

## ON CHOICE SEQUENCES DETERMINED BY SPREADS

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**§1. Introduction.** From the moment choice sequences appear in Brouwer's writings, they do so as elements of a spread. This led Kreisel to take the so-called axiom of spreaddata as the basic axiom in a formal theory of choice sequences (Kreisel [1965, pp. 133–136]). This axiom expresses the idea that to be given a choice sequence means to be given a spread to which the choice sequence belongs. Subsequently, however, it was discovered that there is a formal clash between this axiom and closure of the domain of choice sequences under arbitrary (lawlike) continuous operations (Troelstra [1968]). For this reason, the formal system CS was introduced (Kreisel and Troelstra [1970]), in which spreaddata is replaced by analytic data. In this system CS, the domain of choice sequences is closed under all continuous operations, and therefore it provides a workable basis for intuitionistic analysis. But the problem whether the axiom of spreaddata is compatible with closure of the domain of choice sequences under the continuous operations from a restricted class, which is still rich enough to validate the typical axioms of continuous choice, remained open. It is precisely this problem that we aim to discuss in this paper.

Recall that a *spread* is a (lawlike, inhabited) decidable subtree  $S$  of the tree  $\mathbb{N}^{<\mathbb{N}}$  of all finite sequences, having all branches infinite:

- (i)  $\forall u, v (u \in S \ \& \ v \leq u \rightarrow v \in S),$
- (ii)  $\forall u \exists n (u \in S \rightarrow u * \langle n \rangle \in S).$

(Unless otherwise stated, all notational conventions are as in van der Hoeven and Moerdijk [1981], henceforth [HM]; so  $u, v$  range over finite sequences,  $n, m$  over natural numbers,  $\alpha, \beta, \xi, \eta$  over elements of the domain of choice sequences, and  $*$  is used for concatenation.) A spread  $S$  determines a subset of  $\mathbb{N}^{\mathbb{N}}$ , also called  $S$ , by

$$\alpha \in S \Leftrightarrow \forall n \bar{\alpha}(n) \in S.$$

Kreisel's axiom of *spreaddata* now reads

$$A(\alpha) \rightarrow \exists \text{ spread } S (\alpha \in S \ \& \ \forall \beta \in S \ A(\beta)),$$

where  $A(\alpha)$  contains no free variables for choice sequences other than  $\alpha$ ; all other

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parameters should be lawlike. Every spread contains sequences, i.e. we have the *density axiom*

$$\forall \text{ spread } S \exists \alpha \alpha \in S.$$

Other typical axioms are continuity principles for quantifier combinations of the form  $\forall \alpha \exists n$  and  $\forall \alpha \exists \beta$ . (In the system CS the axiom of spreaddata is replaced by the axiom

$$A(\alpha) \rightarrow \exists \text{ lawlike continuous } f (\alpha \in \text{range}(f) \ \& \ \forall \beta \in \text{range}(f) A(\beta))$$

of *analytic data*. In other words, spreads are replaced by images of lawlike continuous functions. In CS, the density axiom is redundant, since the universe of choice sequences is closed under application of an arbitrary continuous operation.)

In this paper we will present a model for a theory of choice sequences containing the axiom of spreaddata. This model has all the desired properties: besides spreaddata and the density axiom, it satisfies  $\forall \alpha \exists n$ -continuity,  $\forall \alpha \exists \beta$ -continuity, bar induction, and the specialization property. Furthermore, the domain of choice sequences is closed under application of all lawlike continuous operations from a certain subclass  $S \subseteq \mathcal{K}$ . Every spread is the image of a function in  $S$ :

$$\forall \text{ spread } S \exists f \in S \text{ im}(f) = S.$$

The density axiom is an immediate consequence of this, and we also get relativized continuity principles for quantifier combinations of the form  $\forall \alpha \in S \exists n$ ,  $\forall \alpha \in S \exists \beta$ , and relativized bar induction. Finally, an axiom of pairing holds in the model. The model will be similar to the models presented in [HM], and we will assume that the reader has some familiarity with the techniques used in §2 of that paper. As shown in [HM], the elimination translation for CS is a special case of such a model. The model we present here, however, does not lead to a similar elimination translation based on spreaddata rather than analytic data. This will be pointed out in a final section, where we will also briefly discuss the relation of this model to other models for spreaddata that have occurred in the literature.

