MATHEMATICS

ITERATIVE SOLUTION OF A DISCRETE AXIALLY SYMMETRIC POTENTIAL PROBLEM

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Abstract

The Dirichlet problem for the axially symmetric potential equation in a cylindrical domain is discretized by means of a five-point difference approximation. The resulting difference equation is solved by point or line iterative methods. The rate of convergence of these methods is determined by the spectral radius of the underlying point or line Jacobi matrix. An asymptotic approximation for this spectral radius, valid for small mesh size, is derived.

1. Introduction

We consider the axially symmetric Poisson equation

$$\Delta\Phi \equiv \frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{\partial^2\Phi}{\partial z^2}=F(r,z) \tag{1.1}$$

in a cylindrical domain D: $0\leqslant r<Mh$, $0<z<Nh$; r, z denote cylindrical coordinates and M, N are positive integers. Along the boundary of D a Dirichlet boundary condition is prescribed. The present potential problem is numerically solved by a finite-difference method. The domain D is covered by a square net of mesh length h and the usual five-point difference approximation is applied. Thus we arrive at the system of linear equations

$$\begin{cases} \Phi_{m,n}=\frac{1}{4}\left(1+\dfrac{1}{2m}\right)\Phi_{m+1,n}+\frac{1}{4}\left(1-\dfrac{1}{2m}\right)\Phi_{m-1,n}+\frac{1}{4}\Phi_{m,n+1}+ \\ \qquad +\frac{1}{4}\Phi_{m,n-1}-\frac{1}{4}h^2F_{m,n}, \quad m=1(1)M-1, \quad n=1(1)N-1; \\ \Phi_{0,n}=\frac{2}{3}\Phi_{1,n}+\frac{1}{6}\Phi_{0,n+1}+\frac{1}{6}\Phi_{0,n-1}-\frac{1}{6}h^2F_{0,n}, \quad n=1(1)N-1; \end{cases} \tag{1.2}$$

where the notation $\Phi_{m,n}=\Phi(mh,nh)$, $F_{m,n}=F(mh,nh)$ is employed. We remark that the five-point difference approximation induces a formal discretization error of order h^2.

The system (1.2) is shortly written as

$$\Phi=B\Phi+a, \tag{1.3}$$

where B is a matrix of order $M(N-1)$ with zero diagonal elements, Φ denotes the vector with elements $\Phi_{m,n}$, and a is a given vector determined by the set of values $F_{m,n}$ and by the Dirichlet data along the boundary of D. In the terminology of VARGA [6], B is the "point Jacobi matrix" associated with the discrete potential problem (1.2). The matrix equation (1.3) is most conveniently solved by one of the point iterative methods going by the names of Jacobi, Gauss-Seidel and successive overrelaxation. In connection with the latter method we notice that the matrix $I-B$ does have Young's property A, hence, the Young-Frankel theory of successive overrelaxation is applicable. It is well known that the rate of convergence of the various iterative methods is determined by the eigenvalues of the point Jacobi matrix B, in particular by its spectral rádius $\varrho(B)$. In Section 2 of this paper the main properties of the eigenvalues of the matrix B are discussed. In Section 3 we derive a three-term asymptotic expansion for the spectral radius $\varrho(B)$, valid for large values of M, N. Section 4 deals with the solution of equation (1.3) by line iterative methods. The eigenvalues and spectral radius of the corresponding "line Jacobi matrix" are investigated.

Boundary-value problems for the axially symmetric potential equation (1.1) arise, e.g., in electron optics, cf. WEBER [7]. The Dirichlet problem stated above is meant as a model problem that allows an easy comparison of the convergence properties of point and line iterative methods. A similar model problem for the two-dimensional Laplace equation in a square domain was considered by FORSYTHE and WASOW [3], Sec. 22.5, and VARGA [6], Sec. 6.5.

