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Analytical Mini–Max Methods for Hamiltonian Brake Orbits of Prescribed Energy

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1. INTRODUCTION

This paper is concerned with explicit characterizations of periodic solutions of second-order autonomous Hamiltonian systems of the form

$$-\ddot{q} = V'(q), \qquad q(t) \in \mathbb{R}^N, \tag{1.1}$$

where the potential V is required to satisfy

$$(V)_0 : \begin{cases} V \in C^2(\mathbb{R}^N),\ V(x) \geqslant V(0) = 0 \\ V \text{ is strictly convex on } \mathbb{R}^N. \end{cases}$$

More specifically, we shall look for *brake orbits* (cf. [19]), i.e., nonconstant solutions of (1.1) which satisfy for some $T > 0$

$$\dot{q}(0) = \dot{q}(T) = 0. \tag{1.2}$$

Upon periodic continuation, a solution of (1.1), (1.2) will produce a periodic solution of (1.1) with period $2T$. The trajectory in configuration space of such a motion is a simple curve connecting the two "restpoints" $q(0)$ and $q(T)$ along which the solution oscillates back and forth. A simple time-scaling according to

$$t = T\tau, \qquad q(t) = x(\tau), \qquad \tau \in [0, 1], \tag{1.3}$$

transforms (1.1), (1.2) into an equivalent problem on the fixed parameter interval $[0, 1]$:

$$\begin{aligned} -\ddot{x} &= T^2 V'(x) \\ \dot{x}(0) &= \dot{x}(1) = 0. \end{aligned} \tag{1.4}$$

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In this paper we shall not prescribe the value of the period but, rather, look for brake orbits with prescribed energy $E > 0$:

$$\tfrac{1}{2}\dot{q}^2 + V(q) = E. \tag{1.5}$$

Since we do not require E to be small, we look for finite-amplitude solutions, and global methods are needed.

The proof of the existence of solutions is a lively topic of research; Seifert [18] in 1948 has proved an existence result for this case, and later in 1978 both Weinstein [19] and Rabinowitz [17] obtained such existence results for more general Hamiltonian systems.

These proofs have in common that they rely on toplogical considerations and are not constructive in any sense. For a somewhat different problem, Berger [2] in 1971 obtained solutions in a more constructive way by minimizing the kinetic energy functional on the set of functions for which the potential energy functional (and thus not the total energy) has a prescribed value. For the prescribed energy case, Ekeland [6] in 1979 was able to characterize a solution as a minimizer for a certain functional which is related to the convex set bounded by the energy surface in the phase space $\mathbb{R}^{2N}$.

The aim of this paper is to present another, constructive, formulation of such a brake orbit. This formulation will deal with a certain functional J_E that depends on the potential V and the prescribed value of the energy E in an explicit way. In particular, no convex conjugation is needed, which is important for an investigation of the dependence of the solution on the parameter E (see [12]) and which is attractive for numerical purposes. The variational principle for J_E that will be used here seems to be new, but is ultimately related to the classical principle of "least" action (see [11] for a complete account).

As is very often the case for variational problems from mathematical physics, the critical points of J_E we are interested in, do not correspond to (local) extrema, but are saddle points. However, we shall show that in this case, we are able to obtain an *analytical* mini-max characterization of one of these saddle points. Such a characterization will lead to the construction of a specific subset N_E of the original domain of definition of the functional. This subset has the properties that (i) it can be described in an analytical way, (ii) it is a *natural constraint* for the original variational problem, and (iii) it is such that at least one saddle point of the original problem turns into a global minimizer of the functional when restricted to N_E. This last property provides a very explicit characterization of the solution. Besides its potential usefulness for a numerical approximation of the solution, it will imply that the corresponding brake orbit has *minimal* period. As a consequence of this, a multiplicity result for the number of distinct brake-orbits can be derived rather easily. (Another consequence is that the relation between prescribed value of the energy and the resulting period of the brake orbit, can be investigated; see [12]).

