EXACT OBSERVABILITY OF DIAGONAL SYSTEMS WITH A ONE-DIMENSIONAL OUTPUT OPERATOR

BIRGIT JACOB*, HANS ZWART**

In this paper equivalent conditions for exact observability of diagonal systems with a one-dimensional output operator are given. One of these equivalent conditions is the conjecture of Russell and Weiss (1994). The other conditions are given in terms of the eigenvalues and the Fourier coefficients of the system data.

Keywords: infinite-dimensional systems, unbounded observation operator, exact observability, Hautus test, Lyapunov equation

1. Introduction

On the Hilbert space Z we consider the following system:

$$\dot{z}(t) = Az(t), \quad y(t) = Cz(t), \tag{1}$$

where we assume that

1. A is a *diagonal operator*, i.e., $Az = \sum_{n=1}^{\infty} \lambda_n \langle z, \phi_n \rangle \phi_n$, with $\mathrm{Re}\,(\lambda_n) < 0$ and $\{\phi_n\}$ being an orthonormal basis of the Hilbert space Z.
2. C is a bounded linear operator from the domain of A, $D(A)$, to $\mathbb{C}$.
3. C is an *infinite-time admissible output operator*, i.e.,

$$\int_0^\infty \|CT(t)z_0\|^2 \,\mathrm{d}t \le \kappa \|z_0\|^2, \tag{2}$$

for all $z_0 \in D(A)$. Here $T(t)$ is the C_0-semigroup generated by A.

A system (1) that satisfies the above conditions will be denoted by $\Sigma(A, C)$.

The admissibility of C, eqn. (2), implies that we can extend the mapping $z_0 \to CT(\cdot)z_0$ to a bounded linear mapping from Z to $L_2(0,\infty)$. We denote this mapping by $\mathcal{C}$. Thus we have that for any initial condition z_0 the solution of (1) is given by

$$z(t) = T(t)z_0, \quad y(\cdot) = \mathcal{C}z_0.$$

* Fachbereich Mathematik, University of Dortmund, D–44221 Dortmund, Germany, e-mail: `birgit.jacob@math.uni-dortmund.de`

** Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands, e-mail: `h.j.zwart@math.utwente.nl`

Furthermore, the output is an element of $L_2(0,\infty)$.

Just as for bounded output operators C, we can define exact observability, see (Curtain and Zwart, 1995, Def. 4.1.12).

Definition 1. We say that the system $\Sigma(A,C)$ is *exactly observable (in infinite time)* if there exists an $m>0$ such that for all $z_0\in Z$ we have

$$\|\mathcal{C}z_0\|_{L_2(0,\infty)} \geq m\|z_0\|.$$

In the literature a variety of necessary and sufficient conditions have been derived which ensure exact observability, see, e.g., (Avdonin and Ivanov, 1995; Grabowski, 1990; Grabowski and Callier, 1996; Jacob and Zwart, 1999; 2000b; 2001; Komornik, 1994; Rebarber and Weiss, 2000; Russell and Weiss, 1994). Related to the equivalent conditions obtained in this article are the results given by Grabowski (1990), Russell and Weiss (1994), and Jacob and Zwart (2001).

In (Grabowski, 1990) it is shown that exact observability is equivalent to the unique solvability of the following Lyapunov equation by a coercive L:

$$\langle Az_1, Lz_2\rangle + \langle Lz_1, Az_2\rangle = -\langle Cz_1, Cz_2\rangle, \quad z_1,z_2\in D(A). \tag{3}$$

Using the Lyapunov equation, Russell and Weiss (1994) showed that a condition which corresponds to the Hautus test in the finite-dimensional situation is necessary for exact observability. Moreover, they proved that for some classes of exponentially stable systems this infinite-dimensional Hautus test is even an equivalent condition, and they conjectured that this holds in general.

In (Jacob and Zwart, 2001) four equivalent conditions for exact observability of diagonal systems with a finite-dimensional output space are given. The infinite-dimensional Hautus test of Russell and Weiss (1994) is one of the four equivalent conditions. For this class the assumption that the system is exponentially stable is not needed; it is only required that the systems satisfy the weaker condition of strong stability, i.e., $\lim_{t\to\infty} T(t)z_0 = 0$. The second condition is given in terms of the solution of the Lyapunov equations for $k+1$ dimensional subsystems. The third condition is stated in terms of the eigenvalues and finite collections of the vectors $\{C_n e^{\lambda_n \cdot}\}$, whereas the last equivalent condition states that the vectors $\{C_n e^{\lambda_n \cdot}\}$ form a Riesz basis in the closure of its span. This last equivalent condition can also be found in (Avdonin and Ivanov, 1995, Thm. III.3.3).

