On Self-Complementation

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ABSTRACT

We prove that, with very few exceptions, every graph of order n, $n \equiv 0, 1 \pmod 4$ and size at most $n - 1$, is contained in a self-complementary graph of order n. We study a similar problem for digraphs.

Throughout the paper, G and D will denote a finite graph and a finite digraph, respectively, without loops or multiple edges, with vertex-sets $V(G)$ and $V(D)$, and edge-sets $E(G)$ and $E(D)$; define $e(G) = |E(G)|$, $e(D) = |E(D)|$. An edge of G joining x and y is denoted by xy, an edge of D from z to t by (z,t), and a symmetric edge of D joining u and v by uv. C_i denotes a cycle of G of length $i \geq 3$. $G \cup H$ will refer to two vertex disjoint graphs G and H, and mG to m disjoint copies of G. If $A \subset V(G)$, then $G - A$ is the subgraph of G induced by $V(G) - A$.

G is said to be a self-complementary graph (or s.c. graph) if it is isomorphic to its complement $\overline{G}$, then, there exists a permutation σ of $V(G)$, called s.c. permutation of G, such that xy is an edge of G if and only if $\sigma(x)\sigma(y)$ is an edge of $\overline{G}$ (for simplicity, we use the notation $\sigma(xy) = \sigma(x)\sigma(y)$). We say that G is contained in a graph G' if there exists a subgraph in G' isomorphic to G.

The same is applicable to D.

It is known [6,7,8] that if G is an s.c. graph of order n, then $n \equiv 0, 1 \pmod 4$, and its s.c. permutation has all its cycles of lengths being multiples of 4 (except one of length one if n is odd), and lengths of cycles of an s.c. permutation of an s.c. digraph D are even (except one of length one if the order of D is odd). Furthermore, if G is an s.c.

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graph with s.c. permutation σ, then $xy \in E(G)$ if and only if $\sigma^m(xy) \in E(G)$ for m even and $\sigma^m \in E(\overline{G})$ for m odd.

In [3], there is proved that every s.c. graph contains a hamiltonian path and s.c. graphs with 2-factors have been characterized in [5]. Using similar arguments [10], one can prove that every s.c. digraph contains a hamiltonian path (some edges can be oriented the wrong way) and that if an s.c. permutation of D has all the cycles of length at least 4, then D contains a 2-factor. If σ is any permutation whose cycles are even, except possibly one of length one, and if one of the cycles of σ has length at most 2, then there exist two digraphs D and D' such that σ is an s.c. permutation of D and D', D has a 2-factor and D' has no 2-factor.

In [4], it was proved that every graph of order n and size at most n is contained in its complement except for some exceptions (see also [1,2,9]).

We need Lemmas 1 and 3 to prove that, with very few exceptions, every graph of order n, $n \equiv 0, 1 \pmod 4$ and size at most $n - 1$ is contained in an s.c. graph of order n, and that every digraph of order n and size at most n is contained in an s.c. digraph of order n.

Lemma 1. Let G be a graph of order n, $n \equiv 0, 1 \pmod 4$. Then, G is contained in an s.c. graph of order n if and only if there exists a permutation σ of $V(G)$ whose cycles have length equal to a multiple of 4 (except one of length one if n is odd) so that $\sigma^{2m+1}(xy) \in E(\overline{G})$ for every $xy \in E(G)$.

Proof. Let σ be a permutation as defined in Lemma 1. We construct two sets of edges E_b and E_r by the following algorithm:

(i) For every $xy \in E(G)$, assign $\sigma^{2m}(xy)$ to E_b, and assign $\sigma^{2m+1}(xy)$ to E_r, for all $m \geq 0$.

(ii) If there are vertices $z,t \in V(G)$ such that $zt \notin E_b \cup E_r$, then assign $\sigma^{2m}(zt)$ to E_b and $\sigma^{2m+1}(zt)$ to E_r, for all $m \geq 0$.

(iii) If, after E_b and E_r have been enlarged by step (ii), there is still another pair $z,t \in V(G)$ such that $zt \notin E_b \cup E_r$, then repeat step (ii), until all edges of the complete graph constructed on $V(G)$ lie in E_b or E_r.

Clearly, the graph G_1 defined by $V(G_1) = V(G)$, $E(G_1) = E_b \cup E_r$ is complete, the graph G' defined by $V(G') = V(G)$, $E(G') = E_b$ is an s.c. graph and σ is an s.c. permutation of G', whose complement is the graph G'' defined by $V(G'') = V(G)$ and $E(G'') = E_r$.