The idea to use natural constraints to turn saddle points into global minimizers, goes back to Poincaré [16] and was abstracted by Berger [3] to a functional analysis setting (see also Berger and Bombieri [5]). Nehari [15], for a special class of scalar, second-order equations, and Berger [2, 3], Berger and Schechter [4], who analyzed natural constraints for rather general classes of semi-linear equations, are among the few who contributed to the development and application of natural constraints. The analytical minimax characterization of this paper seems to be the first one that leads to a natural constraint which is an intersection of a natural constraint of Nehari-type with one of Berger-type. For a more general application of natural constraints in some elliptic problems with rotation symmetry, see [13].

In Section 2 we shall present the new variational principle for brake orbits of energy E. In Section 3, any solution of (1.1), (1.2), (1.5) is shown to correspond to a saddle point of J_E and a natural constraint N_E, of codimension $N+1$, is constructed. Minimizing J_E on N_E provides a brake orbit which has *minimal* period. The multiplicity result will be treated in Section 4; it requires a condition on V that resembles a condition on the Hamiltonian (in the general case) of Ekeland and Lasry [7]; see also [8–10] for the case of normal mode solutions instead of brake orbits.

The results of this paper have been presented on the Delft Nonlinear Analysis Day, May 1984, and announced in [14].

2. A Variational Principle for Prescribed Energy Orbits

In this section we shall introduce a new variational principle for solutions of (1.1), (1.2) which have energy E, i.e., satisfy (1.5). To that end, let $X = W_{1,2}([0, 1], \mathbb{R}^N)$ be the usual Sobolev space of N-vector functions. For given $E > 0$, let J_E be the functional defined on X by

$$J_E(x) = \left[\tfrac{1}{2}\int \dot{x}^2\right] \cdot \left[E - \int V(x)\right], \tag{2.1}$$

where, here and in the following, $\int$ denotes integration with respect respect to τ over $[0, 1]$. Furthermore, for $x \in X$ with $J_E(x) > 0$, let T be defined as

$$T = \left\{\frac{\frac{1}{2}\int \dot{x}^2}{E - \int V(x)}\right\}^{1/2}. \tag{2.2}$$

The first result states that brake orbits of energy E are in an one-to-one correspondence with critical points of J_E.

PROPOSITION 2.1. *Let* $E > 0$.

(i) *Let* x *be a critical point of* J_E *on* X *with* $J_E(x) > 0$. *Define* T *by* (2.2), *and let then* q *be defined by the transformation* (1.3). *Then* q *satisfies* (1.1), (1.2), (1.5).

(ii) *Let* q *satisfy* (1.1), (1.2), (1.5). *Then the function* x *defined by the transformation* (1.3) *is a critical point of* J_E *on* X.

Proof. (i) Multiplying the Euler–Lagrange equation of J_E by $(E - \int V(x))^{-1}$ (which is finite since $J_E(x) > 0$) shows that a critical point x of J_E satisfies Eq. (1.4) with T given by (2.2); the boundary conditions at $\tau = 0$, $\tau = 1$ arise as natural boundary conditions. Under the transformation (1.3) a solution q of (1.1), (1.2) is obtained. In order to calculate the energy of this solution, say $\hat{E}$, note that upon integrating the expression for energy conservation, it follows that

$$\int_0^T \tfrac{1}{2}\dot{q}^2\,dt + \int_0^T V(q)\,dt = T\hat{E}.$$

Since

$$\int_0^T V(q)\,dt = T\int_0^T V(x) \qquad \text{and} \qquad \int_0^T \tfrac{1}{2}\dot{q}^2\,dt = \frac{1}{T}\int_0^T \tfrac{1}{2}\dot{x}^2, \tag{2.3}$$

it follows from the expression (2.2) for T, that $\hat{E} = E$. Hence q satisfies (1.5).

