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On some intriguing problems in hamiltonian graph theory—a survey

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Abstract

We survey results and open problems in hamiltonian graph theory centered around three themes: regular graphs, t-tough graphs, and claw-free graphs. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

As this survey paper is the outcome of an invited lecture at the eighth *Workshop* on Cycles and Colourings (Stará Lesná, Slovakia, 1999), the presentation of results is motivated by *open* problems in hamiltonian graph theory rather than the intention to write an exhaustive concise survey on this topic. Of course the choice is biased by the preferences of the author. The presented results and problems are centred around three themes: regular graphs, t-tough graphs, and claw-free graphs. Namely, for all these three graph classes there exist some intriguing ‘long-standing’ conjectures on hamiltonicity, as well as a number of recent developments towards proving or refuting these conjectures. It is our aim to stimulate and inspire the reader to continue the work in this fascinating area of graph theory.

We use Bondy and Murty’s book [15] for terminology and notation not defined here, and consider finite simple graphs only. To avoid irrelevant technicalities we will

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consider only graphs with at least three vertices. This implies, e.g., that all complete graphs considered are hamiltonian.

A graph G is *hamiltonian* if it contains a Hamilton cycle (a cycle containing every vertex of G). The number of vertices of a graph will be denoted by n.

It is well-known that the problem of deciding whether a given graph is hamiltonian, is NP-complete, and that (up to now) there exists no easily verifiable necessary and sufficient condition for the existence of a Hamilton cycle. This fact gave rise to a growing number of conditions that are either necessary or sufficient. We refer to papers by Bermond [7], Bermond and Thomassen [8], Bondy [11,12], Chvátal [25], and Gould [34] for more background and general surveys.

Before we turn to our three graph classes, we mention a few results that inspired most of today's work, and give some recent developments that cannot be found in the most recent survey [34].

1.1. Early degree conditions and a closure operation

Most of the sufficient conditions for hamiltonicity are based on the intuitive idea that a Hamilton cycle is likely to exist if all vertices have many neighbors. The earliest degree condition is based on the minimum degree $\delta(G)$ of the graph G.

Theorem 1.1 (Dirac [26]). *If $\delta(G) \geqslant n/2$, then G is hamiltonian.*

The lower bound in Theorem 1.1, often referred to as Dirac's Theorem, cannot be relaxed without destroying the conclusion of the theorem (unless we add an extra condition, e.g., that G is regular, G is t-tough, or G is claw-free, as we will see in the next sections). Nevertheless, Dirac's Theorem has been generalized in several directions.

Denote by $d(v)$ the degree of a vertex v in the graph G. We will refer to the next generalization of Theorem 1.1 as Ore's Theorem.

Theorem 1.2 (Ore [64]). *If $d(u) + d(v) \geqslant n$ for every pair of distinct nonadjacent vertices u and v of G, then G is hamiltonian.*

For further generalizations of Ore's Theorem in terms of vertex degrees we refer to the aforementioned surveys.

As remarked in [25], the closure concept introduced by Bondy and Chvátal [13] was found in an attempt to find a constructive proof for a sufficient condition for hamiltonicity based on degree sequences. It exploits the following variation on Ore's Theorem.

Theorem 1.3. *Let u and v be distinct nonadjacent vertices of a graph G such that $d(u) + d(v) \geqslant n$. Then, G is hamiltonian if and only if $G + uv$ is hamiltonian.*

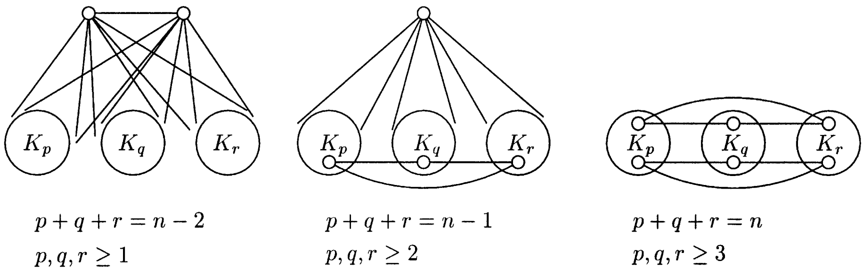

Fig. 1. Three classes of exceptional graphs.

The closure technique based on this result generalized many known degree conditions, and opened a new horizon for the research on hamiltonian and related properties of graphs. We refer to [21] for a recent survey on closure concepts and their applications. We will consider a different kind of closure concept for claw-free graphs in the last section.

1.2. Recent generalizations of Ore's Theorem

Around 10 years ago, sufficient conditions for hamiltonicity appeared in which certain vertex sets are required to have large neighborhood unions instead of large degree sums. Many of these new results do not generalize Ore's Theorem. The following more recent result in [18] uses a neighborhood type condition, and generalizes Ore's Theorem.

Denote by $N(v)$ the set of neighbors of a vertex v in the graph G.

Theorem 1.4 (Broersma et al. [18]). *If G is a 2-connected graph and $|N(u) \cup N(v)| \geqslant n/2$ for every pair of distinct nonadjacent vertices u and v of G, then either G is hamiltonian, or G is the Petersen graph, or G is in one of the three classes of exceptional graphs of connectivity 2 shown in Fig. 1.*

The three classes of exceptional graphs shown in Fig. 1 will be described more formally in the next section, and play an important role in many recent developments in hamiltonian graph theory.

Theorem 1.4 has been further generalized by Liu and Wang [57] and Liu et al. [58].

2. Hamiltonicity of regular graphs

It is likely that the minimum degree bound in Dirac's Theorem (Theorem 1.1) can be relaxed if we add the condition that the graph under consideration is regular (and 2-connected). This is indeed the case. In this section, we will discuss several results, conjectures, and partial solutions on minimum degree conditions for regular graphs to be hamiltonian. As the graphs are regular, we formulate a degree condition as an upper bound on n in terms of the degree of regularity. We will also describe a variant of an important technique known as 'hopping', since it has been a key ingredient in most of the proofs of the results described in this section.

We start with a result of Jackson [42], and refer to [42] for earlier results.