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**§2. Description of the model.** Our model will be similar to the sheafmodels for the systems CS( $M$ ) of [HM]. These systems contain an axiom of “relativized analytic data”,

$$A(\alpha) \rightarrow \exists f \in M (\alpha \in \text{range}(f) \ \& \ \forall \beta \in \text{range}(f) A(\beta)),$$

where  $M$  is a fixed monoid of lawlike continuous operations. In these models, “lawlike” is interpreted as “external” or “constant” (that is, as lying in the image of the “constant sets functor”  $\Delta: \text{Sets} \rightarrow \text{Sh}(\mathbf{C})$ ,  $\mathbf{C}$  a site,  $\Delta$  left adjoint to the global sections functor).

Assuming this interpretation of lawlike objects, a lawlike spread is just a spread given in Sets, and when working in a classical metatheory, in the models all lawlike spreads will automatically be decidable. In Sets, spreads correspond to closed

nonempty subspaces of Baire-space: every spread  $S \subseteq \mathbb{N}^{<\mathbb{N}}$  determines a closed subspace  $\{x \in \mathbb{N}^{\mathbb{N}} \mid \forall n \bar{x}(n) \in S\}$  of Baire-space, and conversely, to each closed set  $T \subseteq \mathbb{N}^{\mathbb{N}}$  we can assign a spread  $\{\bar{x}(n) \mid n \in \mathbb{N}, x \in T\}$ . These processes are inverse to each other.

We will begin the construction of our model by describing a class of mappings from Baire-space to itself which map spreads to spreads (i.e. are closed mappings) by *retracting* every spread onto its image:

DEFINITION 1. A closed continuous function  $f: B \rightarrow B$  is called a *CHR-mapping* (closed-hereditary retraction mapping) if for any closed subset  $F \subseteq B$ , the restriction  $f \upharpoonright F: F \rightarrow f(F)$  has a continuous right inverse  $i_F: f(F) \rightarrow F$ ; that is,  $f \circ i_F = \text{id}_{f(F)}$ .

Note that if  $f: B \rightarrow B$  is CHR, each inverse  $i_F: f(F) \rightarrow F$  is also a closed mapping. For if  $G$  is a closed subset of  $f(F)$  and  $\{x_n\}_n$  is a sequence of points from  $G$  such that  $\{i_F(x_n)\}_n$  converges to  $p$ , then  $\{f i_F(x_n)\}_n = \{x_n\}_n$  converges to  $f(p)$ , so  $f(p) \in G$ , and  $i_F f(p) = \lim_n i_F f i_F(x_n) = \lim_n i_F(x_n) = p$ ; hence also  $p \in i_F(G)$ .

Examples of CHR-mappings are closed homeomorphic embeddings and constant functions. In fact, the CHR-mappings form a monoid:

LEMMA 2. *The composition of two CHR-mappings is again a CHR-mapping.*

PROOF. If  $f$  and  $g$  are CHR and  $F \subseteq B$  is a closed set, then we find right inverses  $i_F: f(F) \rightarrow F$  and  $j_{f(F)}: g f(F) \rightarrow f(F)$  for  $f$  and  $g$  respectively, so  $i_F \circ j_{f(F)}$  is a right inverse for  $g \circ f$ . □

The key property of CHR-mappings is expressed by the following lemma.

LEMMA 3 (FACTORIZATION LEMMA). *Let  $f$  and  $g: B \rightarrow B$  be CHR-mappings, and suppose  $\text{im}(g) \subseteq \text{im}(f)$ . Then there exists a CHR-mapping  $h: B \rightarrow B$  such that  $g = f \circ h$ .*

PROOF. Let  $i: f(B) \rightarrow B$  be a right inverse for  $f: B \rightarrow f(B)$ , and define  $h$  to be the function  $i \circ g$ .

$$\begin{array}{ccc}
 B & \xrightarrow{g} & g(B) \\
 \downarrow h & & \cap \\
 B & \xrightleftharpoons[i]{f} & f(B)
 \end{array}$$

Obviously  $f \circ h = g$ . To show that  $h$  is a CHR-mapping, choose a closed subset  $H \subseteq B$ . Since  $g$  is CHR, we can find a map  $k: g(H) \rightarrow H$  such that the composite  $g(H) \xrightarrow{k} H \xrightarrow{g \upharpoonright H} g(H)$  is the identity map. Now let  $j: h(H) = ig(H) \rightarrow H$  be the composite  $k \circ f: ig(H) \rightarrow fg(H) = g(H) \xrightarrow{k} H$ . Then  $h \circ j = \text{id}_{h(H)}$ , for if  $x \in h(H)$ , then  $x = i(y)$  for some  $y \in g(H) \subseteq f(B)$ , so  $hj(x) = igkf(x) = igkfi(y) = igk(y) = i(y) = x$ . Thus  $h$  is a CHR-mapping. □

In order to get a model which has the properties as described in the Introduction, we need a sufficient supply of CHR-mappings. For each spread  $S$  we will define a CHR-mapping  $\tilde{S}$  which retracts  $B$  onto  $S$ .