(ii) To prove part (ii), first note that x satisfies (1.4). Furthermore, from (1.5) and (2.3) it follows that the relation (2.2) between T and x holds. From this the result is obtained in a standard way. ∎

Remark. From the relations (2.3) it follows that the value of J_E at a critical point x can be expressed in terms of the corresponding function q as:

$$J_E(x) = T\cdot\left(\int_0^T \tfrac{1}{2}\dot{q}^2\,dt\right)\cdot\left(E - \frac{1}{T}\int_0^T V(q)\,dt\right) = \left[\tfrac{1}{2}\int_0^T \dot{q}^2\,dt\right]^2. \tag{2.4}$$

This means that the critical value is the square of the *total kinetic energy* of the physical motion.

As presented here, the origin of the functional J_E may be somewhat obscure. In [11] it is shown, for more general Hamiltonian systems too, how a functional like (2.1) is obtained in a natural way from a modified version of the clasical principle of "least"-action (the Euler–Maupertius principle). Since the derivation (and formulation) of this classical principle is often obscure in the literature (see [11]), and in order to elucidate the modification, first consider the following (formal) derivation of the classical result. The starting point is the variational problem of looking for critical points of the action functional on phase space on the set of functions that satisfy the energy constraint in a pointwise fashion. Hence, with y the variable canonically conjugate to x (with respect to the parametrization with τ), this problem can be written as

$$\text{stat}\left\{\int \dot{x}y \,\middle|\, \tfrac{1}{2}y^2(\tau) + V(x(\tau)) = E\right\}, \tag{2.5}$$

where "stat" refers to taking stationary (i.e., critical) points. Then, taking for fixed configuration curve $x(\tau)$ which satisfies $V(x(\tau)) \leqslant E$ for each τ, the supremum with respect to y (satisfying $\frac{1}{2}y^2(\tau) = E - V(x(\tau))$), there results the usual variational principle in the configuration space; i.e., the problem of finding critical points x of the functional

$$\int \sqrt{2}\,\sqrt{E - V(x)}\,|\dot{x}|\,d\tau. \tag{2.6}$$

Now, the modification that leads to the functional J_E consists of prescribing the energy constraint in an integrated way, instead of in a pointwise fashion. Hence, instead of (2.5), consider

$$\text{stat}\left\{\int y\dot{x} \,\middle|\, \tfrac{1}{2}\int y^2 + \int V(x) = E\right\}. \tag{2.7}$$

Taking, for fixed $x \in X$ with $\int V(x) \leqslant E$, the supremum with respect to $y \in L_2([0, 1], \mathbb{R}^N)$, the variational problem (2.7) in phase-space reduces to a variational problem in the configuration space X for the functional

$$K_E(x) = \sqrt{2}\cdot\sqrt{E - \int V(x)}\,\sqrt{\int \dot{x}^2} \tag{2.8}$$

which can be written as

$$K_E(x) = 2\sqrt{J_E(x)} \tag{2.9}$$

with J_E given by (2.1). Note the essential difference between the functional (2.6), which is a distance functional for the (degenerate) metric $(E - V(x))\, dx \cdot dx$, and the functional (2.8) that is a product of functionals. Since K_E and J_E have the same critical points, we can, for the aims of this paper, simplify the notation by working with J_E instead of with K_E.

3. The Analytical Mini-Max Formulation

In this section we shall look for critical points of the functional J_E on X. Since J_E is neither bounded from below nor from above on X we will look for saddle points. A natural decomposition of the space X will lead to a description of X as the union of sets M_ξ over the space of normalized functions with mean value zero. Each M_ξ is a halfspace in $\mathbb{R}^{N+1}$ and J_E attains its maximum on M_ξ at an unique point.

The restriction of the functional to these constrained maximizers leads us to look for critical points (and, in particular, a minimizer) of J_E on a manifold of codimension $N+1$.