If G is contained in an s.c. graph, the existence of σ is given by definition. ∎

If the permutation σ of Lemma 1 exists, then σ will be also called s.c. permutation of G. So, the s.c. permutations below will refer to σ of Lemma 1 and we denote by $\sigma(G)$ an arbitrary s.c. permutation (if it exists) of G.

Theorem 1. Let G be a graph of order n, $n \equiv 0, 1 \pmod 4$. If $e(G) \leqslant n - 1$, then G is contained in an s.c. graph of order n, unless G is isomorphic to $K_{1,n-1}$, $C_3 \cup K_1$, $C_3 \cup K_{1,n-4} (n \geqslant 5)$, or $C_4 \cup K_1$.

Proof. We shall prove a stronger result, that is: G is contained in an s.c. graph whose s.c. permutation has all its cycles of length 4 (unless one of length one if n is odd), except if G is one of the forbidden graphs.

The proof is by induction on n. The theorem is true for $n = 4$ and for $n = 5$, so, let $n \geqslant 8$, $n \equiv 0, 1 \pmod 4$ and assume it is true for every graph of order $p < n$ and size at most $p - 1$, $p \equiv 0, 1 \pmod 4$. Consider a graph G of order n and size at most $n - 1$.

We assume that G has precisely $n - 1$ edges.

We easily verify that $K_{1,n-1}$, $C_3 \cup K_1$, $C_3 \cup K_{1,n-4} (n \geqslant 5)$, $C_4 \cup K_1$ are not contained in their complement, a fortiori in any s.c. graph. Then, we assume that G is not isomorphic to any of these graphs.

Property (P) and Lemma 2 given below are very useful to prove most of the cases.

Property (P). G has four vertices x, y, z, t such that if $G' = G - \{x,y,z,t\}$, then $d(x,G') = d(t,G') = 0$ [resp. $d(y,G') = d(z,G') = 0$], and xz, xt, $yt \notin E(G)$, and there exists an s.c. permutation of G'. (Observe that if $e(G') \leqslant n - 6$ then an s.c. permutation of G' exists.)

Lemma 2. If for x, y, z, $t \in V(G)$ Property (P) holds, then there is an s.c. permutation of G.

Proof of Lemma 2. We use Lemma 1 to show that for $\sigma' = \sigma(G')$, $\sigma = \sigma' o(x,y,t,z)$ is an s.c. permutation of G.

Indeed, if $d(x,G') = d(t,G') = 0$, then $\sigma(xy) = \sigma^3(zt) = yt \notin E(G)$, $\sigma(zt) = \sigma^3(xy) = xz \notin E(G)$, $\sigma(yz) = \sigma^3(yz) = xt \notin E(G)$, and for every $u \in V(G')$, $\sigma(yu) = t\sigma'(u) \notin E(G)$, $\sigma^3(yu) = x\sigma'^3(u) \notin E(G)$, $\sigma(zu) = x\sigma'(u) \notin E(G)$, $\sigma^3(zu) = t\sigma'^3(u) \notin E(G)$.

The argument is similar if $d(y,G') = d(z,G') = 0$. ∎

In particular, the components of G which are cycles will be assumed to have 3 or 4 vertices and we shall assume that there is at most one isolated vertex in G.

Suppose first that G is connected. For $e(G) = n - 1$, G is a tree, and since $G \neq K_{1,n-1}$, there are two endvertices p and q joined by a path $(p,x_1, \ldots, x_k, q)$ $k \geqslant 2$. So, either $G' - \{p,q,x_1,x_k\}$ admits an s.c. permutation and hence Property (P) holds for p,x_1,x_k,q or $G' = K_{1,n-5}$ and Property (P) holds for p,x_1, the vertex y of G' of maximum degree in G' and an endneighbor of y.

So assume G is not connected, and let $T_1, \ldots, T_\alpha$ ($\alpha \geqslant 1$) be the tree-components of G, and $H_1, \ldots, H_\beta$ ($\beta \geqslant 1$) the other components.

Case 1. G contains an isolated edge ab.

Then every vertex of degree at least 3 [resp. at least 2] is joined to every vertex of degree at least 1 (except a,b) [resp. at least 2], otherwise G has Property (P) and Lemma 2 applies.

Since $n \geq 8$, there is a vertex x of degree at least 3. Then $d(x,G) = n - 3 - k$, where k is the number of isolated vertices ($k \leq 1$). G has Property (P) for the vertices a,b,x and every vertex $t \neq a,b$ of degree at most 1.