The points of  $B$  carry a natural linear ordering given by  $x < y$  iff  $x(n) < y(n)$  for the smallest  $n$  at which  $x$  and  $y$  differ. If  $x < y$  we will say that  $x$  is to the left of  $y$ .

Let  $S$  be a closed subspace of  $B$ . As noted earlier,  $S$  can also be regarded as a set of finite sequences  $\{u \mid \exists x \in S \underline{x} \in u\}$ . We define the function  $\tilde{S}$  as follows (for each  $x \in B$  we define initial segments  $\tilde{S}(x)(n)$  of length  $n$  by induction).  $\tilde{S}(x)(0) = \langle \rangle$  of course,

and

$$\overline{\tilde{S}(x)}(n+1) = \begin{cases} \bar{x}(n+1) & \text{if } \bar{x}(n+1) \in S, \\ \overline{\tilde{S}(x)(n) * \langle m \rangle} & \text{otherwise, where } m \text{ is the least number} \\ & \text{for which } \overline{\tilde{S}(x)(n) * \langle m \rangle} \in S. \end{cases}$$

Thus, when we think in terms of the tree  $\mathbb{N}^{<\mathbb{N}}$ ,  $\tilde{S}(x)$  is that path in  $S$  which is equal to  $x$  as long as this is possible, and then picks out the leftmost branch in  $S$ . (Later on,  $\tilde{S}$  will also give an internal function from choice sequences to choice sequences, and in the model it will hold that  $\alpha \in S \leftrightarrow \exists \beta \alpha = \tilde{S}(\beta)$ .)

LEMMA 4. For each closed  $S \subseteq B$ ,  $\tilde{S}$  is a uniformly continuous closed retraction of  $B$  onto  $S$ .

PROOF. Uniform continuity of  $\tilde{S}$  is clear, since we need only the initial segment  $\bar{x}(n)$  of  $x$  to define  $\overline{\tilde{S}(x)}(n)$ . And if  $x \in S$ ,  $\tilde{S}(x) = x$ , so  $\tilde{S}$  retracts  $B$  onto  $S$ .

To see that  $\tilde{S}$  is closed, suppose  $F \subseteq B$  is closed, and  $\{y_n\}_n$  is a sequence of points in  $F$  such that  $\{\tilde{S}(y_n)\}_n$  converges to a point  $p$ . We need to show that  $p \in \tilde{S}(F)$ . Since  $\tilde{S}(y_n) \rightarrow p$ ,

$$(*) \quad \forall k \exists n_k \forall n \geq n_k \overline{\tilde{S}(y_n)}(k) = \bar{p}(k).$$

We now distinguish two cases:

1) If  $\forall k \exists m_k \forall m \geq m_k \bar{y}_k(k) \in S$ , then the sequence  $\{y_n\}_n$  also converges to  $p$ . So  $p$  must lie in the closed set  $F$ , and  $\tilde{S}(p) = p$ .

2) Otherwise there exists a  $k_0$  such that the set  $M = \{m \mid \bar{y}_m(k_0) \notin S\}$  is infinite. Since  $S$  is a tree, also for each  $k \geq k_0$  and  $m \in M$ ,  $y_m(k) \notin S$ . By (\*) we find for this  $k_0$  that  $\forall n \geq n_{k_0} \overline{\tilde{S}(x_n)}(k_0) = \bar{p}(k_0)$ . But then for  $m \in M$ ,  $m \geq n_{k_0}$ , and  $k \geq k_0$ ,  $\overline{\tilde{S}(x_m)}(k)$  is the leftmost extension of  $\bar{p}(k_0)$  in  $S$ , and hence no longer depends on  $m$ . Thus the sequence  $\{\tilde{S}(y_n)\}_n$  contains a constant subsequence, necessarily having value  $p$ . Therefore also in this case,  $p \in \tilde{S}(F)$ .  $\square$

PROPOSITION 5. For every closed  $S \subseteq B$ , the function  $\tilde{S}$  is a CHR-mapping.