In this paper we consider the same class of systems as in (Jacob and Zwart, 2001), but now we deal with one-dimensional outputs. The results obtained here are contained in (Jacob and Zwart, 2001) as well. However, the methods presented to prove the results are different. Here the proofs are easier and more direct.

We remark that the assumption that $\{\phi_n\}$ form an orthonormal basis is essential. In (Jacob and Zwart, 2000a) we present an example of a system similar to (1), but there the eigenfunctions $\{\phi_n\}$ form a conditional basis. We show that this system satisfies the Hautus test of Russell and Weiss but is not exactly observable.

2. Main Result

Consider the system $\Sigma(A,C)$ as introduced in the previous section. For this system we can obtain the following four equivalent conditions for exact observability. Here we define $c_n = C\phi_n$.

Theorem 1. *For the system $\Sigma(A,C)$ the following conditions are equivalent:*

1. *The system $\Sigma(A,C)$ is exactly observable.*
2. *There exists an $m_1 > 0$ such that for all $z_0 \in D(A)$ and for all s with* $\mathrm{Re}(s) < 0$
$$\|(sI - A)z_0\|^2 + |\mathrm{Re}(s)|\,\|Cz_0\|^2 \geq m_1 \mathrm{Re}(s)^2 \|z_0\|^2. \tag{4}$$
3. *There exists an $m_2 > 0$ such that*
$$m_2\,|\mathrm{Re}(\lambda_k)| \leq |c_k|^2, \tag{5}$$
and λ_n are properly spaced, *i.e.,*
$$\inf_{n \neq k} \left| \frac{\lambda_n - \lambda_k}{\mathrm{Re}(\lambda_n)} \right| > 0. \tag{6}$$
4. *There exists an $m_3 > 0$ such that for any pair n, k with $n \neq k$ we have that the solution $L_{n,k}$ of the Lyapunov equation associated with the system*
$$A_{n,k} = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_k \end{pmatrix}, \quad C_{n,k} = \begin{pmatrix} c_n & c_k \end{pmatrix}$$
satisfies $L_{n,k} \geq m_3 I_2$, i.e., the matrices $L_{n,k}$ are uniformly coercive.
5. *The set $\{c_n e^{\lambda_n \cdot}, n \in \mathbb{N}\}$ is a Riesz basis in the closure of its span in $L^2(0,\infty)$.*

The statement in item 2 is the infinite-dimensional Hautus test as introduced by Russell and Weiss (1994). They conjectured that this condition would be sufficient for exact observability for any exponentially stable system. Here we prove this conjecture for our class of diagonal systems. Note that our systems are in general not exponentially stable.

For the proof of this theorem we need the following result.

Lemma 1. *Consider a sequence λ_n which satisfies* $\mathrm{Re}(\lambda_n) < 0$. *Then the following statements are equivalent:*

1. *The sequence λ_n is properly spaced, i.e., (6) holds.*

2. We have

$$\inf_{k\neq n} \frac{|\lambda_k - \lambda_n|}{|\lambda_k + \overline{\lambda_n}|} > 0. \tag{7}$$

3. The matrices

$$M_{n,k} := \begin{pmatrix} 1 & -2\frac{\sqrt{-\mathrm{Re}(\lambda_n)}\sqrt{-\mathrm{Re}(\lambda_k)}}{\overline{\lambda_n}+\lambda_k} \\ -2\frac{\sqrt{-\mathrm{Re}(\lambda_n)}\sqrt{-\mathrm{Re}(\lambda_k)}}{\lambda_n+\overline{\lambda_k}} & 1 \end{pmatrix} \tag{8}$$

are uniformly coercive.