2. Eigenvalues of the Point Jacobi Matrix B

The eigenvalues λ of the matrix B are determined by the homogeneous system of equations

$$(2.1)\quad \begin{cases} \tfrac{1}{4}\left(1+\dfrac{1}{2m}\right)\Phi_{m+1,n}+\tfrac{1}{4}\left(1-\dfrac{1}{2m}\right)\Phi_{m-1,n}+\tfrac{1}{4}\Phi_{m,n+1}+\tfrac{1}{4}\Phi_{m,n-1}=\lambda\Phi_{m,n}, \\ \qquad\qquad m=1(1)M-1,\ n=1(1)N-1; \\ \tfrac{2}{3}\Phi_{1,n}+\tfrac{1}{6}\Phi_{0,n+1}+\tfrac{1}{6}\Phi_{0,n-1}=\lambda\Phi_{0,n},\ n=1(1)N-1; \\ \Phi_{m,0}=\Phi_{m,N}=0,\ m=0(1)M-1;\ \Phi_{M,n}=0,\ n=0(1)N. \end{cases}$$

The latter system is reduced by separation of variables. Substituting

$$(2.2)\qquad \Phi_{m,n}=\varphi_m \sin(k\pi n/N),\ k=1(1)N-1,$$

the equations (2.1) separate into $N-1$ systems, viz.

$$(2.3)\quad \begin{cases} \tfrac{1}{4}\left(1+\dfrac{1}{2m}\right)\varphi_{m+1}+\tfrac{1}{4}\left(1-\dfrac{1}{2m}\right)\varphi_{m-1}+\tfrac{1}{2}\alpha_k\varphi_m=\lambda\varphi_m,\ m=1(1)M-1; \\ \tfrac{2}{3}\varphi_1+\tfrac{1}{3}\alpha_k\varphi_0=\lambda\varphi_0;\ \varphi_M=0, \end{cases}$$

where $\alpha_k=\cos(k\pi/N)$.

The system (2.3) can be solved in two ways. First of all, it is obvious that the solutions λ of (2.3) coincide with the eigenvalues of a tri-diagonal matrix of order M. By a simple similarity transformation the latter matrix can be reduced to a symmetric tri-diagonal matrix C_k with non-zero elements

$$(2.4)\quad \begin{cases} \operatorname{diag}[1]=\frac{1}{3}\alpha_k,\ \operatorname{diag}[i]=\frac{1}{2}\alpha_k,\ i=2(1)M; \\ \operatorname{codiag}[1]=\frac{1}{6}\sqrt{3},\ \operatorname{codiag}[i]=\dfrac{2i-1}{8\sqrt{i(i-1)}},\ i=2(1)M-1. \end{cases}$$

Let the (real) eigenvalues of C_k be denoted by $\lambda_{j,k}$ numbered in decreasing order, i.e., $\lambda_{1,k} \geqslant \lambda_{2,k} \geqslant \ldots \geqslant \lambda_{M,k}$. Thus the eigenvalues of the matrix B are given by $\lambda_{j,k}$, $j=1(1)M$, $k=1(1)N-1$. We remark that the eigenvalues of a symmetric tri-diagonal matrix can be computed by the highly effective bisection method, cf. WILKINSON [8].

We list some elementary properties of the eigenvalues $\lambda_{j,k}$ of the matrix B:

PROPERTY 1. All eigenvalues $\lambda_{j,k}$ are real.

PROPERTY 2. For fixed j, $\lambda_{j,k}$ decreases when k increases.

PROPERTY 3. $\lambda_{j,k}=-\lambda_{M+1-j,N-k}$.

COROLLARY. $\varrho(B)=\max|\lambda_{j,k}|=\lambda_{1,1}$.

PROPERTY 4. $-\frac{2}{3}+\frac{1}{3}\alpha_k \leqslant \lambda_{j,k} \leqslant \frac{2}{3}+\frac{1}{3}\alpha_k$; hence, all eigenvalues $\lambda_{j,k}$ certainly lie in the interval $(-1, 1)$.

Property 2 follows from the minimax characterization of eigenvalues; notice that for a fixed column vector x the Rayleigh quotient $x^T C_k x / x^T x$ decreases when k increases. Property 3 is obvious when comparing the matrices C_k and C_{N-k}. Property 4 follows from Gerschgorin's theorem applied to the original matrix associated with (2.3). It follows from property 4 that the point iterative methods associated with the matrix B do converge.