PROOF. Let  $S$  be a closed subset of  $B$ . We want to define a right inverse  $i_F: \tilde{S}(F) \rightarrow F$  for each restriction  $\tilde{S} \upharpoonright F: F \rightarrow \tilde{S}(F)$  of  $\tilde{S}$  to a closed set  $F$ .

If  $x \in S$ , we call  $x$  a leftmost point in  $u$  if  $u$  is an initial segment of  $x$  and for each  $n > 1\text{th}(u)$ ,  $x(n)$  is the smallest  $m$  such that  $\bar{x}(n) * \langle m \rangle \in S$  (in other words,  $x$  is the leftmost branch in the tree  $\{v \mid u \leq v \text{ \& } v \in S\}$ ). To define  $i_F$  we consider three types of points in  $\tilde{S}(F)$ .

(1) If  $x \in \tilde{S}(F)$  and for none of its initial segments  $u$ ,  $x$  is a leftmost point in  $u$ , then  $\tilde{S}^{-1}(x)$  consists of precisely one point, viz.  $x$  itself, so  $x \in F$ , and putting  $i_F(x) = x$  is the only thing we can do.

(2) If  $x \in \tilde{S}(F)$  and  $x$  is a leftmost point in one of its initial segments  $u$ , while  $x$  is not isolated in  $\tilde{S}(F)$ , we also put  $i_F(x) = x$ . Indeed,  $x \in F$  in this case, since from the fact that  $x$  is not isolated in  $\tilde{S}(F)$  we conclude that there is a sequence  $\{y_n\}_n$  of points in  $F$  such that  $\{\tilde{S}(y_n)\}_n$  converges to  $x$ , while for no  $n$  do we have  $\tilde{S}(y_n) = x$ . In particular, no subsequence of  $\{\tilde{S}(y_n)\}_n$  is constant. Therefore it follows as in the proof of Lemma 4 that the points  $y_n$  also converge to  $x$ . Each  $y_n$  is in the closed set  $F$ , hence so is  $x$ .

(3) The remaining case:  $x \in \tilde{S}(F)$ ,  $x$  is the leftmost point in one of its initial segments  $u$ , and  $x$  is isolated in  $\tilde{S}(F)$ . Then we let  $i_F(x)$  be the leftmost point in  $\tilde{S}^{-1}(x) \cap F$ .

Clearly,  $\tilde{S} \circ i_F$  is the identity on  $\tilde{S}(F)$ . We claim that  $i_F$  is continuous at each point  $x \in \tilde{S}(F)$ . To see this, choose a sequence  $\{x_n\}_n$  in  $\tilde{S}(F)$  converging to  $x$ . We have to show that  $i_F(x_n) \rightarrow i_F(x)$  also. If  $x$  is a point of type (3) this is trivial. If  $x$  is a point of type (2), make any choice of points  $y_n \in F$  with  $\tilde{S}(y_n) = x_n$ . Then again as in the proof of Lemma 4 it follows that the sequence  $\{y_n\}$  also converges to  $x$  (provided we assume that for all  $n$ ,  $x_n \neq x$ , which we can do without loss). In particular,  $i_F(x_n) \rightarrow x = i_F(x)$ . Finally, suppose  $x$  is a point of type (1), i.e.  $x$  is never leftmost in  $S$ . Without loss we may assume that the points  $x_n$  are all of the same type. If each  $x_n$  is of type (1) or each  $x_n$  is of type (2), we have  $i_F(x_n) = x_n$  and  $i_F(x) = x$ , so trivially  $i_F(x_n) \rightarrow i_F(x)$ . So suppose all  $x_n$  are of type (3). For each  $n$  there exists a shortest sequence  $v_n$  such that  $x_n$  is the leftmost branch in  $S$  running through  $v_n$ . Choose any points  $y_n$  with  $\tilde{S}(y_n) = x_n$ . Then  $v_n$  must also be an initial segment of  $y_n$ . If  $1\text{th}(v_n)$  converges to infinity, i.e.  $\forall k \exists n_k \forall n \geq n_k \ 1\text{th}(v_n) \geq k$ , then clearly  $y_n \rightarrow x$ . In particular  $i_F(x_n) \rightarrow x$ . Otherwise there exists a subsequence of  $\{x_n\}_n$  on which  $1\text{th}(v_n)$  is constant, hence a subsequence of  $\{x_n\}_n$  is constant. But all  $x_n$  were assumed to be of type (3) while  $x$  is of type (1), so this is impossible.  $\square$

