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NON-κ-CRITICAL VERTICES IN GRAPHS

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Let G be a graph with $\kappa(G)=h$. A vertex v of G is called κ-critical if $\kappa(G-v)=h-1$. Generalizing a result of Chartrand, Kaugars and Lick and one of Hamidoune, respectively, we prove: (1) If $\delta(G)>\frac{3}{2}h-1$, then G contains at least $h+1+\varepsilon(h)$ non-κ-critical vertices, where $\varepsilon(h)=0$ if h is odd and $\varepsilon(h)=1$ if h is even; (2) If G contains at most one vertex of degree not exceeding $\frac{3}{2}h-1$, then G has at least $2+\varepsilon(h)$ noncritical vertices. The results are best possible in the sense that under either condition there exist, for every h, infinitely many graphs containing exactly the specified minimum number of noncritical vertices.

1. Introduction

All concepts not defined here may be found in Harary [3]. A graph G is *h-connected* if for every $S\subset V(G)$ with $|S|<h$, the graph $G-S$ is connected ($G-S$ is an abbreviation of $\langle V(G)-S\rangle$, the subgraph of G induced by $V(G)-S$). The *connectivity* of G, denoted $\kappa(G)$, is the maximum value of h for which G is h-connected. A vertex v of G is called *κ-critical* (or just *critical*) if $\kappa(G-v)=\kappa(G)-1$. The number of noncritical vertices of G is denoted by $\tau(G)$. We say that G is *critically h-connected* if $\kappa(G)=h$ and $\tau(G)=0$. If $A\subset V(G)$, then $N(A)$ denotes the set of all vertices of $G-A$ adjacent to vertices in A. Furthermore, $\hat{A}$ is defined as $V(G)-(A\cup N(A))$. A subset T of $V(G)$ is a *vertex cut* of G if $G-T$ is disconnected. A *k-vertex cut* is a vertex cut of k elements. A *minimum vertex cut* is a $\kappa(G)$-vertex cut. Following Hamidoune, we define a subset A of $V(G)$ to be a *fragment* of G if $\hat{A}\neq\emptyset$ and $N(A)$ is a minimum vertex cut of G. If A is a fragment and $N(A)=T$, then A is said to be a fragment *with respect to* T. An *endfragment* is a fragment that contains no other fragment as a proper subset.

Lemma 1. *If A is an endfragment of a graph G, then $N(A)$ is the only minimum vertex cut of G contained in $A\cup N(A)$.*

Proof. By contradiction. Assume that A is an endfragment of G such that $A\cup N(A)$ contains, next to $N(A)$, another minimum vertex cut T of G. If H is a component of $\langle\hat{A}\rangle$, then, $N(A)$ being a minimum vertex cut, each vertex of $N(A)-T$ is adjacent to at least one vertex of H. This is true for all components of

$\langle \hat{A} \rangle$, implying that $\langle \hat{A} \cup (N(A) - T) \rangle$ is connected. Hence there exists a fragment B with respect to T with $B \supset \hat{A} \cup (N(A) - T)$. The fragment $\hat{B}$ is then properly contained in the endfragment A, which is impossible. □

A vertex cut T_1 *interferes with* a vertex cut T_2 if at least two components of $G - T_1$ contain vertices of T_2.

Lemma 2. *If T_1 and T_2 are distinct minimum vertex cuts of a graph G, then T_1 interferes with T_2 if and only if T_2 interferes with T_1.*

Proof. Assume that T_1 does not interfere with T_2. Let H_1 be the component of $G - T_1$ that contains all vertices of $T_2 - T_1$ and let H_2 be another component of $G - T_1$. Since T_1 is a minimum vertex cut, each vertex of $T_1 - T_2$ is adjacent to one or more vertices of H_2, so $\langle V(H_2) \cup (T_1 - T_2) \rangle$ is connected. Thus all vertices of $T_1 - T_2$ are in the same component of $G - T_2$. In other words, T_2 does not interfere with T_1. The argument of symmetry completes the proof. □

In view of Lemma 2 we may speak without ambiguity of *interfering* minimum vertex cuts.