Considered from a second point of view, the system of equations (2.3) is equivalent to the following recurrence relation. Setting

$$(2.5)\qquad \lambda=\tfrac{1}{2}z+\tfrac{1}{2}\alpha_k,$$

we introduce the sequence of polynomials $\varphi_m(z)$ determined by the recurrence relation

$$(2.6)\qquad (2m+1)\varphi_{m+1}(z)-4mz\varphi_m(z)+(2m-1)\varphi_{m-1}(z)=0,\ m=1, 2, \ldots,$$

and the initial conditions

$$(2.7)\qquad \varphi_0(z)=1,\ \varphi_1(z)=\tfrac{3}{4}z+\tfrac{1}{4}\alpha_k.$$

For simplicity the dependence on k is suppressed in the notation $\varphi_m(z)$.

The polynomials $\varphi_m(z)$ are related to the discrete Bessel functions intro duced by BOYER [1]. The sequence $\{\varphi_m(z)\}$ is a Sturm sequence, hence, the polynomial $\varphi_m(z)$ (of degree m) has m distinct, real zeros which are separated by the zeros of $\varphi_{m-1}(z)$. A comparison of (2.6), (2.7) with (2.3) shows that the eigenvalues λ are related through (2.5), to the zeros z of $\varphi_M(z)$.

Finally, we deduce a lemma to be used in Section 3. The relation (2.6) with $z=1$ is rearranged as

$$(2m+1)[\varphi_{m+1}(1)-\varphi_m(1)]=(2m-1)[\varphi_m(1)-\varphi_{m-1}(1)],$$

thus permitting a simple evaluation of $\varphi_m(1)$, viz.

$$\varphi_m(1)=1-\tfrac{1}{4}(1-\alpha_k)\sum_{r=1}^{m}\frac{1}{2r-1}. \tag{2.8}$$

Let M_k be the smallest integer such that

$$\sum_{r=1}^{M_k}\frac{1}{2r-1}>\frac{4}{1-\alpha_k}. \tag{2.9}$$

Then the number of sign changes in the Sturm sequence $\varphi_0(1), \varphi_1(1), \ldots, \ldots, \varphi_m(1)$ is either zero or one, depending on $m<M_k$ or $m\geqslant M_k$, respectively. Hence, according to a well-known property of Sturm sequences the following lemma holds true:

LEMMA. For $m<M_k$ all zeros of $\varphi_m(z)$ are less than 1. For $m\geqslant M_k$ the largest zero of $\varphi_m(z)$ is greater than or equal to 1 whereas the remaining zeros are less than 1.

3. Asymptotic Expansion of the Spectral Radius $\varrho(B)$

In this section $\{\varphi_m(z)\}$ is the sequence of polynomials determined by (2.6), (2.7) with $k=1$. It was observed by BOYER [1] that the Legendre functions $P_{m-\frac{1}{2}}(z)$ and $Q_{m-\frac{1}{2}}(z)$ satisfy the recurrence relation (2.6) as well. Therefore $\varphi_m(z)$ can be expressed in terms of Legendre functions, viz.

$$\left\{\begin{aligned}\varphi_m(z)=&\tfrac{1}{8}(3z+\alpha_1)[P_{m-\frac{1}{2}}(z)Q_{-\frac{1}{2}}(z)-P_{-\frac{1}{2}}(z)Q_{m-\frac{1}{2}}(z)]\\&-\tfrac{1}{2}[P_{m-\frac{1}{2}}(z)Q_{\frac{1}{2}}(z)-P_{\frac{1}{2}}(z)Q_{m-\frac{1}{2}}(z)],\end{aligned}\right. \tag{3.1}$$

where we used the relation (cf. [2], form. 3.4 (25), 3.8 (10))

$$P_{\frac{1}{2}}(z)Q_{-\frac{1}{2}}(z)-P_{-\frac{1}{2}}(z)Q_{\frac{1}{2}}(z)=2.$$

In Section 2 it was shown that the spectral radius $\varrho(B)$ is connected with the largest zero ζ_M of $\varphi_M(z)$ (compare (2.5)),

$$\varrho(B)=\tfrac{1}{2}\,\zeta_M+\tfrac{1}{2}\,\alpha_1. \tag{3.2}$$

According to the lemma one has $\zeta_M<1$ $(\zeta_M\geqslant 1)$ when $M<M_1$ $(M\geqslant M_1)$ where M_1 is the integer introduced in (2.9). We shall now investigate the asymptotics of the zero ζ_M when M is large.