Let us write  $\mathbf{S}$  for the monoid of CHR-mappings.  $\mathbf{S}$  can be equipped with a Grothendieck topology, as follows: A sieve of functions  $\mathcal{W} \subseteq \mathbf{S}$  is defined to be a cover if for some open cover  $\{V_{u_i} \mid i \in I\}$  of Bairespace (i.e.  $\forall x \in B \exists i \in I \ u_i$  is an initial segment of  $x$ ), every function  $\tilde{V}_{u_i}$  is a member of  $\mathcal{W}$ . To show that this indeed defines a Grothendieck topology, we need to verify that

(i) (transitivity) if  $\mathcal{W}$  is a cover, and  $\mathcal{R}$  is a sieve such that for each  $f \in \mathcal{W}$ ,  $f^*(\mathcal{R}) = \{g \mid f \circ g \in \mathcal{R}\}$  covers, then  $\mathcal{R}$  also covers; and

(ii) (stability) if  $\mathcal{W}$  is a cover and  $f \in \mathbf{S}$  then  $f^*(\mathcal{W}) = \{g \mid f \circ g \in \mathcal{W}\}$  is a cover.

The proof of (i) uses the cancellation property of the mappings of the form  $\tilde{S}$ : if  $S$  and  $T$  are spreads and  $S \subseteq T$ , then  $\tilde{S} \circ \tilde{T} = \tilde{T} \circ \tilde{S} = \tilde{S}$ . If  $\mathcal{W}$  is a cover, there is an open cover  $\{V_{u_i} \mid i \in I\}$  of  $B$  with each  $\tilde{V}_{u_i} \in \mathcal{W}$ . By assumption, for each fixed  $i$ ,  $\tilde{V}_{u_i}^*(\mathcal{R})$  covers, so there is a cover  $\{V_{v_j} \mid j \in J\}$  of Bairespace such that for each  $j$ ,  $\tilde{V}_{u_i} \circ \tilde{V}_{v_j} \in \mathcal{R}$ . If  $w$  is an extension of some  $v_j$  which also extends  $u_i$ , then by cancellation  $\tilde{V}_w = \tilde{V}_{u_i} \circ \tilde{V}_{v_j} \circ \tilde{V}_w \in \mathcal{R}$ , so there exists an open cover  $\{V_w\}_w$  of  $V_{u_i}$  with each corresponding  $\tilde{V}_w \in \mathcal{R}$ . This holds for each  $i \in I$ , so  $\mathcal{R}$  is a covering sieve. Thus (i) holds.

To show (ii), pick  $f \in \mathbf{S}$  and suppose  $\{V_{u_i} \mid i \in I\}$  covers  $B$  and each  $\tilde{V}_{u_i} \in \mathcal{W}$ . By continuity of  $f$ , there exists an open cover  $\{V_{v_j} \mid j \in J\}$  of  $B$  such that each  $f(V_{v_j}) \subseteq \text{some } V_{u_i}$ . From the factorization lemma it then follows that each  $\tilde{V}_{v_j}$  is in  $f^*(\mathcal{W})$ .

This Grothendieck topology makes  $\mathbf{S}$  into a site (also denoted by  $\mathbf{S}$ ), and we can interpret the higher order logic in  $\text{Sh}(\mathbf{S})$  as in [HM, §2]. Thus the natural numbers appear in the model as the sheaf  $N = \text{Cts}(B, \mathbb{N})$ , and internal Bairespace  $N^N$  is the sheaf  $\text{Cts}(B, B)$ . In both cases restrictions are given by composition,  $x \upharpoonright f = x \circ f$ . As in [HM], the lawlike sequences are interpreted as the subsheaf  $B_L \subseteq \text{Cts}(B, B)$  of locally constant functions, while the choice sequences are interpreted by the subsheaf  $B_C$  of  $\text{Cts}(B, B)$  generated by the identity. Thus our internal choice sequences are

precisely the external functions from Bairespace to itself which are *locally* elements of  $\mathbf{S}$ .

