

# Shape Preserving Interpolatory Subdivision Schemes for Nonuniform Data

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This article is concerned with a class of shape preserving four-point subdivision schemes which are stationary and which interpolate nonuniform univariate data  $\{(x_i, f_i)\}$ . These data are functional data, i.e.,  $x_i \neq x_j$  if  $i \neq j$ . Subdivision for the strictly monotone  $x$ -values is performed by a subdivision scheme that makes the grid locally uniform. This article is concerned with constructing suitable subdivision methods for the  $f$ -data which preserve convexity; i.e., the data at the  $k$ th level,  $\{x_i^{(k)}, f_i^{(k)}\}$  is a convex data set for all  $k$  provided the initial data are convex. First, a sufficient condition for preservation of convexity is presented. Additional conditions on the subdivision methods for convergence to a  $C^1$  limit function are given. This leads to explicit rational convexity preserving subdivision schemes which generate continuously differentiable limit functions from initial convex data. The class of schemes is further restricted to schemes that reproduce quadratic polynomials. It is proved that these schemes are third order accurate. In addition, nonuniform *linear* schemes are examined which extend the well-known linear four-point scheme to the case of nonuniform data. Smoothness of the limit function generated by these linear schemes is proved by using the well-known smoothness criteria of the *uniform* linear four-point scheme. © 2002 Elsevier Science (USA)

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## 1. INTRODUCTION

Subdivision schemes for interpolation of nonuniform univariate data are examined in this article. Such a nonuniform data set is given by  $\{(x_i, f_i) \in \mathbb{R}^2\}$  where the  $x$ -data are strictly monotone, i.e.,  $x_j < x_{j+1}$ ,  $\forall j$ . The basic idea in the construction is to distinguish between subdivision of the  $x$ -data and the  $f$ -data. First, a suitable scheme for the  $x$ -values is

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defined. Then a subdivision scheme for the  $f$ -values is constructed, which depends on the choice of the  $x$ -subdivision. This leads to nonuniform interpolatory subdivision schemes, i.e., schemes for interpolation of non-uniform data. We require that these schemes are stationary and that they use (at most) four points; i.e., the schemes are local. Overviews on subdivision can be found in e.g., [CM89, CDM91, DD89, DL90, DGL91].

As the data are univariate and nonuniform, the  $x$ -values have to be subdivided preserving monotonicity. This is discussed in Section 2.3. The class of monotonicity preserving interpolatory subdivision schemes examined in [KvD99] is attractive for this purpose, since it is capable of generating grids  $\{x_i^{(k)}\}$  that, in the limit, become locally uniform. Although it is only necessary for convergence that the grid becomes dense; see, e.g., [GQ96, DGS99], this stronger property turns out to be helpful for the convergence analysis of the limit function.

Having treated the subdivision scheme for the  $x$ -data, a class of nonuniform subdivision schemes that possess some natural invariances; see, e.g., [CD94], is characterised. This class of schemes is further restricted to a class of nonuniform subdivision schemes that preserve convexity. As is also mentioned in [KvD98a], the only linear interpolatory subdivision scheme that preserves convexity, is the two-point scheme that generates the piecewise linear interpolant, but this limit function is not  $C^1$ , however. We therefore consider nonlinear subdivision schemes that preserve convexity and that produce continuously differentiable limit functions. The construction is a generalization of the approach for equidistant data in [KvD98a], and leads to stationary rational subdivision schemes that preserve convexity. Convexity preserving interpolatory subdivision algorithms have also been discussed in [DLL92, LU94]. These methods are purely geometric but are only second order accurate and much more involved, however.