First assume $M < M_1$ then $\zeta_M < 1$. We shall derive an asymptotic expansion for $\varphi_M(z)$ valid for large values of M, N. Simultaneously, the variable z will be close to 1 such that

$$z = 1 - \frac{x^2}{2M^2}, \tag{3.3}$$

where x is bounded. From ROBIN [5], Sec. 87, we quote the following expansion for $P_{M-\frac{1}{2}}(z)$ in terms of Bessel functions of the first kind:

$$P_{M-\frac{1}{2}}(z) = J_0(x) + \frac{x^2}{4M^2}\left[\frac{x}{6}J_3(x) - J_2(x) + \frac{1}{2x}J_1(x)\right] + O(1/M^4). \tag{3.4}$$

By means of the recurrence relation for Bessel functions, (3.4) can be reduced to

$$P_{M-\frac{1}{2}}(z) = J_0(x) + \frac{1}{M^2}\left[\frac{x^2}{12}J_0(x) - \frac{x^3+x}{24}J_1(x)\right] + O(1/M^4). \tag{3.5}$$

A similar expansion in terms of Bessel functions of the second kind holds for $Q_{M-\frac{1}{2}}(z)$, see ROBIN [5], Sec. 87,

$$Q_{M-\frac{1}{2}}(z) = -\frac{\pi}{2}\left\{Y_0(x) + \frac{1}{M^2}\left[\frac{x^2}{12}Y_0(x) - \frac{x^3+x}{24}Y_1(x)\right] + O(1/M^4)\right\}. \tag{3.6}$$

Expansions for $P_{\pm\frac{1}{2}}(z)$, $Q_{\pm\frac{1}{2}}(z)$ are adopted from ERDÉLYI [2], form. 3.2 (14), 3.6.1 (11), viz.

$$P_{-\frac{1}{2}}(z) = F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{1-z}{2}\right) = 1 + \frac{x^2}{16M^2} + O(1/M^4), \tag{3.7}$$

$$P_{\frac{1}{2}}(z) = F\left(-\tfrac{1}{2}, \tfrac{3}{2}; 1; \frac{1-z}{2}\right) = 1 - \frac{3x^2}{16M^2} + O(1/M^4), \tag{3.8}$$

$$(3.9)\quad \begin{cases} Q_{-\frac{1}{2}}(z) = \tfrac{1}{2}P_{-\frac{1}{2}}(z)\left[\log\dfrac{1+z}{1-z} - 2\psi(\tfrac{1}{2})\right] \\ \qquad + \dfrac{1}{\pi}\displaystyle\sum_{l=0}^{\infty}\frac{\Gamma(l+\frac{1}{2})\Gamma(l+\frac{1}{2})}{l!\,l!}\psi(l+1)\left(\frac{1-z}{2}\right)^l = \\ \qquad = -\left[1 + \dfrac{x^2}{16M^2}\right]\log\left(\dfrac{x}{8M}\right) - \dfrac{x^2}{16M^2} + O(\log M/M^4), \end{cases}$$

$$(3.10)\quad \begin{cases} Q_{\frac{1}{2}}(z) = \tfrac{1}{2}P_{\frac{1}{2}}(z)\left[\log\dfrac{1+z}{1-z} - 2\psi(\tfrac{3}{2})\right] \\ \qquad - \dfrac{1}{\pi}\displaystyle\sum_{l=0}^{\infty}\frac{\Gamma(l-\frac{1}{2})\Gamma(l+\frac{3}{2})}{l!\,l!}\psi(l+1)\left(\frac{1-z}{2}\right)^l = \\ \qquad = -\left[1 - \dfrac{3x^2}{16M^2}\right]\log\left(\dfrac{x}{8M}\right) - 2 + \dfrac{x^2}{16M^2} + O(\log M/M^4), \end{cases}$$

where F stands for the hypergeometric function. The previous expansions are substituted into the representation (3.1) for $\varphi_M(z)$ and α_1 is replaced by

$$\alpha_1 = \cos(\pi/N) = 1 - \frac{\pi^2}{2N^2} + O(1/N^4). \tag{3.11}$$

Thus we ultimately obtain the following expansion for $\varphi_M(z)$:

$$\left\{\begin{aligned} \varphi_M(z) &= J_0(x) + \frac{1}{48M^2}[x^2J_0(x) - (2x^3+2x)J_1(x) \\ &\quad + 3x^2\{J_0(x)\log\left(\frac{x}{8M}\right) - \frac{\pi}{2}Y_0(x)\}] \\ &\quad + \frac{\pi^2}{16N^2}[J_0(x)\log\left(\frac{x}{8M}\right) - \frac{\pi}{2}Y_0(x)] + O(\log M/P^4), \end{aligned}\right. \tag{3.12}$$

where $P = \min(M, N)$.