Thus, in the following cases, we can assume that the number of vertices in T_i is different from 2, for $i = 1, \ldots, \alpha$.

Case 2. G contains an isolated vertex a.

We consider the following subcases:

Subcase 2.1. $d(x,G) \leq 2$ for every $x \in V(G)$. Then $\alpha = 1$ and $H_i = C_3$ or C_4 for $i = 1, \ldots, \beta$.

Subcase 2.1.a. $n = 8$ or $n = 9$, then G contains a cycle $C = (x_1, \ldots, x_4, x_1)$. Let y_1, y_2, y_3 be three consecutive vertices of the second cycle C'. Then $\sigma(G) = \mathscr{C} o (x_1,x_2,y_1,y_2) o (x_3,x_4,y_3,y_4)$, where $\mathscr{C} = \emptyset$ and $y_4 = a$ if $n = 8$ and $\mathscr{C} = (a)$ and y_4 is the fourth vertex of C' if $n = 9$.

Subcase 2.1.b. $n \geq 12$ and G contains a cycle $C = (x_1, \ldots, x_4, x_1)$ of length 4. Consider three vertices y_1, y_2, y_3 of a cycle $C' \neq C$ of G and a vertex $y_4 \notin V(C) \cup V(C')$ of degree at least 2.

Choose $\sigma(G) = \sigma(G') o (x_1,y_1,x_2,y_4) o (x_4,x_3, y_2,y_3)$, where $G' = G - \{x_1, \ldots, x_4, y_1, \ldots, y_4\}$.

Subcase 2.1.c. $n \geq 12$ and (x_1,x_2,x_3,x_1), (y_1,y_2,y_3,y_1) and (z_1,z_2,z_3,z_1) are three triangles of G.

Put $\sigma(G) = \sigma(G') o (x_1,x_2,y_1,z_1) o (y_2,y_3,x_3,z_2)$, where $G' = G - \{x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2\}$.

Subcase 2.2. G contains a vertex of degree at least 3. Then it contains an endvertex b. Let x be the neighbor of b, thus $d(x,G) \geq 2$.

If $d(x,G) = n - 2$ then (P) holds for a,b,x, and another endpoint.

If $4 \leq d(x,G) < n - 2$ then G has Property (P) for a,t,x,b, where t is a vertex nonadjacent to x.

If $d(x,G) = 3$ one can prove that there exists in G a vertex y such that $e(G - \{a,b,x,y\}) \leq n - 6$ so that (P) holds.

If $d(x,G) = 2$, then we can assume that the neighbor c ($\neq b$) of x has degree 3, otherwise (P) applies to a,b,x,c or a,b,x and a vertex of degree at least 3. Then (P) holds for c,x,b and a vertex y ($\neq a$) not joined to c.

Observe that we assume there is exactly one tree-component in G. Indeed, suppose that $\alpha \geq 2$. Then T_i contains an endvertex p_i with neighbor x_i, $d(x_i,G) \geq 2$. $G' = G - \{x_1,p_1,p_2,x_2\}$ is a graph with $n - 4$ vertices and at most $(n - 4) - 1$ edges. G' is not a forbidden graph, so there is an s.c. permutation of G' and $\sigma(G)$ exists by Lemma 2.

Hence, the only not yet considered case is the following

Case 3. G is a union of a tree T having at least 3 vertices and disjoint cycles H_j, $j = 1, \ldots, \beta$ of length 3 or 4.

Let $m = |V(T)|$.

Subcase 3.1. $T \neq K_{1,m-1}$. Then there exists in T a path $(p,x_1, \ldots, x_k,q)$, $k \geq 2$, joining two endvertices p,q. If the degree of x_1 [resp. x_k] is equal to 2, then (P) holds for v, p, x_1, x_2 [resp. x_{k-1},x_k,q,v] where $v \notin V(T)$. If the degrees of x_1 and x_k are both greater than 2, G has Property (P) for p, x_1, x_k, q.

Subcase 3.2. $T = K_{1,m-1}$, $\beta \geq 2$. If G contains a cycle $(x_1, \ldots, x_4,x_1)$ of length 4, then an s.c. permutation is similar to those of subcase 2.1.b.