Theorem 2.1 (Jackson [42]). *Every* 2*-connected* k*-regular graph on at most* $3k$ *vertices is hamiltonian.*

As noted in [42], Theorem 2.1 is best possible for $k=3$ in view of the Petersen graph, and essentially best possible for $k \geqslant 4$. For future reference also, we define three classes $\mathscr{G}$, $\mathscr{H}$ and $\mathscr{J}$ of graphs (see Fig. 1) illustrating the latter assertion. For a positive integer t, let $\mathscr{K}_t$ denote the set of all graphs consisting of three disjoint complete graphs, where each of the components has order at least t. Now $\mathscr{G}$ is the class of all spanning subgraphs of graphs that can be obtained as the join of K_2 and a graph in $\mathscr{K}_1$. The class $\mathscr{H}$ is the set of all spanning subgraphs of graphs that can be obtained from the join of K_1 and a graph G in $\mathscr{K}_2$ by adding the edges of a triangle between three vertices from distinct components of G. The class $\mathscr{J}$ is the set of all spanning subgraphs of graphs that can be obtained from a graph G in $\mathscr{K}_3$ by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each component of G. It is easy to check that all graphs in $\mathscr{G} \cup \mathscr{H} \cup \mathscr{J}$ are nonhamiltonian. (Indeed, $\mathscr{G}$, $\mathscr{H}$ and $\mathscr{J}$ were first obtained by Watkins and Mesner [73] in a characterization of the 2-connected graphs that have three vertices which are not contained in a common cycle.) Furthermore, each of the classes $\mathscr{G}$, $\mathscr{H}$ and $\mathscr{J}$ contains 2-connected k-regular graphs on $3k+4$ vertices for even $k \geqslant 4$, and $3k+5$ vertices for all $k \geqslant 3$. (Note that $\mathscr{G}$, $\mathscr{H}$ and $\mathscr{J}$ are not pairwise disjoint.) We set $\mathscr{F} = \mathscr{G} \cup \mathscr{H} \cup \mathscr{J}$.

Theorem 2.1 has been extended in several papers, e.g., by Bondy and Kouider [14], Hilbig [39] and Zhu et al. [77]. The strongest among these extensions is in [39]. Let Π denote the Petersen graph and Π^{Δ} the 3-regular graph obtained from Π by replacing one vertex by a triangle.

Theorem 2.2 (Hilbig [39]). *Let* G *be a* 2*-connected* k*-regular graph on at most* $3k+3$ *vertices. Then,* G *is hamiltonian if and only if* $G \notin \{\Pi, \Pi^{\Delta}\}$.

In a paper by Jackson et al. [46], the following improvement of Theorem 2.1 for 3-connected graphs is conjectured. (Note that no graph in $\mathscr{F}$ is 3-connected.)

Conjecture 2.3 (Jackson et al. [46]). *For* $k \geqslant 4$, *every* 3*-connected* k*-regular graph on at most* $4k$ *vertices is hamiltonian.*

Conjecture 2.3 is a special case of Häggkvist's Conjecture, appearing in [42], that every m-connected k-regular graph ($k \geqslant 4$) on at most $(m+1)k$ vertices in hamiltonian. However, for $k \equiv 0 \pmod 4$, the graph $\overline{K_k} \vee (\overline{K_{k-1}} + 2K_{k+1})$, where $\vee$ denotes the join and $+$ denotes the union of two disjoint graphs, contains a nonhamiltonian

$\frac{1}{2}k$-connected k-regular spanning subgraph G_k, showing that Häggkvist's Conjecture is not true in general. The graphs G_k were independently found by Jung and Jackson. For a more detailed description we refer to [46] or a paper by Min Aung [61]. The graphs G_k also show that Conjecture 2.3 would be best possible.

A first step towards proving Conjecture 2.3 was made in [46].

A cycle C of a graph G is called a *dominating cycle* if $V(G)\backslash V(C)$ is an independent set of G.

Theorem 2.4 (Jackson et al. [46]). *Let G be a 3-connected k-regular graph on at most $4k$ vertices. Then for $k \geqslant 63$, every longest cycle of G is a dominating cycle.*

In the graph G_k, every longest cycle is dominating. Still Theorem 2.4 is essentially best possible: for even $k \geqslant 8$, the graph $K_3 \vee (2K_k + 2K_{k+1})$ of order $4k+5$ has a 3-connected k-regular spanning subgraph containing no dominating cycle.

In a paper by Zhu and Li [76], Theorem 2.4 was used to obtain another result in the direction of Conjecture 2.3.

Theorem 2.5 (Zhu and Li [76]). *For $k \geqslant 63$, every 3-connected k-regular graph on at most $\frac{22}{7}k$ vertices is hamiltonian.*

This was improved by Broersma et al. [16].

Theorem 2.6 (Broersma et al. [16]). *Let G be a 2-connected k-regular graph on at most $\frac{7}{2}k - 7$ vertices. Then, G is hamiltonian if and only if $G \notin \mathscr{F}$.*

Since no graph in $\mathscr{F}$ is 3-connected, the following improvement of Theorem 2.5 is an immediate consequence of Theorem 2.6.

Corollary 2.7 (Broersma et al. [16]). *Every 3-connected k-regular graph on at most $\frac{7}{2}k - 7$ vertices is hamiltonian.*

The necessity of the condition for hamiltonicity in Theorem 2.6 is obvious. The sufficiency is an immediate consequence of the following two results that are proved in [16].

Theorem 2.8 (Broersma et al. [16]). *Let G be a k-regular graph on at most $\frac{7}{2}k - 7$ vertices. If G contains a dominating cycle, then G is hamiltonian.*

Theorem 2.9 (Broersma et al. [16]). *Let G be a 2-connected k-regular graph on at most $4k - 3$ vertices. Then, G contains a dominating cycle or $G \in \mathscr{F}$.*

The proof of Theorem 2.9 is based on ideas from [18,71].

The proof of Theorem 2.6 (via Theorems 2.8 and 2.9) uses several ideas from [17], where a relatively short proof of (an extension of) Theorem 2.1 occurs. In particular, the idea of breaking the proof into two parts in the way reflected by Theorems 2.8 and 2.9, stems from [17].

In view of the above results the following strengthening of Conjecture 2.3 was proposed in [16].