Any external continuous function  $F: B \rightarrow B$  reappears internally as a continuous operation on Bairespace  $N^N$ , by  $F(f) = F \circ f$ . All internal lawlike continuous functions are (locally) of this form. In particular, if  $\alpha: B \rightarrow B$  is an element of  $B_C$ ,  $\alpha$  induces a lawlike function  $N^N \rightarrow N^N$  in the model, and it follows easily from the stability property (ii) above that this lawlike function restricts to a map of  $B_C$  into itself. Let us write  $\hat{\mathbf{S}}$  for the subsheaf of lawlike continuous operations on  $N^N$  generated by the elements of  $B_C$  in this way. That is, if  $F$  is an internal function  $N^N \rightarrow N^N$  induced by an external continuous  $F: B \rightarrow B$ , then  $\Vdash F \in \hat{\mathbf{S}}$  iff  $F \in B_C$ , i.e.  $F$  is locally in  $\mathbf{S}$ . Then in the model it holds that  $B_C$  is closed under the lawlike operations from  $\hat{\mathbf{S}}$ ,

$$\Vdash \forall \alpha \forall f \in \hat{\mathbf{S}} \exists \beta f(\alpha) = \beta$$

( $\alpha$  and  $\beta$  range over  $B_C$ ). Note that the functions in  $\hat{\mathbf{S}}$  do not map spreads to spreads, but they do so locally, i.e.

$$\begin{aligned} \Vdash \forall f \in \mathbf{S} \forall \text{spread } S \exists e \in K \forall u (e(u) \neq 0 \\ \rightarrow \exists \text{spread } S' \forall \alpha (\alpha \in S' \leftrightarrow \exists \beta \in u (\beta \in S \ \& \ \alpha = f(\beta))))). \end{aligned}$$

Every function  $F \in \mathbf{S}$  appears in particular as an internal operation on  $N^N$  which is in  $\hat{\mathbf{S}}$ , and we will also write  $\mathbf{S}$  for the subsheaf of  $\hat{\mathbf{S}}$  generated by these internal mappings coming from an  $F \in \mathbf{S}$ ; so  $\Vdash \mathbf{S} \subseteq \hat{\mathbf{S}}$ .

Let us consider the relevant properties of this model. Many of the arguments that follow are analogous to the arguments in [HM, §2], and will only be indicated briefly.

First of all, as noted the universe of choice sequences is closed under operations from  $\mathbf{S}$ , and this gives a *pairing axiom*

$$\Vdash \forall \alpha, \beta \exists \gamma \exists f, g \in \mathbf{S} (\alpha = f(\gamma) \ \& \ \beta = g(\gamma)).$$

(Proof: if  $\alpha, \beta \in B_C \subset N^N$ , they are restrictions of the identity  $\gamma$  (locally), say  $\alpha = \gamma \upharpoonright f$ ,  $\beta = \gamma \upharpoonright g$ , so  $\Vdash \alpha = f(\gamma)$ ,  $\beta = g(\gamma)$ .)

Since  $\mathbf{S}$  contains all constant functions, every  $\alpha \in B_C$  has a restriction which is lawlike, so the model satisfies the *specialization property*,

$$\Vdash A(\alpha) \rightarrow \exists \text{ lawlike } a \ A(a),$$

where all parameters other than  $\alpha$  in  $A(\alpha)$  are lawlike.

The *density axiom*  $\forall \text{spread } S \exists \alpha \alpha \in S$  holds in the model: again by using constant functions, or alternatively, by observing that  $\Vdash \hat{\mathbf{S}} \in \mathbf{S}$ . We will come back to this below, and formulate an axiom of *strong density*.

If  $\alpha \in B_C$ ,  $\alpha$  is externally given as (locally) an element  $f \in \mathbf{S}$ , and every  $f(\beta)$  is a restriction of  $\alpha$ . Hence if  $\Vdash A(\alpha)$  and all other parameters in  $A$  are lawlike, it follows that  $\Vdash \forall \beta A(f(\beta))$ . As in [HM] this yields *analytic data relativized* to  $\mathbf{S}$ ,

$$\Vdash A(\alpha) \rightarrow \exists f \in \mathbf{S} (\forall \beta A(f(\beta)) \ \& \ \alpha \in \text{im}(f)).$$

From this, we immediately obtain *spreaddata* by an application of the factorization lemma: if  $\alpha \in B_C$ , then, on a suitable cover,  $\text{im}(\alpha) = S$  is a closed subset of  $B$ , and

every other  $\beta \in B_C$  with  $\text{im}(\beta) \subseteq S$  can be written as  $\alpha \circ \gamma$  for some  $\gamma$ , i.e. as a restriction of  $\alpha$ . Thus

$$(\text{spreaddata}) \quad \Vdash A(\alpha) \rightarrow \exists \text{ spread } S(\alpha \in S \ \& \ \forall \beta \in S \ A(\beta)).$$