On the basis of (3.12) the smallest x-zero of $\varphi_M(z)$ can easily be determined, viz.

$$x = j_0 - \frac{j_0}{48M^2}[2j_0^2 + 2 + 3/J_1^2(j_0)] - \frac{\pi^2}{16N^2}\frac{1}{j_0J_1^2(j_0)} + O(\log M/P^4), \tag{3.13}$$

where j_0 is the smallest positive zero of $J_0(x)$, $j_0 = 2.4048 \ldots$. Some elementary properties of Bessel functions, like

$$J_0'(x) = -J_1(x), \quad Y_0(j_0) = 2/[\pi j_0 J_1(j_0)],$$

were intermediate in the derivation of (3.13).

Finally, by means of (3.2), (3.3) we accomplish the asymptotic expansion of the spectral radius $\varrho(B)$,

$$\left\{\begin{aligned} \varrho(B) &= \tfrac{1}{2} - \frac{x^2}{4M^2} + \tfrac{1}{2}\cos(\pi/N) = \\ &= 1 - \left(\frac{j_0^2}{4M^2} + \frac{\pi^2}{4N^2}\right) + \left(\frac{\beta_1}{M^4} + \frac{\beta_2}{M^2N^2} + \frac{\pi^4}{48N^4}\right) + O(\log P/P^6), \end{aligned}\right. \tag{3.14}$$

with

$$\beta_1 = \frac{j_0^2}{96}[2j_0^2 + 2 + 3/J_1^2(j_0)] = 1.48781579 \ldots,$$

$$\beta_2 = \pi^2/[32J_1^2(j_0)] = 1.14437467 \ldots.$$

Weber [7] derived the two-term approximation

$$\varrho(B) \approx 1 - \left(\frac{j_0^2}{4M^2} + \frac{\pi^2}{4N^2}\right) \tag{3.15}$$

by means of a method of Garabedian [4].

In the case $M \geqslant M_1$ one has $\zeta_M \geqslant 1$. From the discussion in Section 2 it is obvious that the sequence of largest zeros ζ_m is increasing and bounded. Hence, $\lim \zeta_m = \zeta_\infty$ does exist and $1 \leqslant \zeta_M < \zeta_\infty$. Consider the polynomial $\varphi_m(z)$ as given by (3.1). For fixed $z > 1$, $Q_{m-\frac{1}{2}}(z)/P_{m-\frac{1}{2}}(z) \to 0$ when $m \to \infty$, cf. ROBIN [5], Sec. 83, 84. Thus for $m \to \infty$ the equation $\varphi_m(z) = 0$ passes into

$$\tfrac{1}{8}(3z+\alpha_1)Q_{-\frac{1}{2}}(z) - \tfrac{1}{2}Q_{\frac{1}{2}}(z) = 0 \tag{3.16}$$

and the latter equation has ζ_∞ as its unique root greater than 1. Replacing $Q_{\pm\frac{1}{2}}(z)$ by its expansion around $z=1$ (compare (3.9), (3.10)), the limit equation (3.16) is approximated by

$$1 + \frac{1-\alpha_1}{16} \log \frac{z-1}{32} = 0, \tag{3.17}$$

from which we derive

$$\zeta_\infty \approx 1 + 32 \exp\left[-16/(1-\alpha_1)\right] \approx 1 + 32e^{-8/3} \exp(-32N^2/\pi^2). \tag{3.18}$$

In consequence, we obtain the asymptotic results

$$\zeta_M = 1 + O(\exp(-32N^2/\pi^2)), \tag{3.19}$$

$$\varrho(B) = \tfrac{1}{2}\zeta_M + \tfrac{1}{2}\alpha_1 = 1 - \frac{\pi^2}{4N^2} + \frac{\pi^4}{48N^4} + O(1/N^6). \tag{3.20}$$

The integer M_1 was introduced in (2.9). By some simple asymptotic analysis it can be shown that

$$M_1 \sim K \exp(16N^2/\pi^2), \quad N \to \infty, \tag{3.21}$$

where K is a certain constant. Therefore, since $M \geqslant M_1$, the expansion (3.20) is just a special case of (3.14) and the latter expansion holds true for any large values of M, N.