If every cycle of G has length 3, then let $C = (x_1,x_2,x_3,x_1)$, $C' = (y_1,y_2,y_3,y_1)$ be two cycles of G. If there exist two vertices $z_1,z_2 \notin V(C) \cup V(C')$, so that $d(z_1,G) + d(z_2,G) \geq 4$, then an s.c. permutation is similar to those of subcase 2.1.c. If such z_1,z_2 do not exist, then $G = K_{1,2} \cup 2C_3$ and an s.c. permutation is also similar.

Subcase 3.3 $T = K_{1,m-1}$, $\beta = 1$. Then $G = K_{1,n-5} \cup C_4$. Let x_1 be the vertex of maximum degree in $K_{1,n-5}$, x_2, x_3, x_4 three neighbors of x_1 and $C_4 = (x_5, \ldots, x_8,x_5)$. We choose $\sigma(G) = \sigma(G')o(x_2,x_1,x_5,x_6)o(x_3,x_4 x_7,x_8)$, where $G' = G - \{x_1, \ldots, x_8\}$. ∎

In [2], Burns and Schuster showed that every graph G of order n and size at most $n - 1$ is contained in its complement unless G is isomorphic to one of $K_{1,n-1}, C_3 \cup K_1$, $C_3 \cup K_{1,n-4}$ $(n \geq 5)$, $C_4 \cup K_1$ or $2C_3 \cup K_1$.

This result is a corollary of Theorem 1 for $n \equiv 0, 1 \pmod 4$. However, it can be proved by exactly the same arguments as Theorem 1.

Lemma 3. Let D be a digraph of order n. Then D is contained in an s.c. digraph of order n if and only if there exists a permutation σ of $V(D)$ whose cycles are even (except one of length one if n is odd) so that $\sigma^{2m+1}(xy) \in E(\overline{D})$ for every $(x,y) \in E(D)$.

Proof. The proof is similar to those of Lemma 1. ∎

Theorem 2. Every digraph with $n \geq 3$ vertices and at most n edges is contained in an s.c. digraph of order n.

Proof. The proof is by induction on n. The Theorem is obvious for $n = 3$ or 4, so assume that $n \geq 5$ and the theorem is true for every digraph of order $p < n$ and size at most p. Let D be a digraph of order n and size at most n. We can assume that D has n edges.

Assume that D contains an isolated vertex a, then there exists a vertex b of degree at least 2, so, the induction applies to $D' = D - \{a,b\}$ and $\sigma(D')o(a,b)$ is an s.c. permutation of D.

Assume now that D contains a vertex c of degree 1; take without loss of generality $(c,e) \in E(D)$. One can assume that $d(e,D) = 1$, otherwise $\sigma(D) = \sigma(D - \{c,e\})o(c,e)$. Let x be a vertex of degree at least 3; then either there exists $y \neq c$, e not joined to x, or x is joined to every vertex (except c,e) in which case there exists an antisymmetric edge incident with x, for example (x,y) [the case where $n = 5$ is trivial], in both cases, $D - \{c,e,x,y\}$ admits an s.c. permutation σ and $\sigma(D) = \sigma o(c,y)o(e,x)$.

Assume finally that $d(x,D) \geq 2$ for every $x \in V(D)$. Clearly, $d(x,D) = 2$ for every $x \in V(D)$. If all the cycles of D have length 2, then let x_iy_i, $i = 1, \ldots, n/2$, be the edges of D, then $\sigma(D) = (x_1,y_2)o(x_2,y_3)o \ldots o(x_{n/2},y_1)$. If $C = (a_1,a_2, \ldots, a_k,a_1)$ is a cycle of D of length $k \geq 3$ (some edges can be oriented the wrong way), then it is easy to see that either there exists a directed path or an antidirected path with vertex-set $\{a_{i-1},a_i,a_{i+1},a_{i+2}\}$ and $\sigma(D) = \sigma(D - \{a_i,a_{i-1}\})o(a_i,a_{i+1})$ or D contains a source a_i and a sink a_j, $j \neq i, i+1, i-1$ and $\sigma(D) = \sigma(D - \{a_i,a_j\})o(a_i,a_j)$. ∎

Theorem 2 implies that every digraph of order n and size at most n is contained in its complement.

We conjecture that every digraph of order n and size at most $2n - 3$ is contained in an s.c. digraph of order n unless n is even and D is isomorphic to the digraph D' or it's converse, where D' is defined by $V(D') = \{x_1, \ldots, x_n\}$ and $E(D') = \{x_1x_2,x_1x_3, \ldots, x_1x_{n-2},(x_1,x_{n-1}),(x_1,x_n),(x_{n-1},x_n)\}$.

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