Chartrand, Kaugars and Lick proved in [1] that a critically h-connected graph G satisfies $\delta(G) \leq \frac{3}{2}h - 1$. An equivalent statement is

Theorem A (Chartrand, Kaugars and Lick [1]). *If G is a graph with $\kappa(G) = h$ and $\delta(G) > \frac{3}{2}h - 1$, then $\tau(G) \geq 1$.*

In Section 2 (Corollary 1) we improve this result by showing that under the conditions of Theorem A, $\tau(G) \geq h + 1 + \varepsilon(h)$, where

$$\varepsilon(h) = \begin{cases} 0 & \text{if } h \text{ is odd,} \\ 1 & \text{if } h \text{ is even.} \end{cases}$$

Another generalization of Theorem A was stated by Hamidoune [2]. He proved that a critically h-connected graph contains at least two vertices of degree not exceeding $\frac{3}{2}h - 1$, or equivalently,

Theorem B (Hamidoune [2]). *Let G be a graph with $\kappa(G) = h$. If G contains at most one vertex of degree not exceeding $\frac{3}{2}h - 1$, then $\tau(G) \geq 1$.*

This result, too, is sharpened in Section 2 (Theorem 2): if the conditions of Theorem B are met, then $\tau(G) \geq 2 + \varepsilon(h)$.

At the end of Section 2 our results are shown to be best possible (Propositions 1 and 2).

2. Results

Theorem 1. *If G is a graph with $\kappa(G)=h$ and $\delta(G)>\frac{3}{2}h-1$, then no endfragment of G contains a critical vertex.*

Proof. By contradiction. Let G be a graph with $\kappa(G)=h$ and $\delta(G)>\frac{3}{2}h-1$ and assume there exists an endfragment A of G containing at least one critical vertex. Put $N(A)=T_1$. Since A contains a critical vertex, there exists an h-vertex cut T_2 satisfying $A\cap T_2\neq\emptyset$ and, by Lemma 1, $\hat{A}\cap T_2\neq\emptyset$. Put $h_1=|A\cap T_2|$, $h_2=|\hat{A}\cap T_2|$, $h_3=|T_1\cap T_2|$, so that $h_1+h_2+h_3=h$. Denote by $\mathscr{B}$ the collection of all connected fragments with respect to T_2 that contain a vertex of T_1. Since T_1 and T_2 interfere, we conclude that $|\mathscr{B}|\geq 2$. Let B be a fragment in $\mathscr{B}$ for which $|B\cap T_1|$ is minimum. Distinguishing two cases, we show first that $|B\cap T_1|<\frac{1}{2}h$ and next that $B\subset T_1$. It then follows that the vertices of B have degree smaller than $\frac{1}{2}h-1+h$, contradicting the assumption that $\delta(G)>\frac{3}{2}h-1$.

Case 1: $h_1\leq h_2$. The assumption $\delta>\frac{3}{2}h-1$ implies that $|A|>\frac{1}{2}h$. Since $h_1\leq\frac{1}{2}h$, there exists a vertex $u\in A-T_2$. By Lemma 1, T_1 is the only h-vertex cut of G contained in $A\cup T_1$, so the graph H obtained from $\langle A\cup T_1\rangle$ by joining each pair of nonadjacent vertices of T_1 by an edge is $(h+1)$-connected, so that $H-T_2$ is $(h+1-h_1-h_3)$-connected. In $H-T_2$ there exist, by a variation on Menger's theorem, $h+1-h_1-h_3$ paths with origin u and terminus in T_1 having only u in common. These paths can be chosen in such a way that none of them has an internal vertex belonging to T_1. The paths are then subgraphs of $\langle (A\cup T_1)-T_2\rangle$ too. We conclude the existence of a fragment B_1 in $\mathscr{B}$ that contains at least $h+1-h_1-h_3$ vertices of T_1. Consequently,

$$|B\cap T_1|\leq(h-h_3)-(h+1-h_1-h_3)=h_1-1<\tfrac{1}{2}h.$$

To show that $B\subset T_1$ or, equivalently, $B\cap A=B\cap\hat{A}=\emptyset$, assume the contrary. Suppose first that $B\cap A\neq\emptyset$. Then $(A\cap T_2)\cup(B\cap T_1)\cup(T_1\cap T_2)$ is a vertex cut of G satisfying

$$\begin{aligned}|(A\cap T_2)\cup(B\cap T_1)\cup(T_1\cap T_2)|&\leq h_1+(h_1-1)+h_3\\&\leq h_1+h_2+h_3-1=h-1,\end{aligned}$$

which is impossible, since G is h-connected.