Continuity principles follow as in [HM] by considering the generic element  $\tilde{B} = \text{id} \in B_C$ . For example, for  $\forall \alpha \exists \beta$ -continuity suppose  $\Vdash \forall \alpha \exists \beta \ A(\alpha, \beta)$  (all non-lawlike parameters in  $A(\alpha, \beta)$  shown). Then in particular there is a  $\beta \in B_C$  such that  $\Vdash A(\text{id}, \beta)$ .  $\beta$  acts internally by composition as a lawlike operation  $F$  on  $B_C$  which is in  $\hat{S}$ , and we obtain

$$\Vdash \forall \alpha \exists \beta \ A(\alpha, \beta) \rightarrow \exists F \in \hat{S} \ \forall \alpha \ A(\alpha, F(\alpha)).$$

A similar argument gives  $\forall \alpha \exists n$ -continuity.

Since  $B_C$  contains all constant functions, Bar Induction holds in the form BI\* (see [HM, §2]).

Note that in the axiom of *spreaddata* as just formulated, we cannot economize on spreads, i.e. there is no proper subclass  $P$  of the class of all spreads such that *spreaddata* holds with “ $\exists \text{ spread } S$ ” replaced by “ $\exists \text{ spread } S \in P$ ”. This follows by taking  $A(\alpha)$  to be  $\alpha \in S$ , and choosing  $\alpha = \tilde{S}$ .

This is how it should be, since given any spread  $S$ , there is no a priori reason why  $S$  cannot occur as “complete information at a certain stage”, i.e. why we cannot construct a sequence  $\alpha$  such that at a certain stage of its construction the only information we have about  $\alpha$  is that  $\alpha \in S$ . One way of formalizing this as an axiom is to say that for any spread  $S$  there is a step in a construction process consisting of a single application of a lawlike continuous operation  $f$  (under which the universe of choice sequences is closed), such that after applying this step to the universal sequence  $\alpha$  about which we have not yet gained any knowledge, we know that  $\alpha \in S$  and nothing more. We call this axiom the axiom of *strong density*, since it is a strengthening of the ordinary density axiom ( $\forall S \exists \alpha \alpha \in S$ ).

**STRONG DENSITY AXIOM.**  $\forall \text{ spread } S \ \exists f \in \mathbf{S} \ \forall \alpha (\alpha \in S \leftrightarrow \exists \beta \alpha = f(\beta)).$

Observe that the strong density axiom is satisfied in our model, since

$$\Vdash \forall \text{ spread } S \ \text{im}(\tilde{S}) = S.$$

Since the mappings  $\tilde{S}$  are retractions, i.e.  $\Vdash \tilde{S} \circ \tilde{S} = \tilde{S}$ , we obtain relativized forms of continuity,

$$\begin{aligned} (\text{relativized } \forall \alpha \exists \beta \text{-continuity}) \quad & \Vdash \forall \alpha \in S \ \exists \beta \ A(\alpha, \beta) \\ & \rightarrow \exists \text{ lawlike continuous } F: S \rightarrow B_C \ \forall \alpha \in S \ A(\alpha, F\alpha), \end{aligned}$$

$$\begin{aligned} (\text{relativized } \forall \alpha \exists n \text{-continuity}) \quad & \Vdash \forall \alpha \in S \ \exists n \ A(\alpha, n) \\ & \rightarrow \exists \text{ lawlike continuous } F: S \rightarrow N \ \forall \alpha \in S \ A(\alpha, F\alpha). \end{aligned}$$

(By definition, a lawlike continuous operation  $F: S \rightarrow B_C$  comes from a neighbourhood function  $N^{<N} \xrightarrow{F} N^{<N}$  such that for all  $n$ ,  $\{u \mid \text{length } F(u) \geq n\}$  is a (n inductive) bar for  $S$ . Similarly for functions  $S \rightarrow N$ . The relativized versions follow easily from the global ones together with the fact that

$$\Vdash \forall \text{ spread } S \ \exists f \in \mathbf{S} (\text{im}(f) = S \ \& \ f \circ f = f.)$$

We also conclude that a relativized form of Bar Induction holds in the model: for any spread  $S$ ,

$$(BI_S) \quad \Vdash \forall P \subseteq N^{<N} (P \text{ is a monotone inductive bar for } S \rightarrow \langle \rangle \in P).$$

**PROOF.** This follows from the global version  $BI^*$  and strong density. Suppose  $P$  is monotone ( $u \geq v \in P \rightarrow u \in P$ ), inductive ( $\forall n(u * \langle n \rangle \in S \rightarrow u * \langle n \rangle \in P) \rightarrow u \in P$ ) and bars  $S$  ( $\forall \alpha \in S \exists n \bar{\alpha}(n) \in P$ ). Let  $f \in S$  be such that  $\text{im}(f) = S$ , let  $P' = \{v \mid \forall \alpha \in v f(\alpha)(1\text{th}(v)) \in P\}$ , and apply  $BI^*$  to  $P'$  to conclude that  $\langle \rangle \in P$ .