Conjecture 2.10 (Broersma et al. [16]). *Let G be a 2-connected k-regular graph on at most $4k$ vertices. Then for $k \geqslant 4$, G is hamiltonian if and only if $G \notin \mathscr{F}$.*

The proof of Theorem 2.8 as well as most of the other results in this section, uses a variation of Woodall's Hopping Lemma [74]. To demonstrate the general idea of 'hopping', we will describe the variant that was used to prove Theorem 2.8. For other variants of this important lemma and their applications we refer to [6,37, Chapter 4].

2.1. The hopping lemma

The 'hopping' technique is based on sets of vertices defined iteratively by 'hopping' around a given cycle or path. In order to describe this technique and its premisses, we first develop some additional terminology and notation.

Let C be a cycle of a graph G. We call C *extendable* if there exists an *extension* of C, i.e., a cycle C' with $V(C) \subseteq V(C')$ and $V(C) \neq V(C')$. For $v \in V(G)\backslash V(C)$, the cycle C is *v-extendable* if there exists a *v-extension* of C, i.e., an extension with vertex set $V(C) \cup \{v\}$.

We denote by $\overrightarrow{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor. If $Z \subseteq V(C)$, then $Z^+ = \{z^+ \mid z \in Z\}$ and $Z^- = \{z^- \mid z \in Z\}$. Similar notation is used for paths. When more than one cycle or path is under consideration, we write u^{+C}, u^{-C} instead of just u^+, u^- in order to avoid ambiguity.

In this variation of Woodall's Hopping Lemma [74], Lemma 2.11 below, we use the following hypotheses and definitions.

Let G be a graph, $\overrightarrow{C}$ a cycle of G with $V(C) \neq V(G)$, and a a vertex in $V(G)\backslash V(C)$. Assume C is not a-extendable. Set

$$X_1 = N(a) \cap V(C)$$

and for $i \geqslant 1$,

$$Y_i = X_i^+ \cap X_i^-,$$

$$U_i = X_i^+ \backslash Y_i,$$

$$W_i = X_i^- \backslash Y_i,$$

$$X'_{i+1} = \{v \in V(C) \mid \text{there exist six neighbors } w_1, u_1, w_2, u_2, w_3, u_3 \text{ of } v \text{ such that}$$

$$w_j \in W_i,\ u_j \in U_i \text{ and } w_j^+ \overrightarrow{C} u_j^- \subseteq X_i \cup Y_i\ (j = 1,2,3)\},$$

$$X''_{i+1} = N(Y_i) \cap V(C),$$

$$X_{i+1} = X_i \cup X'_{i+1} \cup X''_{i+1}.$$

Then, $X_1 \subseteq X_2 \subseteq \cdots$ and $Y_1 \subseteq Y_2 \subseteq \cdots$. Set

$$X = \bigcup_{i=1}^{\infty} X_i,$$

$$Y = \bigcup_{i=1}^{\infty} Y_i.$$

Then,

(1) $N(Y) \cap V(C) \subseteq X$.

Many results on the existence of long cycles do not use the above iteration, but are based on observations on the sets X_1, U_1, and W_1 only; as an example, it is easy to show that no two vertices of X_1 are adjacent on C, and that no two vertices of U_1 (or W_1) are connected by a path internally disjoint from C; otherwise, we can obtain a longer cycle containing $V(C)$. Similar observation can be made for the sets X_j, U_j, and W_j. The details depend on the assumptions and goals.

The name 'hopping' reflects the fact that we obtain the sets X''_{i+1} from vertices in Y_i by 'hopping' around the cycle, i.e., considering their neighbors on the cycle, and by iterating this process in the way described above. In other variants the main differences are the assumptions on C and a, the choice of X_1, and the definition of X_{i+1}.

The *height* $h(x)$ of $x \in X$ is defined by

$$h(x) = \min\{i \mid x \in X_i\}.$$

A path $P = x_1 \overrightarrow{P} x_2$ is called a *hopping path* if each of the following conditions is satisfied:

(2) $x_1, x_2 \in X$;
(3) $V(P) = V(C)$;
(4) if $1 \leqslant i < \max\{h(x_1), h(x_2)\}$ and $y \in Y_i \backslash \{x_1, x_2\}$, then $\{y^{-P}, y^{+p}\} = \{y^{-C}, y^{+C}\}$;
(5) if $1 \leqslant i < \max\{h(x_1), h(x_2)\}$, then $X_i \backslash \{x_1, x_2\}$ contains at most one vertex x for which $\{x^{-P}, x^{+P}\} \neq \{x^{-C}, x^{+C}\}$.

These definitions differ from those given in [74] in that

- we do not require $N(a) \subseteq V(C)$;
- we add the sets X_i' to X;
- the conditions (4) and (5) for a hopping path are more restrictive.

The following lemma is crucial for the proof of Theorem 2.8 in [16].

Lemma 2.11 (Broersma et al. [16]). *There exists no hoping path.*

2.2. How the hopping lemma is used

To conclude the section on regular graphs, we shall now briefly indicate how Lemma 2.11 has been used in [16] to prove Theorem 2.8. This also reflects the way in which variants of the hopping technique have been applied in other proofs.

First Lemma 2.11 has been applied in [16] to obtain several other lemmas concerning the (non)existence of edges with one end in $X^+ \cup X^-$. In the proof of Theorem 2.8 given there, G is supposed to be a nonhamiltonian graph satisfying the hypotheses of Theorem 2.8. Then, the lemmas derived from Lemma 2.11 have been applied to a nonextendable dominating cycle in G, in order to restrict the number of edges between $X^+ \cup X^-$ and $V(G)\backslash X$. The regularity condition then implies that there must be 'many' edges between $X^+ \cup X^-$ and X. Finally, a contradiction to k-regularity is obtained by showing that this number of edges is greater than $k|X|$.

3. Hamiltonicity of *t*-tough graphs

The number of components of a graph G is denoted by $\omega(G)$. The graph G is *t-tough* ($t \in R, t \geqslant 0$) if $|S| \geqslant t\omega(G-S)$ for every subset S of $V(G)$ with $\omega(G-S) > 1$. The *toughness* of G, denoted by $\tau(G)$, is the maximum value of t for which G is t-tough (for K_n we define $\tau(K_n) = \infty$).