Proof. We have

$$\frac{2\mathrm{Re}(\lambda_n)}{\lambda_n-\lambda_k} = \frac{\lambda_n - \lambda_k + \lambda_k + \overline{\lambda_n}}{\lambda_n-\lambda_k} = 1 + \frac{\lambda_k + \overline{\lambda_n}}{\lambda_n - \lambda_k}.$$

From this we see that the supremum of $|\mathrm{Re}(\lambda_n)|/|\lambda_n - \lambda_k|$ is finite if and only if the supremum of $|\lambda_k + \overline{\lambda_n}|/|\lambda_n - \lambda_k|$ is finite. This proves the equivalence of Statements 1 and 2. Thus it remains to show that (7) is equivalent to the uniform coercivity of $M_{n,k}$. The trace of $M_{n,k}$ is 2, and the determinant is given by

$$\det(M_{n,k}) = 1 - 4\frac{\mathrm{Re}(\lambda_n)\mathrm{Re}(\lambda_k)}{|\overline{\lambda_n}+\lambda_k|^2} = \frac{|\lambda_k-\lambda_n|^2}{|\overline{\lambda_n}+\lambda_k|^2}.$$

From this we see that the matrix $M_{n,k}$ is non-negative. The non-negative matrices $M_{n,k}$ are uniformly coercive if and only if the determinants are uniformly bounded away from zero. From this observation we conclude that (7) is equivalent with the uniform coercivity of $M_{n,k}$. ■

Proof of Theorem 1. We shall prove this theorem via the chain $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 3 \Rightarrow 5 \Rightarrow 1$. However, first we see that admissibility implies

$$-\frac{|c_n|^2}{2\mathrm{Re}\lambda_n} = \int_0^\infty |c_n e^{\lambda_n t}|^2 \,\mathrm{d}t = \|CT(t)\phi_n\|^2_{L_2(0,\infty)}$$
$$= \|\mathcal{C}\phi_n\|^2 \le \|\mathcal{C}\|^2\|\phi_n\|^2 = \|\mathcal{C}\|^2. \tag{9}$$

1 ⇒ 2. This can be found in (Russell and Weiss, 1994).

2 ⇒ 3. Choosing $s = \lambda_k$ and $z_0 = \phi_k$ in (4), we get

$$0 + |\mathrm{Re}(\lambda_k)|\, \|C\phi_k\|^2 \ge m_1 \mathrm{Re}(\lambda_k)^2 \cdot 1,$$

and thus (5) holds with $m_2 = m_1$.

In order to prove that the eigenvalues of A are properly spaced, we take $k \neq n$. Then there exist $\alpha_{k,n}, \beta_{k,n} \in \mathbb{C}$ such that

$$C(\alpha_{k,n}\phi_k + \beta_{k,n}\phi_n) = 0, \quad |\alpha_{k,n}|^2 + |\beta_{k,n}|^2 = 1.$$

We define

$$z_{k,n} := \alpha_{k,n}\phi_k + \beta_{k,n}\phi_n.$$

Choosing $z_0 = z_{k,n}$ and $s = \lambda_k$ in (4), we get that

$$|\beta_{k,n}|^2|\lambda_k - \lambda_n|^2 \geq m_1 \mathrm{Re}(\lambda_k)^2.$$

This is equivalent to

$$\frac{|\lambda_k - \lambda_n|^2}{\mathrm{Re}(\lambda_k)^2} \geq \frac{m_1}{|\beta_{k,n}|^2}.$$

Since $|\beta_{k,n}| \leq 1$, we see that this inequality implies (6).

3 ⇒ 4. It is easy to calculate that $L_{n,k}$ equals

$$L_{n,k} = \begin{pmatrix} -\dfrac{|c_n|^2}{2\mathrm{Re}\lambda_n} & -\dfrac{\overline{c_n}c_k}{\overline{\lambda_n} + \lambda_k} \\ -\dfrac{c_n\overline{c_k}}{\lambda_n + \overline{\lambda_k}} & -\dfrac{|c_k|^2}{2\mathrm{Re}\lambda_k} \end{pmatrix}. \tag{10}$$

We define the matrix

$$D_{n,k} := \begin{pmatrix} \dfrac{\sqrt{-2\mathrm{Re}\lambda_n}}{c_n} & 0 \\ 0 & \dfrac{\sqrt{-2\mathrm{Re}\lambda_k}}{c_k} \end{pmatrix}.$$

By (5), we have

$$\|x\|^2 \leq \frac{2}{m_2}\|D_{n,k}^{-1}x\|^2. \tag{11}$$

A straightforward calculation gives

$$D_{n,k}^* L_{n,k} D_{n,k} = M_{n,k}. \tag{12}$$

Since (6) holds, we see by Lemma 1 that the matrices $M_{n,k}$ are uniformly coercive, and thus we have

$$\langle L_{n,k}x, x\rangle = \langle M_{n,k}D_{n,k}^{-1}x, D_{n,k}^{-1}x\rangle \geq \tilde{m}\|D_{n,k}^{-1}x\|^2 \geq m_4\|x\|^2,$$

where we have used (11).

