F}(\tau))$$

is inf div *if and only if*

$$\log x - \textstyle\sum_{k=1}^\infty k^{-1}F^{*k}(x) \quad \textit{is non-decreasing.} \tag{5}$$

PROOF. A necessary and sufficient condition for the infinite divisibility of $\tilde{G}(\tau)$ is the complete monotonicity of $-(d/d\tau)\log \tilde{G}(\tau)$ (see [1], p. 425). Now, using Lemma 1, for all $\tau > 0$ we have

$$\begin{aligned}-(d/d\tau)\log \tilde{G}(\tau) &= (1 - \tilde{F}(\tau))^{-1}(d/d\tau)\tilde{F}(\tau) + \tau^{-1}\\ &= \textstyle\sum_{k=1}^\infty (\tilde{F}(\tau))^{k-1}(d/d\tau)\tilde{F}(\tau) + \tau^{-1}\\ &= -\textstyle\sum_{k=1}^\infty k^{-1}\int_0^\infty e^{-\tau x}x\, dF^{*k}(x) + \int_0^\infty e^{-\tau x}\, dx\\ &= \int_0^\infty e^{-\tau x}x\,d(\log x - L(x)).\end{aligned}$$

By the uniqueness theorem for Laplace-transforms (see [7], p. 63) and the representation theorem for completely monotone functions ([1], p. 416) it follows that $-(d/d\tau)\log \tilde{G}(\tau)$ is completely monotone if and only if $\log x - L(x)$ is non-decreasing.

From (4) it follows that $\gamma(t)$ is not inf div if $\phi(t)$ is the cf of a lattice distribution. In that case we would have $\phi(t_0) = 1$ for some $t_0 \neq 0$ and therefore $\gamma(t_0) = 0$ contradicting the infinite divisibility. From Theorem 1 it follows that $\tilde{G}$ is not inf div if $F(x)$ is discontinuous for $x > 0$. The possibility of a jump in $x = 0$ can be excluded as $F(x)$ and $p + (1 - p)F(x)$ have the same renewal distribution. Theorem 1, however, also implies, (this was pointed out to me by Prof. J. F. C. Kingman), the following corollary.

COROLLARY. $\tilde{G}(\tau)$ *is* inf div *if and only if* $F(x)$ *is absolutely continuous and if the inequality*

$$\sum_1^\infty k^{-1} f^{*k}(x) \leqq x^{-1} \qquad (x > 0) \tag{6}$$

holds almost everywhere.

PROOF. As both $F(x)$ and $\log x - F(x)$ must be non-decreasing it follows that $F(x)$ must be absolutely continuous with respect to the measure $d \log x$. As $F(x)$ is supposed to be continuous in $x = 0$ this implies that $F(x)$ must be absolutely continuous with respect to Lebesgue measure. The inequality (6) then follows from (5). Condition (5) is not easily verified, therefore it is more useful for proving that a certain cf is not inf div than for proving that it is inf div. If we take $F_n'(x) = (n^n/(n-1)!)\, x^{n-1} e^{-nx}$ having mean 1, we have

$$S_n(x) = \sum_{k=1}^\infty k^{-1} x f_n^{*k}(x) = e^{-nx} \sum_{k=1}^\infty n (nx)^{nk}/(nk)!$$
$$= e^{-nx}\{\sum_{k=1}^n \exp(z_k n x) - n\},$$

where $z_1, \cdots, z_n$ are the roots of $z^n = 1$. We find $S_1 = 1 - \exp(-x)$, $S_2 = \{1 - \exp(-2x)\}^2$. It is easily verified that $S_3 \leqq 1$ and $S_4 \leqq 1$. For $n \geqq 5$ we have $S_n > 1$ for part of the large values of x, as then $\operatorname{Re} z_1 > 0$. For large values of n we have $f_n(1) \sim n^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}}$ which contradicts the necessary condition $xf(x) \leqq 1$. It can also be seen without computation that the renewal distribution corresponding to $F_n'(x) = n^n\{(n-1)!\}^{-1} x^{n-1} \exp(-nx)$ cannot be inf div for all n. As the latter distribution tends to the degenerate distribution it would follow from the closure property of inf div distributions that the uniform distribution is inf div, which is contradicted by the fact that it has a bounded support (see e.g. [1] p. 174). It follows that an inf div distribution need not yield an inf div renewal distribution. We will now give an example of a distribution, which has an inf div renewal distribution though it is not inf div itself. The cf (cf. [5])

$$\phi(t) = 2/(1 - it) - 6/(3 - it) + 5/(5 - it)$$
$$= (15 - t^2)(1 - it)^{-1}(3 - it)^{-1}(5 - it)^{-1}$$

is not inf div as it has real zeros. For the renewal cf we have

$$(\phi - 1)(i\mu t)^{-1} = \mu^{-1}(23 - 8it - t^2)(1 - it)^{-1}(3 - it)^{-1}(5 - it)^{-1},$$

which has a canonical representation (2), where $\Theta(x)$ satisfies

$$\Theta'(x) = \exp(-x) + \exp(-3x) + \exp(-5x) + 2\cos 7^{\frac{1}{2}}x \exp(-4x),$$

which is positive (compare [5]). It follows that $(\phi - 1)(i\mu t)^{-1}$ is inf div.