The concept of toughness of a graph was introduced by Chvátal [24]. It is an easy exercise to show that 1-toughness is a necessary condition for hamiltonicity, but that it is not sufficient. Jung [49] proved that in the degree bounds in Dirac's Theorem and Ore's Theorem (Theorems 1.1 and 1.2) n can be replaced by $n-4$ if the graphs are assumed to be 1-tough and n is large enough, and this is essentially best possible. In a paper by Bauer et al. [2] it is shown that the bound $n/2$ in Theorem 1.1 can be replaced by roughly $n/(t+1)$ if the graphs are assumed to be t-tough. We refer to [2] for the details. It is a natural question whether we need a degree bound at all if we require a high toughness. In fact, in [24] the following conjecture is stated.

Conjecture 3.1 (Chvátal [24]). *There exists t_0 such that every t_0-tough graph is hamiltonian.*

The stronger conjecture that every t-tough graph with $t > \frac{3}{2}$ is hamiltonian, also occurring in [24], was first disproved by Thomassen (see [7]). Enomoto et al. [27] showed that every 2-tough graph contains a 2-factor (a 2-regular spanning subgraph), while for arbitrary $\varepsilon > 0$ there exist $(2-\varepsilon)$-tough graphs without a 2-factor, and hence without a Hamilton cycle. Therefore the following conjecture, usually attributed to Chvátal, appeared to be both reasonable and best possible.

Conjecture 3.2. *Every* 2*-tough graph is hamiltonian.*

Since every 2-tough graph is 4-connected, the conjecture is true for planar graphs by a result of Tutte [69]. By a result of Fleischner [32], the conjecture also holds for squares of 2-connected graphs. We refer to [69,32] for additional terminology and details.

A graph G is *traceable* if G contains a Hamilton path (a path containing every vertex of G); G is *hamiltonian-connected* if for every pair of distinct vertices x and y of G there is a Hamilton path with endvertices x and y.

In a paper by Bauer et al. [1] a construction of a nontraceable graph from non-hamiltonian-connected building blocks was used to show that Conjecture 3.2 is equivalent to several other statements, some (seemingly) weaker, some (seemingly) stronger than Conjecture 3.2. This construction was inspired by examples of graphs of high toughness without 2-factors by Bauer and Schmeichel [5].

In [3] the same construction was used to obtain $(\frac{9}{4} - \varepsilon)$-tough nontraceable graphs for arbitrary $\varepsilon > 0$, thereby refuting Conjecture 3.2. We will give a brief outline of the construction of these counterexamples in the next section.

Conjecture 3.1 remains open, but we do not believe that $\frac{9}{4}$-tough graphs are hamiltonian. In fact, we hope that constructions will be found yielding counter-examples to Conjecture 3.1 for arbitrary t_0.

3.1. Counterexamples to Conjecture 3.2

For a given graph H and two vertices x and y of H we define the graph $G(H, x, y, \ell, m)(\ell, m \in N)$ as follows. Take m disjoint copies $H_1, \dots, H_m$ of H, with x_i, y_i the vertices in H_i corresponding to the vertices x and y in H $(i = 1, \dots, m)$. Let F_m be the graph obtained from $H_1 \cup \cdots \cup H_m$ by adding all possible edges between pairs of vertices in $\{x_1, \dots, x_m, y_1, \dots, y_m\}$. Let $T = K_\ell$ and let $G(H, x, y, \ell, m)$ be the join $T \vee F_m$ of T and F_m.

The proof of the following theorem occurs in [3] and almost literally also in [1].

Theorem 3.3 (Bauer et al. [3]). *Let H be a graph and x, y two vertices of H which are not connected by a Hamilton path of H. If $m \geqslant 2\ell + 3$, then $G(H, x, y, \ell, m)$ is nontraceable.*

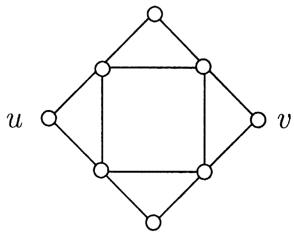

Fig. 2. The graph L.

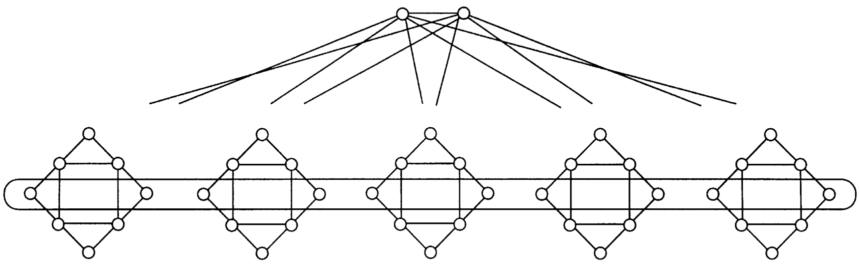

Fig. 3. The graph $G(L, u, v, 2, 5)$.

Consider the graph L of Fig. 2. There is obviously no Hamilton path in L between u and v. Hence, $G(L, u, v, \ell, m)$ is nontraceable for every $m \geqslant 2\ell + 3$. The toughness of these graphs has been established in [3].

Theorem 3.4 (Bauer et al. [3]). *For $\ell \geqslant 2$ and $m \geqslant 1$,*

$$\tau(G(L, u, v, \ell, m)) = \frac{\ell + 4m}{2m + 1}.$$

Combining Theorems 3.3 and 3.4 for sufficiently large values of m and ℓ, one obtains the next result.

Corollary 3.5 (Bauer et al. [3]). *For every $\varepsilon > 0$ there exists a $(\frac{9}{4} - \varepsilon)$-tough nontraceable graph.*

It is easily seen from the proof in [3] that Theorem 3.3 remains valid if '$m \geqslant 2\ell + 3$' and 'nontraceable' are replaced by '$m \geqslant 2\ell + 1$' and 'nonhamiltonian', respectively. Thus, the graph $G(L, u, v, 2, 5)$ is a nonhamiltonian graph, which by Theorem 3.4 has toughness 2. This graph is sketched in Fig. 3. It follows that a smallest counterexample to Conjecture 3.2 has at most 42 vertices. Similarly, a smallest nontraceable 2-tough graph has at most 58 ($|V(G(L, u, v, 2, 7))|$) vertices.

