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REGIONAL ALLOCATION OF INVESTMENT AS A HIERARCHICAL OPTIMIZATION PROBLEM

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A new formulation is given for the well-known problem of investment allocation between regions in the framework of a planning model. The case of a dual economy with a Cobb–Douglas production function is worked out in detail with an illustrative numerical example. The corresponding problem for an industrial economy with income disparity between two regions is also discussed.

1. Introduction

The problem of investment allocation between regions has been studied in the sixties using the techniques of modern control theory [e.g., Rahman (1963), Intriligator (1964), Takayama (1974)]. All these studies used constant capital–output ratio for the production function and a finite horizon optimization problem. Pitchford (1977) discussed extension to general production function and infinite planning horizon. The solutions discussed in Takayama (1974) under various criteria functionals were 'bang-bang' in character, with the possibility of one switch. This strategy is somewhat academic, particularly in the context of planning in an underdeveloped country where dual economy prevails. If the purpose of the central planning board is to maximize the percapita consumption over the whole planning period, it may quite well be optimal to invest all the savings in the advanced sector of the economy. The recent debate on development planning convincingly shows the fallacy of such a policy. The work of Todaro (1969) is particularly illustrative in this context. Myrdal (1968) and Singer (1975) advocated different type of criteria which would reflect some kind of transfer of actual consumption from the advanced to the traditional sector of the economy. The transfer of too much of the development fund to the traditional sector is, however, not politically possible when one realizes that most of this capital comes out of the savings incurred in the advanced sector of the economy. To introduce this aspect to the problem, we do not consider the savings rate of the advanced sector fixed, as assumed in Takayama (1974). We reformulate the problem where the advanced region has the flexibility to choose its savings strategy once it knows the allocation decision of the central planning board. This leads to a hierarchical optimization problem. We consider a short-term (finite horizon) planning problem and ask the question as to what is the optimum proportion of savings that should be allocated to the two regions to achieve a socially desirable goal. For simplicity and also because abrupt change of this proportion of savings to be allocated to the regions are difficult to implement, we consider this quantity to be constant throughout the planning period. In the terminology of the hierarchical control theory, we have a Stackelberg problem where the central planning board is the leader and the agency regulating savings decision of the advanced sector is the follower. This is analyzed in the framework of standard dynamical game theory. The basic concepts and methods of handling such problems may be found in Simaan and Cruz (1973). Finally, an extension of this problem is formulated where both the regions have savings policies. This would correspond to the situation of an industrial economy with income gaps between the regions.

2. Regional allocation problem

We denote the advanced sector by suffix 1 and the traditional sector by suffix 2.

K_i denotes the capital, L_i the working population and Y_i the output of the region i, $i=1, 2$. All these are, of course, time functions. Assuming F_i to be the production function in the region i,

$$Y_i = F_i(K_i, L_i), \qquad i=1,2. \tag{1}$$

We consider the rate of growth of working population in each region as exogenously determined and is a constant n. The case of different rates of growth of working population poses no technical difficulty. Thus

$$\dot{L}_i = nL_i, \qquad n \text{ constant},$$

$$L_i(0) = L_{i0} \quad \text{given} \quad i=1,2. \tag{2}$$

We denote the percapita quantities by the corresponding small letter and denote $F_i(k_i, 1)$ by $f_i(k_i)$, $i=1,2$. Let c_1 be the fraction of the percapita output of the region 1 used for percapita consumption in that region, and assume that region 2 consumes all it produces. Let β be the proportion of savings of the region 1 (and therefore, of the economy as a whole) that is reinvested in the same region, the rest being allocated to the region 2. The system dynamics is then given by

$$\dot{k}_1 = \beta f_1(k_1)(1-c_1) - nk_1, \qquad 0 \leqq c_1 \leqq 1, \tag{3}$$

$k_1(0) = k_{10}$, given,

$$\dot{k}_2 = (1-\beta) f_1(k_1)(1-c_1)\frac{L_{10}}{L_{20}} - nk_2, \tag{4}$$

$k_2(0) = k_{20}$, given.