Suppose next that $B\cap\hat{A}\neq\emptyset$. Then $(\hat{A}\cap T_2)\cup(B\cap T_1)\cup(T_1\cap T_2)$ is a vertex cut of G with

$$|(\hat{A}\cap T_2)\cup(B\cap T_1)\cup(T_1\cap T_2)|\leq h_2+(h_1-1)+h_3=h-1.$$

This contradiction settles Case 1.

Case 2: $h_1>h_2$. From the assumption $\delta>\frac{3}{2}h-1$ we deduce that $|\hat{A}|>\frac{1}{2}h$. Since $h_2<\frac{1}{2}h$, the set $\hat{A}-T_2$ is nonempty, $v\in\hat{A}-T_2$ say. An argument similar to the one used in Case 1 yields that in the subgraph $\langle(\hat{A}\cup T_1)-T_2\rangle$ of G there exist $h-h_2-h_3$ paths with origin v and terminus in T_1 having only v in common. Thus in $\mathscr{B}$ there is a fragment B_2 containing at least $h-h_2-h_3$ vertices of T_1. It follows that

$$|B\cap T_1|\leq (h-h_3)-(h-h_2-h_3)=h_2<\tfrac{1}{2}h.$$

It remains to be shown that $B\subset T_1$.

Assume first that $B\cap A\neq\emptyset$. Then $(A\cap T_2)\cup(B\cap T_1)\cup(T_1\cap T_2)$ is a vertex cut of G contained in $A\cup T_1$. Moreover,

$$|(A\cap T_2)\cup(B\cap T_1)\cup(T_1\cap T_2)|\leq h_1+h_2+h_3=h.$$

G being h-connected, this inequality cannot be strict. However, since A is an endfragment of G, Lemma 1 implies that T_1 is the only h-vertex cut contained in $A\cup T_1$, a contradiction.

Now suppose that $B\cap\hat{A}\neq\emptyset$. Then $(\hat{A}\cap T_2)\cup(B\cap T_1)\cup(T_1\cap T_2)$ is a vertex cut of G satisfying

$$|(\hat{A}\cap T_2)\cup(B\cap T_1)\cup(T_1\cap T_2)|\leq h_2+h_2+h_3$$
$$<h_1+h_2+h_3=h.$$

This contradiction completes the proof. □

Corollary 1. *If G is a graph with $\kappa(G)=h$ and $\delta(G)>\frac{3}{2}h-1$, then $\tau(G)\geq h+1+\varepsilon(h)$.*

Proof. Since $\delta(G)>\frac{3}{2}h-1$, every endfragment of G has at least $\{\frac{1}{2}(h+1)\}$ vertices, where $\{x\}$ denotes the smallest integer n for which $n\geq x$. G has at least two disjoint endfragments, so that, by Theorem 1,

$$\tau(G)\geq 2\cdot\{\tfrac{1}{2}(h+1)\}=h+1+\varepsilon(h). \qquad \square$$

Theorem 2. *Let G be a graph with $\kappa(G)=h$. If G contains at most one vertex of degree not exceeding $\frac{3}{2}h-1$, then $\tau(G)\geq 2+\varepsilon(h)$.*

Proof. If $\delta(G)>\frac{3}{2}h-1$, then the result follows from Theorem 1. Thus assume that G has a unique vertex v of degree at most $\frac{3}{2}h-1$. We distinguish two cases.

Case 1: v is noncritical. Let F be a fragment containing v and A an endfragment contained in $\hat{F}$. All vertices of A have degree greater than $\frac{3}{2}h-1$, so that $|A|>\frac{1}{2}h$. Two subcases are now distinguished.