A graph G is *neighborhood-connected* if the neighborhood of each vertex of G induces a connected subgraph of G. In [24] Chvátal also states the following weaker version of Conjecture 3.2: every 2-tough neighborhood-connected graph is hamiltonian.

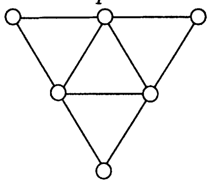

Fig. 4. The graph M.

Since all counterexamples to Conjecture 3.2 described above are neighborhood-connected, this weaker conjecture is also false.

Most of the ingredients used in the above counterexamples to Conjecture 3.2 were already present in [1]. It only remained to observe that using the specific graph L as a 'building block' produced a graph with toughness at least 2. We hope that other building blocks and/or smarter constructions will lead to counterexamples with a higher toughness.

3.2. Chordal graphs

A graph G is *chordal* if it contains no induced cycles of length at least 4. Chvátal [24] obtained $(\frac{3}{2}-\varepsilon)$-tough graphs without a 2-factor for arbitrary $\varepsilon > 0$. These examples are all chordal. Recently, it was shown by Bauer et al. [4] that every $\frac{3}{2}$-tough chordal graph has a 2-factor. Based on this, Kratsch [51] raised the question whether every $\frac{3}{2}$-tough chordal graph is hamiltonian. Using Theorem 3.3 in [3] it has been shown that this conjecture, too, is false. A key observation in this context is that most of the graphs $G(H,x,y,\ell,m)$ are chordal whenever H is chordal, as is easily shown.

Consider the graph M of Fig. 4.

The graph M is chordal and has no Hamilton path with endvertices p and q. Hence, by Theorem 3.3 the chordal graph $G(M,p,q,\ell,m)$ is nontraceable whenever $m \geqslant 2\ell+3$. By arguments as used in the proof of Theorem 3.4 (in [3]) its toughness is $(\ell+3m)/(2m+1)$ if $\ell \geqslant 2$. Hence, for $\ell \geqslant 2$ the graph $G(M,p,q,\ell,2\ell+3)$ is a chordal nontraceable graph with toughness $(7\ell+9)/(4\ell+7)$. This gives the following result.

Theorem 3.6 (Bauer et al. [3]). *For every $\varepsilon > 0$ there exists a $(\frac{7}{4}-\varepsilon)$-tough chordal nontraceable graph.*

On the other hand, Chen et al. [23] recently proved a 'positive' result.

Theorem 3.7 (Chen et al. [23]). *Every 18-tough chordal graph is hamiltonian.*

This means that Conjecture 3.1 is true when restricted to chordal graphs. We expect that the lower bound 18 on the toughness can be considerably decreased. In fact, for chordal planar graphs it has been proved by Böhme et al. [10] that a toughness strictly larger than one implies hamiltonicity, and this is best possible. For the class of split graphs toughness at least $\frac{3}{2}$ suffices and is best possible, as shown by Kratsch et al. [52]. We refer to the sources for additional terminology and details.

4. Hamiltonicity of claw-free graphs

During the last two decades many results on hamiltonian properties of claw-free graphs (i.e., graphs that do not contain $K_{1,3}$ as an induced subgraph) have appeared. We refer the reader to [29] for a recent survey. Most of these results involve sufficient conditions in terms of degrees, neighborhoods, forbidden subgraphs or (local) connectivity. In this section, we will discuss several recent developments on hamiltonicity of claw-free graphs. We start with the earliest minimum degree condition. As shown by Matthews and Sumner [60], the minimum degree bound in Theorem 1.1 can be relaxed if we add the condition that the graph under consideration is claw-free (and 2-connected).

Theorem 4.1 (Matthews and Sumner [60]). *Every 2-connected claw-free graph G with $\delta(G) \geqslant \frac{1}{3}(n-2)$ is hamiltonian.*

Theorem 4.1 has been generalized in several directions. We refer to [29] for a survey, and come back with the most recent developments on minimum degree conditions later.

4.1. On two conjectures and a closure technique

Most of the results in this section are motivated by the following two conjectures.

Conjecture 4.2 (Thomassen [67]). *Every 4-connected line graph is hamiltonian.*

Conjecture 4.3 (Matthews and Sumner [59]). *Every 4-connected claw-free graph is hamiltonian.*

A smallest 3-connected claw-free graph was obtained by Matthews and Sumner [59]. It is the line graph of the graph obtained from the Petersen graph by subdividing each edge of a perfect matching, and has 20 vertices.

Both conjectures are special cases of Conjecture 3.2, since every line graph is claw-free and the toughness of a (noncomplete) claw-free graph is half its connectivity (an easy exercise, see [59]).

A recent result on closures due to Ryjáček [65] (Theorem 4.4 below) implies that Conjectures 4.2 and 4.3 are equivalent.

We first introduce some terminology and notation. The *neighborhood* of a vertex v of a graph G is the subgraph of G induced by the set $N(v)$ of neighbors of v in G. The *local completion of a graph G at a vertex v* is the operation of joining all pairs of nonadjacent vertices in $N(v)$, i.e., replacing the neighborhood of v by the complete graph on $N(v)$.

In [65] the following has been proved.

Theorem 4.4 (Ryjáček [65]). *Let G be a claw-free graph, v a vertex of G whose neighborhood is connected, and G' the graph obtained from G by local completion at v. Then,*

(i) *G' is claw-free, and*
(ii) *for every cycle C' of G' there exists a cycle C of G such that $V(C') \subseteq V(C)$.*

For a claw-free graph G, we define the *closure* $cl(G)$ of G as the graph obtained from G by iteratively performing local completions at vertices with connected neighborhoods until no more edges can be added. As shown in [65], $cl(G)$ is uniquely determined by G, and $cl(G)$ is the line graph of a triangle-free graph. Moreover, in [65] it is shown that Theorem 4.4 has the following consequences. Let $c(G)$ denote the *circumference* of G, i.e., the length of a longest cycle of G. A *factor* of G is a spanning subgraph of G.