Given β, the policy-making agency in region 1 chooses c_1 so as to maximize the criterion

$$\mathscr{J}_1 = \frac{1}{1-v}\int_0^T \{c_1 f_1(k_1(t))\}^{1-v}\,\mathrm{d}t + b_1 k_1(T), \qquad v \text{ a constant,}$$

where the weighting factor b_1 is chosen close to the price of capital at the golden rule path. Discussions on this and other types of criterion can be found in Chakravarty (1969). We use the present criterion for simplicity. The main objective is to reflect the natural behavioural pattern that as β decreases from 1, region 1 will have less incentive to save, as the savings will then be mostly appropriated to the other region. On the other hand, with β getting very close to zero, overconsumption will drastically reduce the capital of region 1 and it will have a natural tendency to give more and more weight to its final capital. Thus, with very small β, consumption of the region 1 will be rather low throughout the planning period. These are discussed again in connection with a numerical example.

We now discuss the policy issue confronting the central planning board. It wants to choose β so that the overall satisfaction of both the regions is maximized. If the planning board chooses an average of the percapita consumption of the two regions as its criterion, one might obtain a maximum with large disparity in the consumption pattern between the two regions. One possible criterion which is less defective from the standpoint of this article is the product of the percapita consumption of the two regions. Thus, the criterion to be maximized at the higher level, assuming that region 2 consumes all it produces, is taken to be

$$\mathscr{J} = \frac{1}{(1-v)^2}\int_0^T \{c_1 f_1(k_1) f_2(k_2)\}^{1-v}\,\mathrm{d}t.$$

We, therefore, have a two-level optimization problem. Given β, we have to determine c_1 at the lower level that maximizes $\mathscr{J}_1$. This optimal c_1, denoted

c_1^*, will naturally depend on β. Using this c_1^*, we have to determine that β which maximizes $\mathscr{J}$.

3. Method of solution

We first solve the optimization problem at the lower level. Thus, assume that β is given. The state equation is

$$\dot{k}_1 = \beta f_1(k_1)(1-c_1) - nk_1, \qquad 0 \leqq c_1 \leqq 1, \tag{5}$$

$$k_1(0) = k_{10}, \quad \text{given.}$$

The criterion to be maximized is

$$\mathscr{J}_1 = \frac{1}{1-v} \int_0^T \{c_1 f_1(k_1(t))\}^{1-v} \, dt + b_1 k_1(T). \tag{6}$$

We assume Cobb–Douglas production function:

$$f_i = A_i k_i^{\alpha_i}.$$

We use the minimum principle of Pontryagin to obtain a necessary condition for optimum. The Hamiltonian for the problem is

$$H = -\frac{1}{1-v}\{c_1 A_1 k_1^{\alpha_1}\}^{1-v} + p_1 \beta [A_1 k_1^{\alpha_1}(1-c_1)] - p_1 n k_1$$

where the costate p_1 satisfies

$$\dot{p}_1 = \alpha_1 (c_1 A_1)^{1-v} k_1^{\alpha_1(1-v)-1} + p_1\{n - \alpha_1 \beta A_1 k_1^{\alpha_1 - 1}(1-c_1)\},$$

$$p_1(T) = -b_1. \tag{7}$$

The c_1 dependent part of H is

$$H_1 = -\frac{1}{1-v}\{c_1 A_1 k_1^{\alpha_1}\}^{1-v} - p_1 \beta A_1 k_1^{\alpha_1} c_1.$$

If $p_1 > 0$, H_1 is monotonically decreasing as c_1 increases from 0, while if $p_1 < 0$, H_1 has a unique minimum for $c_1 > 0$. Thus, the optimal $c_1^* = 0$ in any subinterval is ruled out. The control c_1 is constrained to be in $0 \leqq c_1 \leqq 1$. Since H_1 monotonically decreases as c_1 increases when $p_1 > 0$, it follows that

H_1 will be minimum in that case if $c_1^* = 1$. For $p_1 < 0$, H_1 has a unique minimum for

$$c_1 = (-p_1 \beta A_1^v k_1^{\alpha_1 v})^{-1/v}.$$

If the right-hand side is less than 1, then it gives the optimal c_1^*, while if it is greater than or equal to 1, the optimal $c_1^* = 1$ again. Combining all these cases, we see that the optimal

$$c_1^* = \overline{\mathrm{sat}}[(-p_1 \beta A_1^v k_1^{\alpha_1 v})^{-1/v}], \tag{8}$$

where

$$\overline{\mathrm{sat}}(x) = x \quad \text{if} \quad 0 \leqq x < 1,$$

$$= 1 \quad \text{if} \quad \text{otherwise}.$$

We have yet to specify b_1. For this, we consider the steady-state values of k_1 and p_1, denoted k_{1st} and p_{1st}, respectively. The corresponding c_1^* we denote by c_{1st}^*. The state and costate equations in steady state become