The asymptotic approximation (3.14) can be employed for the actual computation of $\varrho(B)$. The approximate value thus obtained has been compared with the exact value of $\varrho(B)$ computed as the largest eigenvalue of the matrix C_1 given by (2.4). It was found that over the range $M, N = 10(10)100$, the accuracy of the approximation (3.14) increases from 5 to 11 decimal places. Over the same range the corresponding approximation for the optimum relaxation factor ω_0 (see (3.23) below) has an accuracy that increases from 5 to 9 decimal places.

Following VARGA [6], the convergence behaviour of an iteration matrix A is described by the asymptotic rate of convergence $R_\infty(A)$ defined by

$$R_\infty(A) = -\log \varrho(A).$$

Thus the Jacobi iterative method has a rate of convergence

$$R_\infty(B) = -\log \varrho(B) \sim \tfrac{1}{4}\left[\frac{j_0^2}{M^2} + \frac{\pi^2}{N^2}\right], \quad M, N \to \infty. \tag{3.22}$$

The successive overrelaxation method attains a maximal rate of convergence when the relaxation factor is equal to the optimum value ω_0 given by

$$\omega_0 = 2/[1+\sqrt{1-\varrho^2(B)}]. \tag{3.23}$$

Let the corresponding successive relaxation matrix be denoted by $\mathscr{L}_{\omega_0}$, then (cf. VARGA [6])

$$R_\infty(\mathscr{L}\omega_0) = -\log(\omega_0 - 1) \sim \sqrt{2}\left[\frac{j_0^2}{M^2}+\frac{\pi^2}{N^2}\right]^{\frac{1}{2}}, \quad M, N \to \infty, \tag{3.24}$$

which shows the superior convergence of the overrelaxation method.

4. Eigenvalues and Spectral Radius of the Line Jacobi Matrix

This section deals with the solution of the difference equation (1.2) by line iterative methods. For a general discussion of line iterative methods, see FORSYTHE and WASOW [3], Sec. 22.3, and VARGA [6], Sec. 6.4. In the present case the system (1.2) is partitioned into M subsystems numbered by $m = 0(1)M-1$, each consisting of $N-1$ equations for the unknown components $\Phi_{m,n}$, $n = 1(1)N-1$. The individual subsystems are solved by Gaussian elimination; since the underlying matrix is tri-diagonal, the latter method is most convenient. Proceeding along these lines, the system (1.2) takes a form similar to (1.3), viz.

$$\Phi = B^L\Phi + a^L, \tag{4.1}$$

where B^L is a matrix of order $M(N-1)$ with zero $M \times M$ diagonal blocks and a^L is a given vector. In accordance with VARGA [6], B^L is called the "line Jacobi matrix". The matrix equation (4.1) is again solved by the iterative methods of Jacobi, Gauss-Seidel or by successive overrelaxation. The rate of convergence of these methods is determined by the spectral radius $\varrho(B^L)$ of the matrix B^L. In a treatment that parallels Sections 2 and 3, we shall now investigate the matrix B^L and the asymptotics of its spectral radius.

In view of (1.2) the eigenvalues μ of the matrix B^L are determined by the homogeneous system of equations

$$\begin{cases} \frac{1}{4}\left(1+\frac{1}{2m}\right)\Phi_{m+1,n} + \frac{1}{4}\left(1-\frac{1}{2m}\right)\Phi_{m-1,n} = \\ \qquad = \mu(\Phi_{m,n} - \frac{1}{4}\Phi_{m,n+1} - \frac{1}{4}\Phi_{m,n-1}), \quad m = 1(1)M-1, \quad n = 1(1)N-1; \\ \frac{2}{3}\Phi_{1,n} = \mu(\Phi_{0,n} - \frac{1}{6}\Phi_{0,n+1} - \frac{1}{6}\Phi_{0,n-1}), \quad n = 1(1)N-1; \\ \Phi_{m,0} = \Phi_{m,N} = 0, \quad m = 0(1)M-1; \quad \Phi_{M,n} = 0, \quad n = 0(1)N. \end{cases} \tag{4.2}$$