Let us start with the decomposition of the space X. Let Y be the subspace of vector functions with mean value zero: $Y := \{y \in X \mid \int y = 0\}$. Then Y is a Hilbert space for which we can take as norm $\|y\| := \{\int \dot{y}^2\}^{1/2}$. Let S be the unit sphere in Y: $S = \{\xi \in Y \mid \|\xi\| = 1\}$. Then we can write the L_2-orthogonal direct sum $X = \mathbb{R}^N \oplus Y$ as

$$X = \mathbb{R}^N \oplus \mathbb{R}_+ \times S,$$

and any $x \in X$ can be uniquely written as

$$x = c + \rho\xi, \qquad c \in \mathbb{R}^N, \rho \geqslant 0, \xi \in S.$$

For any $\xi \in S$ consider the halfspace

$$M_\xi := \{c + \rho\xi \mid c \in \mathbb{R}^N, \rho \geqslant 0\}. \tag{3.1}$$

Lemma 3.1. *For each $\xi \in S$ the restriction of the functional J_E to M_ξ has a unique critical point $\hat{x} \in \operatorname{int} M_\xi$ at which it attains a positive maximum*:

$$J_E(\hat{x}) = \max_{x \in M_\xi} J_E(x) > 0. \tag{3.2}$$

Moreover, $\hat{x}$ is uniquely determined by the $N+1$ equations:

$$\int V'(x) = 0 \qquad \text{and} \qquad E = \int V(x) + \tfrac{1}{2} \int V'(x) \cdot x. \tag{3.3}$$

Proof. Let $C_\xi := \{x \in M_\xi \mid \int V(x) \leqslant E\}$. Since V is strictly convex and coercive, C_ξ is a convex, compact subset of M_ξ. Therefore, $J_E(x) \leqslant 0$ for $x \in M_\xi \setminus C_\xi$ and the twice differentiable functional J_E attains a non-negative maximum value on M_ξ. This maximum is positive since $J_E(\rho\xi)$ is positive for ρ sufficiently small, and is attained at some point $\hat{x}$. $\hat{x}$ must be an interior point since J_E vanishes for $\rho = 0$. We shall show that this point $\hat{x} = \hat{c} + \hat{\rho}\xi$ is the unique critical point of J_E on M_ξ. From $(\partial/\partial c)\, J_E(x) = 0$ it follows that $\int V'(x)=0$. Using this, the condition $(\partial/\partial\rho)\, J_E(x)=0$ implies $E=\int V(x)+\frac{1}{2}\int V'(x)\cdot x$. Hence $\hat{x}$ satisfies Eq. (3.3). Now, for any $x\in \operatorname{int} M_\xi$ satisfying these equations, the second derivative of $J_E(c+\rho\xi)$ with respect to ρ and c is given by the $(N+1)\times(N+1)$ matrix:

$$-\left(\tfrac{1}{2}\int \xi^2\right)\cdot\begin{pmatrix} 3\int V'(x)\cdot x+\rho^2\int V''(x)\,\xi\cdot\xi & \rho^2\int V''(x)\,\xi \\ \rho^2\int V''(x)\,\xi & \rho^2\int V''(x) \end{pmatrix}.$$

The strict convexity of V implies that $\int V'(x)\cdot x>0$ for $x\neq 0$, and $\int V''(x)\, y\cdot y>0$ for any function $y\not\equiv 0$. From this it easily follows that the second derivative is negative definite. Consequently, any $x\in \operatorname{int} M_\xi$ which satisfies (3.3) belongs to $\operatorname{int} C_\xi$ and is a strict local maximum. Together with the global behaviour of J_E on M_ξ this implies that there exists only one critical point of J_E on $\operatorname{int} M_\xi$ which must, therefore, necessarily be the point $\hat{x}$ at which J_E is maximal. This completes the proof of the lemma. ∎

An immediate consequence of this result is that any critical point of J_E has the maximizing property of Lemma 3.1:

PROPOSITION 3.2. *Let $\hat{x}=\hat{c}+\hat{\rho}\xi$ be any critical point of J_E on X for which $J_E(\hat{x})\neq 0$. Then, $\hat{x}$ satisfies* (3.2) *and, consequently,* (3.3).