$$\beta A_1 k^{\alpha_1}{}_{1st}(1 - c_{1st}^*) - n k_{1st} = 0,$$

$$\alpha_1 (c_{1st}^* A_1)^{1-v} k_{1st}^{\alpha_1(1-v)-1} + p_{1st}\{n - \alpha_1 \beta A_1 k_{1st}^{\alpha_1 - 1}(1 - c_{1st}^*)\} = 0,$$

$$c_{1st}^* = \overline{\mathrm{sat}}[(-p_{1st} \beta A_1^v k_{1st}^{\alpha_1 v})^{-1/v}].$$

The solution of this set of equations is

$$k_{1st} = (\alpha_1 \beta A_1 / n)^{1/1-\alpha_1},$$
$$p_{1st} = -\beta^{-1+v}(\beta A_1 k_{1st}^{\alpha_1} - n k_{1st})^{-v}. \tag{9}$$

We take $b_1 = -p_{1st}$.

We have to solve a two-point boundary value problem to determine c_1^*. Note that the boundary condition for p_1 depends on β. Thus, the decision of the regulating agency of the region 1 regarding the weight b_1 in its criterion functional depends upon the β chosen by the central planning board. For a given β, we have obtained the optimal c_1^* and the corresponding k_1^*. To perform optimization at the higher level, we have to know the capital accumulation of the region 2. This is given by

$$\dot{k}_2 = (1-\beta)\frac{L_{10}}{L_{20}} A_1 k_1^{*\alpha_1}(1 - c_1^*) - n k_2, \qquad k_2(0) = k_{20}, \quad \text{given.} \tag{10}$$

Let k_2^* denote the solution of this equation. We are, then, left with the simple optimization of finding β that maximizes

$$\mathscr{J} = \frac{1}{(1-v)^2}\int_0^T \{c_1^* A_1 k_1^{*\alpha_1} A_2 k_2^{*\alpha_2}\}^{1-v}\,dt.$$

4. Numerical results

The optimization for β is performed using simple search technique. For numerical experiments, we must specify all the numerical values of the parameters and prescribe the necessary initial conditions. We use data in comparable unit as used by Das (1974) who studied an elaborate optimal investment planning problem in the framework of the Feldman–Mahalanobis model. Our results are illustrative rather than prescriptive. Although the units correspond to a planning problem relevant to the economy of India, the values chosen for the initial conditions are somewhat arbitrary:

L_{10} = 10 (number of workers in millions),
L_{20} = 100 (number of workers in millions),
$n = 0.0488$,
T = 10 (number of years),
$v = 0.6$,
$\alpha_1 = 0.6$, $A_1 = 0.32$,
$\alpha_2 = 0.4$, $A_2 = 0.149$,
k_{10} = 0.1 (in rupees 10,000),
$k_{20} = 0.002$,
= 0.007 (in rupees 10,000),
= 0.02.

The table below shows the dependence of b_1 on β

β	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
b_1	1.00	1.22	1.53	1.97	2.64	3.73	5.71	9.86	21.3	79.5

We see that b_1 increases steadily with decreasing β and increases drastically towards the end when β becomes 0.2 and gets smaller. With very small β, region 1 virtually tries to maximize only its final capital as, otherwise, it would be left with too little capital at the end of the planning period.

The following graph gives the functional relation of $\mathscr{J}$ with β corresponding to the case when $k_{20} = 0.002$, 0.007 and 0.02. With fixed initial capital of the advanced sector, we see that the optimal β is steadily higher with the steadily increasing initial capital of the traditional sector. This is not surprising as lesser capital transfer is expected when the traditional sector is not too worse off in relation to the advanced sector.

The drastic fall of $\mathcal{J}$ after β decreases to 0.2 and less is explained by the fact that for β in the range, b_1 is drastically high so that the consumption is very low in the region 1. On the other hand, if the regulating agency of region 1 takes b_1 to be constant corresponding to $\beta=1$, the consumption of region 1 will be large throughout the planning period and $\mathcal{J}$ will be flat for β smaller than 0.2.