4. Monotone densities. If $g(x)$ is a probability density on $(0, \infty)$ with the properties

(a) $g(x)$ is non-increasing on $(0, \infty)$,

(b) $g(+0) = a < \infty$,

then $g(x)$ can be written as $g(x) = \mu^{-1}(1 - F(x))$, where $F(x)$ is a distribution function on $[0, \infty)$ with mean $\mu = a^{-1}$. The cf of $g(x)$ is then given by $\{1 - \phi(t)\}(-i\mu t)^{-1}$, where ϕ is the cf of F. From Theorem 1 deduce

THEOREM 2. *If $g(x)$ is a probability density satisfying conditions* (a) *and* (b), *then*

(i) *$g(x)$ is not* inf div *if $g(x)$ is not absolutely continuous on* $(0, \infty)$,

(ii) *if $g(x)$ has a derivative $g'(x)$, then a necessary condition for $g(x)$ to be* inf div *is that*

$$-g'(x) \leqq x^{-1}g(+0). \tag{6}$$

Examples of pdf's which by this criterion are not inf div are given by

$$g(x) = c_n \exp(-x^n) \qquad (x > 0)$$

for $n > e$. Here we have $-g'(x) = nx^{n-1}g(x)$ and $g(+0) = c_n = n^{-1}\{\Gamma(n^{-1})\}^{-1}$. The necessary condition of Theorem 2 now reduces to

$$nx^n \leqq \exp(x^n).$$

which is not satisfied for $x^n = \log n$ and $n > e$. In general condition (6) implies that $g(x)$ should not decrease too sharply and in particular that $g'(x)$ should be bounded in every interval $[\delta, \infty)$ with $\delta > 0$.

5. A non-inf div waiting-time distribution. It is well-known that many waiting-time distributions are inf div. As the Lindley case always yields inf div waiting-time distributions (cf. [4], p. 150 and [2], p. 203) we will have to look elsewhere for waiting-time distributions, which are not inf div. In the last-come first-served case as treated by Wishart [8] for the Laplace transform (LT) $\gamma(\tau)$ of the waiting-time distribution we have

$$\gamma(\tau) = 1 - \rho + \lambda\{1 - \Gamma(t)\}\{\tau + \lambda - \lambda\Gamma(\tau)\}^{-1}. \tag{7}$$

Here $\Gamma(\tau)$ is the LT of the busy period distribution determined by $\Gamma(\tau) = \beta(\tau + \lambda - \lambda\Gamma(\tau))$, β denoting the LT of the service time distribution. For details we refer to [8]. The continuous part of the waiting-time distribution has LT

$$\begin{aligned}\gamma_c(\tau) &= \rho^{-1}\lambda\{1 - \Gamma(\tau)\}\{\tau + \lambda - \lambda\Gamma(t)\}^{-1}\\ &= \rho^{-1}\lambda\{1 - \beta(\tau + \lambda - \lambda\Gamma(\tau)\}\{\tau + \lambda - \lambda\Gamma(\tau)\}^{-1}\end{aligned}$$

which is inf div if $\{1 - \beta(\tau)\}\tau^{-1}\mu^{-1}$ is inf div. This follows from the fact that

$\exp -\{\tau + \lambda(1 - \Gamma)\}$ is an inf div LT and that $\phi(-\log \psi(\tau))$ is an inf div LT if both ϕ and ψ are (cf. [1] p. 427). On the other hand if $\beta(\tau)$ is the LT of a lattice distribution then the same holds for $\Gamma(\tau)$. It follows that $\Gamma(\tau) = 1$ for some value $\tau = it$, where t is real and $t \neq 0$. But then $\gamma_c(it) = 0$, which implies that $\gamma_c(\tau)$ is not inf div. This is also true for $\rho = 1$, i.e. for $\gamma(\tau)$ in the case $\rho = 1$. This however means that $\gamma(\tau)$ cannot be inf div for all $\rho < 1$ as this would imply the infinite divisibility for $\rho = 1$. An example of a waiting-time distribution, which is not inf div is provided by the LT (take $\lambda = \rho = 1$ and $\beta(\tau) = \exp(-\tau)$ in (7))

$$\{1 - \Gamma(\tau)\}\{\tau + 1 - \Gamma(\tau)\}^{-1},$$

where by Lagrange's theorem

$$\Gamma(\tau) = \textstyle\sum_{k=1}^{\infty} (1/k!)\, k^{k-1} \exp -k(\tau + 1).$$

In the first-come-first-served case (Pollaczek-Khintchine) we have

$$\gamma(\tau) = 1 - \rho + \lambda(1 - \rho)\{1 - \beta(\tau)\}\{\tau - \lambda + \lambda\beta(\tau)\}^{-1}.$$

Here also the continuous part of the waiting time distribution is not inf div if $\beta(\tau)$ is a lattice LT, although $\gamma(\tau)$ itself is always inf div. The conclusion that $\gamma(\tau)$ is not inf div for all $\rho < 1$ cannot be drawn, because $\gamma(\tau)$ does not converge to the LT of a distribution function for $\rho \to 1$. Again, when $\alpha(\tau) = (\tau\mu)^{-1}(1 - \beta(\tau))$ is inf div, then $\gamma_c(\tau)$ is inf div. In this case $\gamma_c(\tau)$ has the form $\gamma_c(\tau) = \alpha(\tau)p(p + 1 - \alpha(\tau))^{-1}$, which is the product of two inf div LT's.

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