Applying the substitution (2.2), the equations (4.2) separate into $N-1$

systems, viz.

$$(4.3)\quad \begin{cases} \tfrac{1}{4}\left(1+\dfrac{1}{2m}\right)\varphi_{m+1}+\tfrac{1}{4}\left(1-\dfrac{1}{2m}\right)\varphi_{m-1}=\mu(1-\tfrac{1}{2}\alpha_k)\varphi_m, \quad m=1(1)M-1; \\ \tfrac{2}{3}\varphi_1=\mu(1-\tfrac{1}{3}\alpha_k)\varphi_0; \quad \varphi_M=0, \end{cases}$$

where $k=1(1)N-1$ and $\alpha_k=\cos(k\pi/N)$ as before.

The system (4.3) is equivalent to the generalized eigenproblem

$$(4.4)\qquad \det(C-\mu D_k)=0,$$

where C is a tri-diagonal matrix and D_k is a diagonal matrix dependent on k. By some simple transformations, (4.4) can be reduced to the standard eigenproblem for a symmetric tri-diagonal matrix E_k of order M, with zero diagonal elements and codiagonal elements given by

$$(4.5)\quad \begin{cases} \text{codiag}\,[1]=[2(2-\alpha_k)(3-\alpha_k)]^{-\frac{1}{2}}, \\ \text{codiag}\,[i]=\dfrac{2i-1}{4\sqrt{i(i-1)}}(2-\alpha_k)^{-1}, \quad i=2(1)M-1. \end{cases}$$

Let the (real) eigenvalues of E_k be denoted by $\mu_{j,k}$ numbered in decreasing order, i.e., $\mu_{1,k}\geqslant\mu_{2,k}\geqslant\ldots\geqslant\mu_{M,k}$. Thus the eigenvalues of the matrix B^L are given by $\mu_{j,k}$, $j=1(1)M$, $k=1(1)N-1$. Similarly to Section 2 we list some elementary properties of these eigenvalues $\mu_{j,k}$:

Property 1. All eigenvalues $\mu_{j,k}$ are real.

Property 2. For fixed j, $|\mu_{j,k}|$ decreases when k increases.

Property 3. $\mu_{j,k}=-\mu_{M+1-j,k}$.

Corollary. $\varrho(B^L)=\max|\mu_{j,k}|=\mu_{1,1}$.

Property 4. $|\mu_{j,k}|\leqslant 2/(3-\alpha_k)$; hence, all eigenvalues $\mu_{j,k}$ certainly lie in the interval $(-1, 1)$.

Property 2 follows from the minimax characterization in terms of the generalized Rayleigh quotient x^TCx/x^TD_kx corresponding to (4.4); notice that x^TD_kx increases when k increases. The remaining properties are proved in a similar way as in Section 2.

Alternatively, the system of equations (4.3) induces the following recurrence relation. Setting

$$(4.6)\qquad \mu=s/(2-\alpha_k),$$

we introduce the sequence of polynomials $\psi_m(s)$ determined by the recurrence relation

$$(4.7)\qquad (2m+1)\psi_{m+1}(s)-4ms\psi_m(s)+(2m-1)\psi_{m-1}(s)=0, \quad m=1, 2, \ldots,$$

under the initial conditions

$$\psi_0(s)=1,\ \psi_1(s)=\frac{3-\alpha_k}{2(2-\alpha_k)}s. \tag{4.8}$$

As before, the sequence $\{\psi_m(s)\}$ is a Sturm sequence and the eigenvalues μ are related through (4.6), to the zeros s of $\psi_M(s)$. Similarly to (2.8) we evaluate

$$\psi_m(1)=1-\frac{1-\alpha_k}{2(2-\alpha_k)}\sum_{r=1}^{m}\frac{1}{2r-1}. \tag{4.9}$$

Then the lemma of Section 2 also applies to $\psi_m(s)$ provided that M_k is replaced by M_k' which is the smallest integer such that

$$\sum_{r=1}^{M_k'}\frac{1}{2r-1}\geqslant\frac{2(2-\alpha_k)}{1-\alpha_k}. \tag{4.10}$$