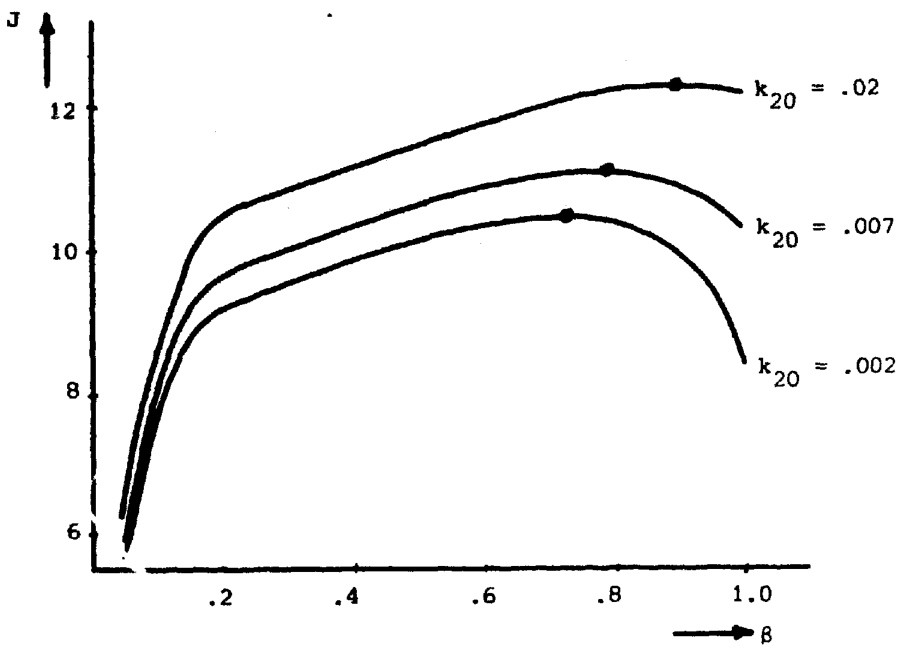

Fig. 1. Dependence of $\mathcal{J}$ on β.

5. Generalization of the model

The situation is somewhat different when one considers the investment allocation problem in an industrialized country with income disparity between two regions. The assumption that region 2 consumes all it produces is not realistic anymore. We, therefore, extend our model to the situation when both the regions save using their own criterion. Thus in the lower level, we have the following problem:

$$\dot{k}_1=\beta f_1(k_1)(1-c_1)+(1-\gamma)\frac{L_{20}}{L_{10}}f_2(k_2)(1-c_2)-nk_1,$$

$$\dot{k}_2=(1-\beta)\frac{L_{10}}{L_{20}}f_1(k_1)(1-c_1)+\gamma f_2(k_2)(1-c_2)-nk_2,$$

$$k_2(0)=k_{20},\quad 0\leqq c_i\leqq 1,\quad i=1,2.$$

Region i wants to maximize

$$\mathscr{J}_i=\frac{1}{1-v}\int_0^T\{c_i f_i(k_i(t))\}^{1-v}\,\mathrm{d}t+b_i k_i(T),$$

where b_i is the price of capital of region i on the golden rule path. The central planning authority, which works as a higher level decision maker, chooses β and γ. For fixed β and γ, we have, thus, a non-zero sum differential game problem at the lower level. Solution concepts, therefore, become crucial. The two regions, for example, may decide to play Nash or, to play Pareto. In the latter situation, the problem reduces to determining c_1 and c_2 so as to maximize

$$\alpha\mathscr{J}_1+(1-\alpha)\mathscr{J}_2$$

for some α, $0<\alpha<1$. In the former case, the problem is even more complicated. This is the situation of interregional conflict. First, it must be decided to which class c_1 and c_2 belong (e.g., open or closed loop class). In principle, one has to solve for optimal c_1 given a fixed c_2 and then solve for optimal c_2 given a fixed c_1. The resulting pair of functional equations will determine the optimal strategies c_1^* and c_2^*. The optimization problem at the higher level remains the same as before. Solution patterns for this problem are being investigated by the authors at present.

6. Conclusion

We posed the investment allocation problem between two regions in a new way, which resulted in a Stackelberg-type hierarchical optimization problem. The idea was to shed some light in the current debate in developing countries with dual economy about the transfer of investment capital from the advanced to the traditional sector of the economy. Generalization of the model to an industrialized country with income disparity between two regions has also been mentioned, although the actual solution was not worked out.

An obvious extension of the present work is to consider time-varying β and pose the higher level optimization also as a control problem. This could be formulated both as open- or closed-loop problem. This is considerably more complicated. For long-term planning problem, one is compelled to consider this situation. On the other hand, the assumption of total consumption of the output of the traditional sector becomes dubious if the planning horizon is very large.

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