Apart from convexity preserving nonuniform subdivision schemes, we examine nonuniform *linear* schemes. Smoothness properties of stationary *linear* subdivision schemes for functional nonuniform data  $\{(x_i, f_i)\}$  are also investigated in [War95]. The schemes discussed there are based on midpoint subdivision for the  $x$ -values. The schemes we examine are linear in the  $f$ -data but are still nonlinear in the parameter values  $x_i^{(k)}$ . A non-uniform extension of the well-known *linear* four-point scheme of Dyn *et al.* [DGL87] is constructed. This generalized linear scheme does not lose accuracy in case of nonuniform data, i.e., the approximation order is still equal to four. The important difference with other articles, e.g., [DGS99], is that subdivision for the  $x$ -values is performed by a simple stationary, rational subdivision scheme. The fact that the scheme for subdivision of the  $x$ -values is nonlinear is not problematic as the scheme for the  $f$ -values is still linear.

The outline of this article is as follows. First, in Section 2 the problem definition is given, some basic definitions are introduced, and the subdivision scheme for the  $x$ -data is discussed. Then a class of nonuniform subdivision schemes that possesses natural invariances is characterised. Sufficient conditions for preservation of convexity are given in Section 3. All subdivision schemes that satisfy this condition automatically generate continuous limit functions from initial convex data.

In Section 4, we first give a condition that is sufficient for convergence to continuously differentiable limit functions of any subdivision scheme in the class constructed in Section 2. In addition, sufficient conditions for convergence to a convex and continuously differentiable limit function are given, provided the data are strictly convex. These conditions lead to an explicit class of subdivision schemes that are rational in the arguments. For equidistant data, all schemes in this class reduce to the convexity preserving subdivision scheme introduced in [KvD97b] and fully discussed in [KvD98a].

The class of convex rational schemes is further restricted by requiring third order accuracy, which is discussed in Section 5. The schemes then reproduce quadratic polynomials, and a relation with rational interpolation is discussed. In Section 6 a convexity preserving midpoint subdivision scheme is proved to generate continuously differentiable limit functions, but this scheme is only second order accurate. The article finally illustrates nonuniform convexity preserving subdivision for some examples.

## 2. NONUNIFORM SUBDIVISION

In this section, the problem definition is given, and some basic definitions are introduced. The method for  $x$ -subdivision is discussed, and a class of nonuniform subdivision schemes with natural invariances is constructed.

### 2.1. Problem Description

Given is a finite bounded data set  $\{(x_i^{(0)}, f_i^{(0)}) \in \mathbb{R}^{2 \times N}\}_{i=0}^N$ , where the data  $\{x_i^{(0)}\}$  are strictly monotone, i.e.,  $x_j^{(0)} < x_{j+1}^{(0)}, \forall j$ .

A class of nonuniform interpolatory subdivision schemes for the data  $(x_i^{(0)}, f_i^{(0)})$  must be characterised. The aim is to construct nonuniform convexity preserving subdivision schemes, i.e., the limit function is convex provided the initial data are convex. This class of schemes has to be restricted to subdivision schemes that generate convex and continuously differentiable limiting functions. The second goal is to obtain maximal order of approximation for these schemes.

First we make a remark how to treat the boundaries; see [KvD98a]. Every initial convex data set  $\{(x_i^{(0)}, f_i^{(0)})\}_{i=0}^N$  can be extended in an arbitrary convexity preserving way to  $\{(x_i^{(0)}, f_i^{(0)})\}_{i=-2}^{N+2}$ , such that the limit function of the subdivision scheme is defined in the whole interval  $I = [x_0^{(0)}, x_N^{(0)}]$ . This means that all relevant properties on the  $k$ th iterate are easily shown to hold for the index set  $\{0, \dots, 2^k N\}$ , i.e., all relevant properties are consistently proved on the original domain  $I = [x_0^{(0)}, x_N^{(0)}] = [x_0^{(k)}, x_{2^k N}^{(k)}]$ .

The approach in this article is to subdivide the  $x$ -values by a monotonicity preserving interpolatory subdivision scheme. The class of schemes examined in [KvD99] is used for this purpose. This choice is discussed in Section 2.3, but first we introduce some basic definitions.