Now, consider the set of points at which J_E is maximized on M_ξ:

$$N_E := \bigcup_{\xi\in S}\,\{\hat{x}\in X \mid J_E(\hat{x})=\max_{x\in M_\xi} J_E(x)\}. \tag{3.4}$$

From Lemma 3.1. it follows that N_E can be written in the following, explicit, analytical way:

$$N_E=\left\{x\in X\,\middle|\,\int V'(x)=0, \qquad E=\int V(x)+\tfrac{1}{2}\int V'(x)\cdot x\right\}. \tag{3.5}$$

The significance of this analytical description is that the variational problem

$$\operatorname*{stat}_{\xi\in S}\max_{x\in M_\xi} J_E(x) \tag{3.6}$$

can now be written as a variational problem for the functional J_E restricted to the set N_E:

$$\operatorname{stat}\{J_E(x)\mid x\in N_E\}. \tag{3.7}$$

Since $X=\bigcup_{\xi\in S} M_\xi$, it follows from Lemma 3.1 that any critical point of J_E belongs to N_E. Among other things, the next result states that the converse is also true. This will be expressed by saying the N_E is natural constraint, where we use the following definition of this notion (Berger [3]): A subset $\hat{X} \subset X$ is called a *natural constraint* for the functional J_E on X if any critical point of the restriction of J_E to $\hat{X}$ is also a critical point of J_E on X.

LEMMA 3.3. *The set N_E given by* (3.6) *is a smooth C^1 manifold of codimension $N+1$, and N_E is a natural constraint for J_E on X.*

Proof. The set N_E as given by (3.5) can be written as

$$N_E = \{x \in X \mid f(x) = 0, F(x) = 0\},$$

where $f: X \to \mathbb{R}$ and $F: X \to \mathbb{R}^N$ are given by

$$f(x) = 2 \int V(x) + \int V'(x) \cdot x - 2E$$

$$F(x) = \int V'(x).$$

Since $V \in C^2$, f and F are differentiable with derivatives:

$$f'(x) = 3V'(x) + V''(x)\, x$$
$$F'(x) = V''(x).$$

To show that N_E has codimension $N+1$ we have to show that, for any $x \in N_E$, $(\sigma, \mu) \in \mathbb{R}^N \times \mathbb{R}$ and

$$F'(x)\, \sigma + \mu f'(x) = 0 \tag{$*$}$$

implies $\sigma = 0$ and $\mu = 0$. Taking the inner product of $(*)$ with the function $\sigma + \mu x$, and using $F(x) = 0$, there results

$$\int V''(x)(\sigma + \mu x)(\sigma + \mu x) + 3\mu^2 \int V'(x) \cdot x = 0. \tag{$**$}$$

Since $x \in N_E$ implies $x \neq 0$, it follows from this equation by the strict convexity of V that $\sigma = 0$ and $\mu = 0$, which shows that N_E is a smooth manifold of codimension $N+1$.

To prove that N_E is a natural constraint, note that Lagrange's multiplier rule applies and states that any critical point of J_E on N_E satisfies for some multipliers $(\sigma, \mu) \in \mathbb{R}^N \times \mathbb{R}$ the equation

$$-\left(E - \int V(x)\right) \ddot{x} - \left(\tfrac{1}{2} \int \dot{x}^2\right) V'(x) = F'(x)\, \sigma + \mu f'(x),$$

and the boundary conditions $\dot{x}(0)=\dot{x}(1)=0$. Taking the innerproduct of this equation with the function $\sigma+\mu x$, the left-hand side vanishes because of the conditions $f(x)=0$ and $F(x)=0$. The resulting vanishing of the right-hand side leads to Eq. (**), from which it follows that $\sigma=0$ and $\mu=0$. Hence, the multipliers vanish, and a critical point of J_E on N_E satisfies (1.4) with T given by (2.2), i.e., N_E is a natural constraint for J_E. ∎