Theorem 4.5 (Ryjáček [65]). *Let G be a claw-free graph. Then,*

(i) $c(cl(G)) = c(G)$.
(ii) *If $cl(G)$ is complete, then G is hamiltonian.*
(iii) *Every nonhamiltonian claw-free graph is a factor of a nonhamiltonian line graph.*

Theorem 4.5(ii) implies the result of Oberly and Sumner [63] that every 2-connected locally connected claw-free graph is hamiltonian. Theorem 4.5(iii) together with a result of Zhan [75] and, independently, Jackson [43] implies that every 7-connected claw-free graph is hamiltonian, showing that Conjecture 3.1 is true for claw-free graphs. Slightly more general results on 6-connected claw-free graphs with some additional conditions were obtained by Fan [28] and Li [56]. Moreover, Theorem 4.5(iii) yields the mentioned equivalence of Conjectures 4.2 and 4.3.

4.2. On factors in 4-connected claw-free graphs

In this section, we give several recent results concerning the existence of certain factors in 4-connected claw-free graphs that were obtained by Broersma et al. [19].

First of all it has been shown there that Conjecture 4.3 holds within the subclass of *hourglass-free* graphs, i.e., graphs that do not contain an induced subgraph isomorphic to the *hourglass*, a graph consisting of two triangles meeting in exactly one vertex. This result also follows from a recent result due to Kriessel [53]. To obtain this result, in [19] the following observation made by several graph theorists is proved. The *inflation* of a graph G is the graph obtained from G by replacing all vertices $v_1, v_2, \ldots, v_n$ of G by disjoint complete graphs on $d(v_i)$ vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,d(v_i)}$, and all edges v_iv_j by disjoint edges $v_{i,p}v_{j,q}$ $(i,j \in \{1,\ldots,n\}; p \in \{1,\ldots,d(v_i)\}; q \in \{1,\ldots,d(v_j)\})$. We use the term *inflation* for a graph that is isomorphic to the inflation of some graph. It is obvious that inflations are claw-free and hourglass-free.

Lemma 4.6 (Broersma et al. [19]). *Every* 4-*connected inflation is hamiltonian.*

The connectivity bound in Lemma 4.6 cannot be decreased, since there are nonhamiltonian 3-connected inflations, e.g., the inflation of the Petersen graph. These graphs also show that the connectivity bound in the next result is best possible.

Theorem 4.7 (Broersma et al. [19]). *Every* 4-*connected claw-free hourglass-free graph is hamiltonian.*

Furthermore, the validity of a weaker form of Conjecture 4.3 has been proved in [19].

By Theorem 3.1 in [47], every connected claw-free graph has a 2-*walk*, i.e., a (closed) walk which passes every vertex at most twice. Clearly, the edges of a 2-walk induce a connected factor of maximum degree at most 4.

In [19] the following related result is proved.

Theorem 4.8 (Broersma et al. [19]). *Every* 4-*connected claw-free graph contains a connected factor which has degree two or four at each vertex.*

By the results of Kriesell [53] it is also possible to prove the related result that between every pair of distinct vertices in a 4-connected line graph there exists a spanning trail which passes every vertex at most twice.

Finally, it has been shown in [19] that Conjectures 4.2 and 4.3 are equivalent to seemingly weaker conjectures in which the conclusion is replaced by the conclusion that there exists a factor consisting of a bounded number of paths.

For convenience we use the term *r-path-factor* for a factor consisting of at most r paths. A *path-factor* is an r-path factor for some r, and its *endvertices* are the vertices of degree less than two of its components. Recall that n denotes the number of vertices of a graph.

Theorem 4.9 (Broersma et al. [19]). *Let* $k \geqslant 2$ *be an integer, and let* $f(n)$ *be a function of* n *with the property than* $\lim_{n\to\infty} f(n)/n = 0$. *Then, the following statements are equivalent*:

(1) *Every k-connected claw-free graph is hamiltonian.*
(2) *Every k-connected claw-free graph has an* $f(n)$*-path-factor.*

(3) *Every k-connected claw-free graph has a* 2*-factor with at most* $f(n)$ *components.*
(4) *Every k-connected claw-free graph has a spanning tree with at most* $f(n)$ *vertices of degree one.*
(5) *Every k-connected claw-free graph on n vertices has a path of length at least* $n - f(n)$.

In particular, Theorem 4.9 shows that Conjecture 4.3 is true if one could show that, e.g., every 4-connected claw-free graph admits a factor consisting of a number of paths which is sublinear in n.

Recently, in [41] it has been shown that a claw-free graph G has an r-path-factor if and only if $cl(G)$ has an r-path-factor. Similarly, in [66] it has been shown that a claw-free graph G has a 2-factor with at most r components if and only if $cl(G)$ has such a 2-factor. These results immediately imply the equivalence of the following statements related to Conjecture 4.2.

Theorem 4.10 (Broersma et al. [19]). *Let* $k \geqslant 2$ *be an integer, and let* $f(n)$ *be a function of n with the property that* $\lim_{n\to\infty} f(n)/n = 0$. *Then, the following statements are equivalent:*

(1) *Every k-connected line graph is hamiltonian.*
(2) *Every k-connected line graph has an* $f(n)$*-path-factor.*
(3) *Every k-connected line graph has a* 2*-factor with at most* $f(n)$ *components.*

In particular, Theorem 4.10 shows that Conjecture 4.2 is true if one could show that, e.g., every 4-connected line graph admits a 2-factor consisting of a number of components which is sublinear in n. The equivalences between (1) and (2) of Theorem 4.9 and of Theorem 4.10 appear also in a sequence of equivalences in [50].

4.3. Back to degree conditions for hamiltonicity

If $G = cl(G)$, then we say that the graph G is *closed* (thus, G is closed if and only if G is the line graph of a triangle-free graph). Using the structural properties of closed claw-free graphs, it is possible to prove that a nonhamiltonian closed claw-free graph with large degrees can be covered by relatively few cliques [30]. Denote by $\theta(G)$ the clique covering number of the graph G, and denote by $\sigma_k(G)$ the minimum degree sum of a set of k distinct mutually nonadjacent vertices of G (or ∞ if such a set does not exist).