4 ⇒ 3. We know that the matrices $L_{n,k}$, as given by (10), are uniformly coercive. In particular, this implies that their left-upper elements are uniformly bounded away from zero or, equivalently, that (5) holds.

Using the matrices $D_{n,k}$ and $M_{n,k}$ as introduced in the proof of (3 ⇒ 4) and Lemma 1, respectively, we see that

$$\langle M_{n,k}x, x\rangle = \langle L_{n,k}D_{n,k}x, D_{n,k}x\rangle \geq m_3\|D_{n,k}x\|^2 \geq m_4\|x\|^2,$$

using in the first inequality the fact that $L_{n,k}$ are uniformly coercive, and in the second inequality that (9) holds. Now Lemma 1 gives (6).

3 ⇒ 5. We first prove that $\{\sqrt{-\mathrm{Re}(\lambda_n)}e^{\lambda_n t}\}$ is a Riesz basis in the closure of its span in $L^2(0,\infty)$.

From (Nikol'skii and Pavlov, 1970, Sec. 10.3), see also (Avdonin and Ivanov, 1995, p. 56), this holds if and only if

$$\inf_k \prod_{n\neq k} \left| \frac{-\lambda_n + \lambda_k}{\overline{\lambda}_n + \lambda_k} \right| > 0.$$

From (Garnett, 1981, Thm. 1.1, p. 287) we have that this condition is equivalent to (6) and

$$\sum_{-\lambda_n \in Q(h,\omega)} -\mathrm{Re}(\lambda_n) \leq \tilde{A}h, \tag{13}$$

where $Q(h,\omega) = \{s = x+iy \in \mathbb{C} \mid 0 < x \leq h, \omega \leq y \leq \omega + h\}$.

So we have that $\{\sqrt{-\mathrm{Re}(\lambda_n)}e^{\lambda_n t}\}$ is a Riesz basis in the closure of its span if we can prove (13). From the Carleson measure criterion of Weiss (1988), see also (Zwart, 1996), we know that the infinite-time admissibility of C implies that

$$\sum_{-\lambda_n \in Q(h,\omega)} |c_n|^2 \leq mh$$

for some m independent of h and ω. Since, by (5), $-\mathrm{Re}(\lambda_n) \leq |c_n|^2/m_2$, we see that this criterion implies (13). Hence we conclude that $\{\sqrt{-\mathrm{Re}(\lambda_n)}e^{\lambda_n t}\}$ is a Riesz basis in its closure.

Combining (5) and (9) gives

$$\frac{1}{2\|\mathcal{C}\|^2}|c_n|^2 \leq |\mathrm{Re}(\lambda_n)| \leq \frac{1}{m_2}|c_n|^2.$$

From this it is easy to see that $\{c_n e^{\lambda_n t}\}$ is a Riesz basis in its closure as well.

5 ⇒ 1. Take $z_0 \in \mathrm{span}_{n=1,\ldots,N}\{\phi_n\}$ for some $n \in \mathbb{N}$. Then we see that $CT(t)z_0 = \sum_{n=1}^{N} c_n \langle z_0, \phi_n\rangle e^{\lambda_n t}$. Thus

$$\begin{aligned} \|\mathcal{C}z_0\|^2_{L_2(0,\infty)} &= \|CT(t)z_0\|^2_{L_2(0,\infty)} \\ &= \left\| \sum_{n=1}^{N} c_n \langle z_0, \phi_n\rangle e^{\lambda_n t} \right\|^2_{L_2(0,\infty)} \geq m_4 \sum_{n=1}^{N} |\langle z_0, \phi_n\rangle|^2. \end{aligned}$$

Here we use the fact that $\{c_n e^{\lambda_n \cdot}\}$ is a Riesz basis. The multiplier m_4 is independent of N. Thus for a dense subset of Z we have

$$\|\mathcal{C}z_0\|^2_{L_2(0,\infty)} \geq m_4 \|z_0\|^2,$$

and this proves exact observability. ■

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