Henceforth, we set $k=1$ in (4.6), (4.8). Similarly to (3.1) the polynomial $\psi_m(s)$ is expressed in terms of Legendre functions, viz.

$$\left\{\begin{aligned}\psi_m(s)=&\frac{3-\alpha_1}{4(2-\alpha_1)}s[P_{m-\frac{1}{2}}(s)Q_{-\frac{1}{2}}(s)-P_{-\frac{1}{2}}(s)Q_{m-\frac{1}{2}}(s)]\\&-\tfrac{1}{2}[P_{m-\frac{1}{2}}(s)Q_{\frac{1}{2}}(s)-P_{\frac{1}{2}}(s)Q_{m-\frac{1}{2}}(s)].\end{aligned}\right. \tag{4.11}$$

Then the spectral radius $\varrho(B^L)$ is connected with the largest zero σ_M of $\psi_M(s)$ (compare (4.6)),

$$\varrho(B^L)=\sigma_M/(2-\alpha_1). \tag{4.12}$$

Assume $M<M_1'$ then $\sigma_M<1$. Proceeding as in Section 3, the polynomial $\psi_M(s)$ is expanded for large values of M, N, and s close to 1 such that

$$s=1-\frac{x^2}{2M^2}, \tag{4.13}$$

where x is bounded. As a result we obtain the expansion

$$\left\{\begin{aligned}\psi_M(s)=&J_0(x)+\frac{1}{48M^2}[x^2J_0(x)-(2x^3+2x)J_1(x)\\&+6x^2\{J_0(x)\log\left(\frac{x}{8M}\right)-\frac{\pi}{2}Y_0(x)\}]\\&+\frac{\pi^2}{8N^2}[J_0(x)\log\left(\frac{x}{8M}\right)-\frac{\pi}{2}Y_0(x)]+O(\log M/P^4),\end{aligned}\right. \tag{4.14}$$

where $P = \min(M, N)$. On the basis of (4.14) an approximation to σ_M can easily be deduced. Omitting all further details we state the final asymptotic expansion for the spectral radius $\varrho(B^L)$, viz.

$$(4.15)\quad \varrho(B^L) = 1 - \left(\frac{j_0^2}{2M^2} + \frac{\pi^2}{2N^2}\right) + \left(\frac{\gamma_1}{M^4} + \frac{\gamma_2}{M^2N^2} + \frac{7\pi^4}{24N^4}\right) + O(\log P/P^6),$$

with

$$\gamma_1 = \frac{j_0^2}{24}[j_0^2 + 1 + 3/J_1^2(j_0)] = 4.3164543 \ldots,$$

$$\gamma_2 = \frac{\pi^2}{8}[2j_0^2 + 1/J_1^2(j_0)] = 18.8469381 \ldots,$$

and j_0 is again the smallest positive zero of $J_0(x)$. It has been verified that the expansion (4.15) remains correct in the case $M \geqslant M_1'$.

A numerical comparison of the approximation (4.15) and the exact value of $\varrho(B^L)$ over the range $M, N = 10(10)100$, discloses that the accuracy increases from 3 to 9 decimal places. The corresponding approximation for the optimum relaxation factor

$$(4.16)\quad \omega_0 = 2/[1 + \sqrt{1 - \varrho^2(B^L)}]$$

has an accuracy that increases from 3 to 7 decimal places over the same range of M, N.

Finally, we determine the asymptotic rate of convergence of the line Jacobi method, viz.

$$(4.17)\quad R_\infty(B^L) = -\log \varrho(B^L) \sim \tfrac{1}{2}\left[\frac{j_0^2}{M^2} + \frac{\pi^2}{N^2}\right], \quad M, N \to \infty,$$

and of the line successive overrelaxation method, viz.

$$(4.18)\quad R_\infty(\mathcal{L}_{\omega_0}^L) = -\log(\omega_0 - 1) \sim 2\left[\frac{j_0^2}{M^2} + \frac{\pi^2}{N^2}\right]^{\frac{1}{2}}, \quad M, N \to \infty.$$

The latter rate of convergence shows a gain by a factor of $\sqrt{2}$ over the corresponding result (3.24) for the point successive overrelaxation method.

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