Now it is time to take full advantage of the set N_E constructed above. On all of X, the functional J_E can take positive and negative values and is strictly indefinite. But its restriction to N_E is positive (from (3.2)), and it makes sense to look for the minimum of J_E on N_E. The set N_E and the restriction of J_E to N_E are so nice that we will be able to show that the minimum value is attained at some nontrivial point. Thus, to obtain the existence of at least one brake orbit, consider the naturally constrained minimization problem:

$$j(E) := \inf \{J_E(x) \mid x \in N_E\}. \tag{3.8}$$

Because of the foregoing, this is an explicit description of the analytical mini-max problem:

$$\inf_{\xi \in S} \sup_{\rho \geqslant 0} \sup_{c \in \mathbb{R}^N} J_E(c+\rho\xi). \tag{3.9}$$

PROPOSITION 3.4. *The minimization problem* (3.8) *has at least one solution* $x \in X$ *with* $J_E(x)>0$.

Proof. We start to investigate the set N_E more closely. First note that, since $E>0$, $0 \notin N_E$. More generally, if $c \in \mathbb{R}^N$, then $V'(c) \neq 0$ if $c \neq 0$. Consequently, constant vector functions $c \in \mathbb{R}^N$ do not belong to N_E. From the convexity of V it follows that

$$V'(x)\cdot x > V(x) \qquad \text{for all} \quad x \neq 0,$$

and

$$V(c) \leqslant \int V(x) \qquad \text{for} \quad x=c+y,\ c \in \mathbb{R}^N,\ y \in Y.$$

Hence, for $x \in N_E$: $E=\int V(x)+\frac{1}{2}\int V'(x)\cdot x \geqslant \frac{3}{2}\int V(x)$, and thus

$$V(c) \leqslant \int V(x) \leqslant \tfrac{2}{3}E \qquad \text{for all} \quad x=c+y \in N_E. \tag{3.10}$$

Among other things, this implies that

$$J_E(x) \geqslant \tfrac{1}{3}E \cdot \int \tfrac{1}{2}\dot{x}^2 \qquad \text{for} \quad x \in N_E. \tag{3.11}$$

Now, let $x_n = c_n + y_n$, $c_n \in \mathbb{R}^N$, $y_n \in Y$, be any minimizing sequence for (3.8): $J_E(x_n) \to j(E)$ for $n \to \infty$ and $x_n \in N_E$. From (3.10) it follows that $\{c_n\}$ is uniformly bounded in $\mathbb{R}^N$, and from (3.11) that $\{y_n\}$ is uniformly bounded in Y. Since Y is compactly embedded in the set of continuous vector functions $C^0([0,1], \mathbb{R}^N)$, it follows that there exists some subsequence, again denoted by x_n, such that x_n converges to some $\hat{x} = \hat{c} + \hat{y}$ in the sense that $c_n \to \hat{c}$ in $\mathbb{R}^N$, $y_n \to \hat{y}$ (weakly) in Y and $y_n \to \hat{y}$ (strongly) in $C^0([0,1], \mathbb{R}^N)$. Then $\hat{y} \in Y$. Since f and F are continuous mappings on $C^0([0,1], \mathbb{R}^N)$, it follows that $f(x_n) \to f(\hat{x})$ and $F(x_n) \to F(\hat{x})$. Hence, $\hat{x} \in N_E$ (i.e., N_E is weakly closed). To investigate $J_E(x_n)$, note that $E - \int V(x_n) \to E - \int V(\hat{x}) \geqslant \frac{1}{3}E > 0$. The functional $\int \dot{x}^2$ being lower semi-continuous with respect to weak convergence, it follows that $J_E(\hat{x}) \leqslant \liminf J_E(x_n) = j(E)$. Since $\hat{x} \in N_E$, we conclude that $\hat{x}$ minimizes J_E on N_E. It follows from (3.11) and the fact that $\hat{x}$ is not constant, since $\hat{x} \in N_E$, that $j(E) = J_E(\hat{x}) > 0$. This completes the proof. ∎