Case 1.1: $\hat{A}=\{v\}$. By Lemma 1, the set $N(A)$ is the only h-vertex cut of G contained in $A\cup N(A)$. The vertex v being contained in no h-vertex cut of G, it follows that $N(A)$ is the only h-vertex cut of G. Thus all vertices in A are noncritical, so that

$$\tau(G)\geq|A|+1\geq\{\tfrac{1}{2}(h+1)\}+1\geq 2+\varepsilon(h).$$

Case 1.2: $\hat{A}-\{v\}\neq\emptyset$. $N(A)$ does not contain v, so all vertices in $N(A)$ have degree greater than $\frac{3}{2}h-1$. Moreover, since all vertices of $\hat{A}-\{v\}$ also have degree greater than $\frac{3}{2}h-1$, it follows that $|\hat{A}|>\frac{1}{2}h$. As in the proof of Theorem 1 one now shows that A, being an endfragment, contains no critical vertex. Thus, again,

$$\tau(G)\geq|A|+1\geq 2+\varepsilon(h).$$

Case 2: v is critical. Let T be an h-vertex cut containing v. Then there exist endfragments A_1, A_2 of G such that A_1, A_2 and T are mutually disjoint. Again we distinguish two subcases.

Case 2.1: No vertex in $A_1\cup A_2$ is critical. Since $v\notin A_1\cup A_2$, all vertices of $A_1\cup A_2$ have degree greater than $\frac{3}{2}h-1$, so that $|A_1|>\frac{1}{2}h$ and $|A_2|>\frac{1}{2}h$. Consequently,

$$\tau(G)\geq|A_1|+|A_2|\geq 2\cdot\{\tfrac{1}{2}(h+1)\}\geq 2+\varepsilon(h).$$

Case 2.2: $A_1\cup A_2$ contains a critical vertex. Assume, without loss of generality, that some vertex in A_1 is critical. In view of Lemma 1 this means that some h-vertex cut T_1 of G interferes with $N(A_1)$. As in the proof of Theorem 1 one now concludes the existence of a fragment B with respect to T_1 all of whose vertices are in $N(A_1)$ and have degree smaller than $\frac{3}{2}h-1$. Note that, in consequence, Case 2.2 does not occur for $h=2$. Since v is the only vertex of G of degree not exceeding $\frac{3}{2}h-1$, it follows that $B=\{v\}$, $T_1=N(\{v\})$ and $\deg v=h$. Moreover, all vertices in $A_1\cup A_2$ not adjacent to v are noncritical, $N(\{v\})$ being the only h-vertex cut interfering with $N(A_1)$ or $N(A_2)$ (or both). The vertices in $A_1\cup A_2$ all have degree greater than $\frac{3}{2}h-1$, implying that $|A_1\cup A_2|\geq 2\cdot\{\frac{1}{2}(h+1)\}=h+1+\varepsilon(h)$. Since $\deg v=h$, at least $1+\varepsilon(h)$ vertices of $A_1\cup A_2$ are not adjacent to v. If no vertex of A_2 is adjacent to v, then all vertices of A_2 are noncritical and hence, since $h\geq 3$,

$$\tau(G)\geq|A_2|\geq\{\tfrac{1}{2}(h+1)\}\geq 2+\varepsilon(h).$$

Now assume that both $N(A_1)$ and $N(A_2)$ contain v. Then, to guarantee that the vertices of $A_1\cup A_2$ which are not adjacent to v have degree greater than $\frac{3}{2}h-1$, it is in fact necessary that $|A_1\cup A_2|\geq h+2+\varepsilon(h)$. Consequently,

$$\tau(G)\geq|A_1\cup A_2|-|N(\{v\})|\geq h+2+\varepsilon(h)-h=2+\varepsilon(h). \qquad \square$$

The lower bounds for $\tau(G)$, as given in Corollary 1 and Theorem 2, are sharp. This assertion is specified in the Propositions 1 and 2 below.

First we define inductively the labelled graph $G_{n,h}$ $(n\geq h)$ with vertex set $\{v_1, v_2, \ldots, v_n\}$: (i) $\langle\{v_1, v_2, \ldots, v_h\}\rangle$ is complete,

(ii) $\langle\{v_1, v_2, \ldots, v_i, v_{i+1}\}\rangle$ is obtained from $\langle\{v_1, v_2, \ldots, v_i\}\rangle$ by joining v_{i+1} to $v_{i-h+1}, v_{i-h+2}, \ldots, v_i$ $(h\leq i\leq n-1)$.