For the record, let us sum up the properties of the model. (The axiom of pairing as formulated below can actually be strengthened by replacing  $\hat{S}$  by  $S$ ; countable choice is proved just as in [HM].)

**THEOREM 6.** *The interpretation in  $\text{Sh}(S)$  described above yields a model in which there is a monoid  $\hat{S}$  of internal lawlike continuous functions, satisfying the following axioms:*

<i>countable choice:</i>	$\forall n \exists m A(n, m) \rightarrow \exists f \in N^N \forall n A(n, fn);$
<i>pairing and closure:</i>	$\forall \alpha, \beta \exists \gamma \exists f, g \in \hat{S} \alpha = f(\gamma) \ \& \ \beta = g(\gamma),$ $\forall \alpha \forall f \in \hat{S} \exists \beta f(\alpha) = \beta;$
<i>specialization:</i>	$\exists \alpha A(\alpha) \rightarrow \exists a A(a);$
<i>spreaddata:</i>	$A(\alpha) \rightarrow \exists \text{spread } S (\alpha \in S \ \& \ \forall \beta \in S A(\beta));$
<i>strong density:</i>	$\forall \text{spread } S \exists f \in \hat{S} \text{im}(f) = S;$
<i>Bar Induction:</i>	$\forall \text{spread } S \forall P \subseteq N^{<N} (P \text{ is a monotone inductive bar for } S \rightarrow \langle \rangle \in P);$
$\forall \alpha \exists n$ -continuity:	$\forall \alpha \in S \exists n A(\alpha, n) \rightarrow \exists \text{lawlike continuous } F: S \rightarrow N \forall \alpha \in S A(\alpha, F\alpha);$
$\forall \alpha \exists \beta$ -continuity:	$\forall \alpha \in S \exists \beta A(\alpha, \beta) \rightarrow \exists F \in \hat{S} \forall \alpha \in S A(\alpha, F\alpha).$

(Except for countable choice, all nonlawlike variables are shown in notation.)  $\square$

**§3. Concluding remarks.** One of the first models for spreaddata seems to be the projection model in van Dalen and Troelstra [1970] (see also Troelstra [1970]). Essentially the same model can be obtained as an analog of the LS-model presented in §5.2 of [HM], which one obtains by replacing Bairespace by the space of decreasing sequences of spreads  $(S_n)_n$  such that  $\bigcap_n S_n$  consists of a single point, with the product topology (finite initial segments topology). Our LS-model from [HM] is essentially equivalent to the LS-model presented in Fourman [1982, §2.2]. If in Fourman's model one replaces basic opens  $U_1 \times \cdots \times U_n \subset B^n$  of finite products of Bairespace by finite products of spreads  $S_1 \times \cdots \times S_n \subset B^n$ , one obtains an analog of Fourman's LS-model in which the axiom of spreaddata holds (this seems to be the model indicated in Fourman [1982, §2.4]).

The main differences between these models and the model presented in this paper are caused by the fact that in the former models, the universe of choice sequences is not closed under application of nontrivial lawlike operations. Consequently,  $\forall \alpha \exists \beta$ -continuity does not hold in the form described in Kreisel [1965]. (In the spreaddata-analog of our LS-model from [HM],  $\forall \alpha \exists \beta$ -continuity holds in the form

$$\forall \alpha \exists \beta A(\alpha, \beta) \rightarrow \exists f \in K \forall u (f(u) \neq 0 \rightarrow (\forall \alpha \in u A(\alpha, \alpha) \vee \exists \beta \forall \alpha \in u A(\alpha, \beta))).$$



On the other hand, these other models can be constructed within a constructive metatheory (IDB), and hence are equivalent to elimination translations into IDB, whereas our present model cannot: the statement that for all spreads  $S$ , the mapping  $\tilde{S}$  is a closed hereditary retraction contradicts Church's thesis. It remains an open question whether a model for spreaddata with the properties as described in Theorem 6 above can be constructed within a constructive metatheory.

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