Theorem 4.11 (Favaron et al. [30]). *Let* $k \geqslant 4$ *be an integer and let G be a* 2*-connected claw-free graph with* $n \geqslant 3k^2 - k - 4$, $\delta(G) \geqslant 3k - 1$ *and* $\sigma_k(G) > n + (k-2)^2$. *Then, either* $\theta(cl(G)) \leqslant k - 1$ *or G is hamiltonian.*

Specifically, Theorem 4.11 implies that, for any integer $k \geqslant 4$, every nonhamiltonian claw-free graph G with $n \geqslant 3k^2 - k - 4$ and $\delta(G) > (n + (k-2)^2)/k$ can be covered by at most $k-1$ cliques. This implies that for proving a minimum degree condition for hamiltonicity of type $\delta(G) > n/k + c$ for any given $k \geqslant 4$, it is enough to list all nonhamiltonian closed claw-free graphs with $\theta(G) \leqslant k-1$.

A characterization of closed nonhamiltonian claw-free graphs with small clique covering number can be achieved by using the correspondence between the graphs and their line graph preimages. The following was proved for $\theta \leqslant 5$ in [30] and independently by Kuipers and Veldman [54]. We refer to [30] for a definition of the classes of graphs contained in $\mathscr{F}'$.

Theorem 4.12 (Favaron et al. [30] and Kuipers and Veldman [54]). *Let G be a 2-connected closed claw-free graph.*

(i) *If* $\theta(G) \leqslant 2$, *then G is hamiltonian.*
(ii) *If* $3 \leqslant \theta(G) \leqslant 5$, *then either G is hamiltonian or G is a spanning subgraph of a graph from* $\mathscr{F}'$.

Combining Theorems 4.11 and 4.12 one can obtain the following result.

Corollary 4.13 (Favaron et al. [30]). *Let G be a 2-connected claw-free graph with* $n \geqslant 77$ *vertices such that* $\delta(G) \geqslant 14$ *and* $\sigma_6(G) > n+19$. *Then, either G is hamiltonian or G is a spanning subgraph of a graph from* $\mathscr{F}'$.

All the nonhamiltonian exceptional graphs have connectivity 2 and hence, under the assumptions of Corollary 4.13, 3-connectedness implies hamiltonicity.

Presently, the best sufficient minimum degree condition for hamiltonicity of 3-connected claw-free graphs we are aware of is due to Favaron and Fraisse [31]; using the claw-free closure and a relationship between properties of cubic graphs and line graphs that will be explained in the next section, they proved that $\delta(G) \geqslant (n+38)/10$ suffices. This is essentially best possible.

Kuipers and Veldman [54] further exploited the fact that the basic idea of finding the exceptional classes of $\mathscr{F}'$ yields a general method for listing these classes for any fixed upper bound on $\theta(G)$. This was a starting point for the proof of the following result. Consider the following two problems.

HAM(c)
Instance: A graph G with $\delta(G) \geqslant cn$.
Question: Is G hamiltonian?

HAMCL(c)
Instance: A claw-free graph G with $\delta(G) \geqslant cn$.
Question: Is G hamiltonian?

Häggkvist [35] proved that HAM$\left(\frac{1}{2}-\varepsilon\right)$ is NP-complete for any fixed $\varepsilon > 0$ (while HAM$\left(\frac{1}{2}\right)$ is trivial by Dirac's Theorem). In claw-free graphs, hamiltonicity is known to be NP-complete [9]. In contrast to these results, the surprising result in [54] says that HAMCL(c) is polynomial for any $c > 0$.

Theorem 4.14 (Kuipers and Veldman [54]). *HAMCL(c) is solvable in polynomial time for any constant $c > 0$.*

The proof of this result in [54] is a clever combination of reduction techniques. Apart from the claw-free closure which opens the possibility to turn to line graphs and their preimages, the key ingredient is a variant of a powerful reduction technique introduced by Catlin [22] and refined by Veldman [72]. These techniques are extremely useful if one is interested in the existence of spanning eulerian subgraphs or eulerian subgraphs that contain at least one endvertex of every edge of the graph, respectively. We show in the next section why such subgraphs are relevant in this context.

4.4. A relationship with properties of cubic graphs

We return to regular graphs in this section, so we are back at the start of our exposition. Despite this, the results and conjectures mentioned below have nothing in common with the former section on regular graphs because, in contrast to the high degrees assumed there, the degree of regularity in this section is just three.

In the sequel we will focus on cyclic and other properties of cubic (i.e., 3-regular) graphs, and show their close relationship with results and conjectures on line graphs and claw-free graphs. We start with some additional terminology and basic facts for general graphs.

4.4.1. Cyclically and essentially k-edge-connected graphs

A graph G is *cyclically k-edge-connected* if there exists no subset E' of $E(G)$ such that $|E'| < k$ and $G - E'$ has at least two components containing cycles. A graph G is *essentially k-edge-connected* if $|E(G)| \geqslant k+1$ and there exists no subset E' of $E(G)$ such that $|E'| < k$ and $G - E'$ has at least two components containing edges.

It is easy to check that the line graph $L(G)$ of a graph G is k-connected if and only if G is essentially k-edge-connected, and that a cubic graph is cyclically 4-edge-connected if and only if it is essentially 4-edge-connected.

A subgraph H of a graph G is *dominating* if every edge of G has at least one end in H.

The following basic result relates the hamiltonicity of a line graph to the existence of a dominating closed trail in its preimage.

Theorem 4.15 (Harary and Nash-Williams [36]). *Let G be a graph with $|E(G)| \geqslant 3$. Then, $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

Some other auxiliary results are related to the existence of edge-disjoint spanning trees, and to their implication for the existence of spanning eulerian subgraphs.