In order to be able to make a statement about the minality of the period of the brake orbit corresponding to the solution found in Proposition 3.4, let, more generally, x be any critical point of J_E. We define, for $k \in \mathbb{N}$ the "kth repetition of x," to be denoted by x_k, as the even periodic continuation of the function $x(kt)$, $t \in [0, 1/k]$. Because of the periodicity, it is easily verified that x_k also belongs to N_E, and that x_k is also a critical point of J_E. Its value is related to $J_E(x)$ according to:

$$J_E(x_k) = k^2 J_E(x). \tag{3.12}$$

Of course, under the transformation (1.3), the kth repetition of x provides the same brake orbit as x itself. From (3.12) the constrained minimizing property of the solution of Proposition 3.4 implies that the period of the corresponding brake orbit is minimal.

In fact, and we shall need this observation in the next section, a simple consequence of (3.12) and the minimizing characterization in (3.8) is the following.

PROPOSITION 3.5. *Let x be any critical point of J_E with $0 < J_E(x) < 4j(E)$, where $j(E)$ is defined in* (3.8). *Then the brake orbit corresponding to x via the transformation* (1.3) *has* minimal *period.*

Restating the results of Propositions 3.4 and 3.5, we get

THEOREM 3.6. *Let x be a solution of* (3.8), *and let T be defined by* (2.2) *and q by* (1.3). *Then q defines a brake orbit of energy E and minimal period* $2T$.

4. A MULTIPLICITY RESULT

As a consequence of the constrained extremizing property of the solution obtained in Section 3, we found that the corresponding brake orbit has a minimal period. In this section we indicate how, as a consequence of that, a multiplicity result for brake orbits of prescribed energy E can be derived in a relatively simple way. Since the arguments are similar to those in [8–10], we shall be brief.

The idea is to apply Lyusternik–Schnirelmann theory, with the invariance group of parameter translations: $S^1 := \{R_\theta \mid \theta \in [0, 1]\}$, where $R_\theta x(\tau) := x(\tau + \theta)$. The functional J_E and set N_E are easily seen to be invariant under the action of this group,

$$J_E(R_\theta x) = J_E(x), \qquad R_\theta x \in N_E, \qquad \text{for all} \quad x \in N_E, \theta \in [0, 1],$$

and the group acts freely on N_E. (Since $R_\theta x = x$ for all θ implies $x = \text{constant}$, so that $x \notin N_E$.) Having verified the Palais–Smale condition, Lyusternik–Schnirelmann theory can be applied using, for instance, the index theory described in [1]. This yields the existence of infinitely many, S^1-distinct critical points of J_E on N_E. Since, together with some critical point x, all its kth repetitions x_k are critical points too, this result alone does not produce multiple brake orbits.

However, as a consequence of Proposition 3.5, the number of distinct brake orbits is at least as large as the number of critical points of J_E with critical value $< 4j(E)$. To exploit this observation, consider the condition:

$(V)_M$: There exist a function $U: \mathbb{R} \to \mathbb{R}_+$ satisfying $(V)_0$ for $N = 1$ and a number $a \in [1, 2)$ such that

$$U(|x|) \leqslant V(x) \leqslant U(a\,|x|) \qquad \text{for all} \quad x \in \mathbb{R}^N.$$

Note that this condition implies that we can compare our original problem with two, closely related, problems which possess rotation symmetry, so essentially with two one-dimensional problems.

With this condition it is possible to construct a set $\Sigma \subset N_E$ of index N with the property that $J_E(\Sigma) < 4j(E)$. (See also [8, 9].) Then Lyusternik–Schnirelmann theory provides the existence of at least N distinct critical points with critical value $< 4j(E)$. Hence, the following result is obtained.

THEOREM 4.1. *Let $E > 0$ and suppose that V satisfies condition $(V)_M$. Then there exist at least N distinct brake orbits of* (1.1) *with energy E.*

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