Proposition 1. *For every h and every $n\geq 2h+1+\varepsilon(h)$ there exists a graph $H_{n,h}$ of order n such that*

(a) $\kappa(H_{n,h}) = h$,
(b) $\delta(H_{n,h}) > \frac{3}{2}h - 1$,
(c) $\tau(H_{n,h}) = h + 1 + \varepsilon(h)$.

Proof. $H_{n,h}$ is defined by performing in succession the following construction steps: (1) $V(H_{n,h}) := \{v_1, v_2, \ldots, v_s, u_1, u_2, \ldots, u_t, w_1, w_2, \ldots, w_t\}$ where $t = \frac{1}{2}(h + 1 + \varepsilon(h))$ and $s = n - 2t$;

(2) $\langle\{v_1, v_2, \ldots, v_s\}\rangle := G_{s,h}$;

(3) $\langle\{u_1, u_2, \ldots, u_t\}\rangle$ is complete and, for $1 \leq i \leq t$, u_i is joined to $v_1, v_2, \ldots, v_h$;

(4) $\langle\{w_1, w_2, \ldots, w_t\}\rangle$ is complete and, for $1 \leq i \leq t$, w_i is joined to $v_{s-h+1}, v_{s-h+2}, \ldots, v_s$.

$H_{n,h}$ satisfies (a) and (b). Furthermore, $u_1, u_2, \ldots, u_t, w_1, w_2, \ldots, w_t$ are the only noncritical vertices of $H_{n,h}$, so that $H_{n,h}$ also satisfies (c). □

Proposition 2. *For every pair of integers (h, n) satisfying either $h = 2$ and $n \geq 5$ or $h \geq 3$ and $n \geq 2h + 2 + \varepsilon(h)$ there exists a graph $L_{n,h}$ of order n such that*
(a) $\kappa(L_{n,h}) = h$,
(b) *$L_{n,h}$ contains a unique vertex of degree at most $\frac{3}{2}h - 1$,*
(c) $\tau(L_{n,h}) = 2 + \varepsilon(h)$.

Proof. We distinguish two cases.

Case 1: $h = 2$. Obtain $L_{n,2}$ from $H_{n,2}$ by deleting the vertex u_1 of $H_{n,2}$. Then $\kappa(L_{n,2}) = 2$ and u_2 is the only vertex of degree 2 of $L_{n,2}$. The noncritical vertices of $L_{n,2}$ are u_2, w_1 and w_2. Thus $L_{n,2}$ satisfies (a), (b) and (c), settling Case 1.

Case 2: $h \geq 3$. Now construct $L_{n,h}$ by going through successively the following steps: (1) $V(L_{n,h}) := \{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_m, w_1, w_2, \ldots, w_{m+1}, z\}$ where $m = \frac{1}{2}(h + 1 + \varepsilon(h))$ and $k = n - 2m - 2$;

(2) $\langle\{v_1, v_2, \ldots, v_k\}\rangle := G_{k,h-1}$;

(3) $\langle\{u_1, u_2, \ldots, u_m\}\rangle$ is complete and, for $1 \leq i \leq m$, u_i is joined to $v_1, v_2, \ldots, v_{h-1}$;

(4) $\langle\{w_1, w_2, \ldots, w_{m+1}\}\rangle$ is complete and, for $1 \leq i \leq m+1$, w_i is joined to $v_{k-h+2}, v_{k-h+3}, \ldots, v_k$;

(5) z is joined to $u_1, u_2, \ldots, u_m, w_1, w_2, \ldots, w_{h-m}$.

It is easily seen that z is the only vertex of $L_{n,h}$ of degree not exceeding $\frac{3}{2}h - 1$ ($\deg z = h$) and $\kappa(L_{n,h}) = h$. The only noncritical vertices of $L_{n,h}$ are $w_{h-m+1}, w_{h-m+2}, \ldots, w_{m+1}$, so that

$$\tau(L_{n,h}) = (m + 1) - (h - m) = 2m + 1 - h = h + 1 + \varepsilon(h) + 1 - h = 2 + \varepsilon(h). \quad \square$$

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