Theorem 4.16 (Nash-Williams [62] and Tutte [70]). *A graph G has k edge-disjoint spanning trees if and only if for every partition $\mathscr{P}$ of $V(G)$ we have $\varepsilon(\mathscr{P}) \geqslant k(|\mathscr{P}| - 1)$, where $\varepsilon(\mathscr{P})$ counts the number of edges of G joining distinct parts of $\mathscr{P}$.*

Theorem 4.17 (Kundu [55]). *Every 4-edge-connected graph has two edge-disjoint spanning trees.*

Theorem 4.18 (Jaeger [48]). *Every graph with two edge-disjoint spanning trees has a spanning eulerian subgraph.*

Combining the above results, we immediately obtain the next corollary.

Corollary 4.19. (i) *Every 4-edge-connected graph has a spanning eulerian subgraph.*
(ii) *Every 4-edge-connected graph has a hamiltonian line graph.*

On the other hand, it is not difficult to show that Conjecture 4.2 is equivalent to the following conjecture.

Conjecture 4.20. *Every essentially 4-edge-connected graph has a hamiltonian line graph.*

At first sight the gap between Corollary 4.19(ii) and Conjecture 4.20 does not look that large. Moreover, in Corollary 4.19(i) we obtain a spanning eulerian subgraph, whereas we would only need a dominating eulerian subgraph in order to prove Conjecture 4.20. Nevertheless, Conjecture 4.20 seems to be very hard.

The next conjecture, that would clearly imply Conjecture 4.20, was put up by Jackson [44]. It resembles the way one can prove that 4-connected planar graphs are hamiltonian by proving assertions on the existence of certain cycles in 2-connected planar graphs.

Conjecture 4.21 (Jackson [44]). *Every 2-edge-connected graph G has an eulerian subgraph H with at least three edges such that each component of $G - V(H)$ is linked by at most three edges to H.*

4.4.2. Cubic graphs

We now turn our attention to related conjectures and results for cubic graphs. The first conjecture is due to Fleischner and Jackson [33] who showed that this conjecture is equivalent to Conjecture 4.2.

Conjecture 4.22 (Fleischner and Jackson [33]). *Every cyclically 4-edge-connected cubic graph has a dominating cycle.*

The main ingredients and observations used to prove the equivalence are sketched here. First, use Theorem 4.15 and the correspondence between 4-connected line graphs and their essentially 4-edge-connected preimages: $L(G)$ is hamiltonian if and only if G contains a dominating eulerian subgraph (which is a dominating cycle if G is cubic); $L(G)$ is 4-connected if and only if G is essentially 4-edge-connected (which is cyclically 4-edge-connected if G is cubic). Secondly, note that one can transform an essentially 4-edge-connected graph into such a graph with minimum degree at least three by deleting the vertices of degree one and suppressing the vertices of degree two. Another transformation can be used to turn the new graph into an essentially and hence cyclically 4-edge-connected *cubic* graph: replace a vertex v of degree $d(v) \geqslant 4$ by a cycle $C_{d(v)}$ and the edges incident with v by edges incident with one vertex of $C_{d(v)}$ each; repeat this for all vertices of degree at least four in such a way that the connectivity requirements remain the same; this is possible, as shown in [33]. We omit the details.

In [33] the following related conjectures are presented.

Conjecture 4.23 (Jaeger; see Fleischner and Jackson [33]). *Every cyclically* 4-*edge-connected cubic graph* G *has a cycle* C *such that* $G - V(C)$ *is acyclic.*

Conjecture 4.24 (Bondy; see Fleischner and Jackson [33]). *Every cyclically* 4-*edge-connected cubic graph has a cycle of length at least cn, for some constant c.*

It is not difficult to show that Conjecture 4.22 implies Conjecture 4.23, while the latter one implies Conjecture 4.24.

The following related problem is mentioned in [68]: Does there exist a natural number m such that every cyclically m-edge-connected cubic graph contains a Hamilton cycle? The Coxeter graph shows that m must be at least 8.

The arguments used to prove the equivalence of Conjectures 4.2 and 4.22 can be combined with the claw-free closure operation to obtain results on cycles in claw-free graphs from results on cycles in cubic graphs. As an example consider the following two results.

A graph G is *k-cyclable* if every set of k vertices of G is contained in a cycle of G.

Theorem 4.25 (Holton et al. [40]). *Every* 3-*connected cubic graph is* 9-*cyclable.*

Theorem 4.26 (Jackson [45]). *Every* 3-*connected claw-free graph is* 9-*cyclable.*

Theorems 4.25 and 4.26 are both best possible, as shown by the inflation of the Petersen graph, which is not 10-cyclable.

The result of Favaron and Fraisse [31] mentioned in Section 4.3 is another example of applying these techniques. Combining these techniques with the reduction methods of Catlin [22] and Veldman [72] gives an opportunity to obtain long cycle results for claw-free graphs, as is done by Broersma and Van der Laag [20].

4.4.3. A possible approach to solving the conjectures

We finish this part on claw-free graphs with an approach to solving the main conjectures proposed by Jackson [45], and some remarks.

Conjecture 4.27 (Jackson [45]). *Every essentially* 6 (*or* 5 *or* 4)-*edge-connected graph has an eulerian subgraph containing all vertices of degree at least* 4.

By Corollary 4.19(i) the conclusion of Conjecture 4.27 holds for 4-edge-connected graphs.

If Conjecture 4.27 is true, then every 6 (or 5 or 4)-connected line graph is hamiltonian. It suffices to prove Conjecture 4.27 for graphs with minimum degree at least 3.

If a graph G contains two edge-disjoint trees T_1 and T_2 such that T_1 is spanning and T_2 contains all vertices of degree at least 4 in G, then G has an eulerian subgraph containing all vertices of degree at least 4 (similarly proved as Theorem 4.18).

We close this section with the following remark that has been made by Van den Heuvel [38]. Let $\mathscr{B}$ be the class of all connected bipartite graphs such that the vertices in one color class all have degree 3 and the vertices in the other color class degree 4. The graphs in $\mathscr{B}$ do not satisfy the above hypothesis concerning the two edge-disjoint trees. The class $\mathscr{B}$ may contain essentially 5-edge-connected graphs. If so, an approach of Conjecture 4.27 via this method could be successful only for essentially 6-edge-connected graphs.

5. Conclusion

We tried to give a flavor of the many open problems, conjectures and recent developments around three themes in hamiltonian graph theory. We hope this inspires and motivates graph theorists to work on these intriguing problems.

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