## 2.2. Preliminaries

Consider a nonuniform univariate initial data set  $\{(x_i^{(0)}, f_i^{(0)})\}$  in  $\mathbb{R}^2$ , where the  $\{x_i^{(0)}\}$  are strictly monotone, i.e.,  $x_i^{(0)} < x_{i+1}^{(0)}$ ,  $\forall i$ . A subdivision scheme now generates  $\{(x_i^{(k)}, f_i^{(k)})\}$  in  $\mathbb{R}^2$ , with  $k \in \mathbb{N}$ .

Define differences  $s_i^{(k)}$  as

$$s_i^{(k)} = x_{i+1}^{(k)} - x_i^{(k)},$$

and ratios of these differences by

$$r_i^{(k)} = \frac{s_{i-1}^{(k)}}{s_i^{(k)}}, \quad R_i^{(k)} = \frac{1}{r_i^{(k)}}. \quad (2.1)$$

Additionally, divided differences  $g_i^{(k)}$  are defined as

$$g_i^{(k)} = \frac{f_{i+1}^{(k)} - f_i^{(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} = \frac{f_{i+1}^{(k)} - f_i^{(k)}}{s_i^{(k)}}, \quad (2.2)$$

and second order differences are in a symmetric way defined as changes in the divided differences (note that our definition of the  $n$ th order divided difference differs by a factor of  $n!$  compared to the commonly used divided differences):

$$d_i^{(k)} = g_i^{(k)} - g_{i-1}^{(k)}. \quad (2.3)$$

Second order divided differences are thus given by

$$\Delta^2 f_i^{(k)} = \frac{d_i^{(k)}}{\frac{1}{2}(s_{i-1}^{(k)} + s_i^{(k)})} = 2 \frac{g_i^{(k)} - g_{i-1}^{(k)}}{s_{i-1}^{(k)} + s_i^{(k)}}.$$

This directly yields

$$d_i^{(k)} = \frac{1}{2} s_i^{(k)} (1 + r_i^{(k)}) \Delta^2 f_i^{(k)}. \quad (2.4)$$

### 2.3. Monotonicity Preservation

The initial data  $\{x_i^{(0)}\}$  are strictly monotone, but as is well-known direct application of the linear four-point scheme [DGL87] to the  $x$ -values does not preserve monotonicity in general. A simple linear scheme that does preserve monotonicity is given by the two-point scheme

$$\begin{cases} x_{2i}^{(k+1)} = x_i^{(k)}, \\ x_{2i+1}^{(k+1)} = \frac{1}{2} (x_i^{(k)} + x_{i+1}^{(k)}). \end{cases}$$

This scheme is called the midpoint scheme, and it satisfies the property that the grid becomes dense; see, e.g., [GQ96, DGS99]. In [GQ96], the authors discuss nonuniform corner cutting and the necessity that the grid becomes dense, but their results cannot directly be used for *interpolatory* subdivision.

Monotonicity of the linear four-point scheme is discussed in [Cai95]. The author determines ranges of the tension parameter such that this scheme applied to given nonuniform functional data is monotonicity preserving. Although this scheme is stationary, it is data-dependent, however.

In [KvD99], we examined four-point interpolatory subdivision schemes for equidistant data that preserve monotonicity. The class of schemes that was examined is characterised by

$$\begin{cases} x_{2i}^{(k+1)} = x_i^{(k)}, \\ x_{2i+1}^{(k+1)} = \frac{1}{2} (x_i^{(k)} + x_{i+1}^{(k)}) + \frac{1}{2} s_i^{(k)} \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)}), \end{cases} \quad (2.5)$$

where  $\mathcal{G}$  is a function that satisfies specific properties examined in [KvD99]. We quote here those possible requirements:

(a) The first condition on  $\mathcal{G}$  is that if the initial data are uniform, they must remain uniform at all levels (this is equivalent to reproduction of linear functions, but also symmetry arguments give the same requirement):

$$\mathcal{G}(1, 1) = 0. \quad (2.6)$$

(b) Second, the function  $\mathcal{G}$  is required to satisfy the condition for preservation of strict monotonicity, namely there exists a  $\mu_{\mathcal{G}}$  such that

$$|\mathcal{G}(r, R)| \leq \mu_{\mathcal{G}} < 1, \quad \forall r, R. \quad (2.7)$$

This property of  $\mathcal{G}$  guarantees that the resulting grid becomes dense, in the sense that the ratios  $r_i^{(k)}, R_i^{(k)}$  remain bounded [KvD99].

(c) A third (and stronger) condition on  $\mathcal{G}$ , that can be further imposed, is that the subdivision scheme (2.5) has the property that it generates grids  $\{x_i^{(k)}\}$  that become locally uniform in the following sense:

$$r^{(k)} = \max_i \max\{r_i^{(k)}, R_i^{(k)}\} \leq 1 + A_0 \rho_0^k, \quad \forall k, \quad 0 \leq \rho_0 < 1, \quad A_0 < \infty. \quad (2.8)$$

This property of generating locally uniform grids is attractive as it turns out to be suited for the convergence analysis of nonuniform subdivision schemes. The initial data are assumed to be strictly monotone, and condition (2.8) then yields that  $1 \leq r^{(k)} < \infty$ . Then, as is proved in [KvD99], there exist  $\mu_{\mathcal{G}} < 1$ ,  $\bar{r} > 0$  and  $\bar{R} < \infty$ , such that (2.7) holds for all  $\bar{r} < r$ ,  $R < \bar{R}$ .

The following example provides explicit monotonicity preserving subdivision schemes that satisfy the required properties (2.6), (2.7) and (2.8).

**EXAMPLE 2.1 (Rational Monotonicity Preserving Subdivision).** Explicit rational subdivision schemes that preserve monotonicity and that satisfy (2.8) are given by the function

$$\mathcal{G}(r, R) = \frac{r - R}{\ell_1 + (1 + \ell_2)(r + R) + \ell_3 r R}, \quad (\ell_1, \ell_2, \ell_3) \in \Omega, \quad (2.9)$$

where  $\Omega$  is defined by

$$\Omega = \{(\ell_1, \ell_2, \ell_3) \mid \ell_1, \ell_2, \ell_3 \geq 0, \ell_1 + 2\ell_2 + \ell_3 = 6\}.$$

This class of monotonicity preserving subdivision schemes has approximation order four when applied to equidistant data  $\{(i, x_i^{(0)})\}$ , see [KvD99]. For this article, a more important property is that one can indeed prove, see [KvD99], that the scheme with (2.9) satisfies (2.8) with  $\rho_0 = \sqrt{3/4}$ .

## 2.4. Nonuniform Subdivision Schemes

In this section we construct a class of subdivision schemes for interpolation of nonuniform functional data  $\{(x_i^{(k)}, f_i^{(k)})\}$ . The  $x$ -values are subdivided using (2.5) for general  $\mathcal{G}$  satisfying (2.6) and (2.7).

The general class of nonuniform subdivision schemes is written as

$$\begin{cases} x_{2i}^{(k+1)} = x_i^{(k)}, \\ x_{2i+1}^{(k+1)} = \frac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + \frac{1}{2}s_i^{(k)}\mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)}), \\ f_{2i}^{(k+1)} = f_i^{(k)}, \\ f_{2i+1}^{(k+1)} = \mathcal{F}_1(f_{i-1}^{(k)}, f_i^{(k)}, f_{i+1}^{(k)}, f_{i+2}^{(k)}, x_{i-1}^{(k)}, x_i^{(k)}, x_{i+1}^{(k)}, x_{i+2}^{(k)}). \end{cases} \quad (2.10)$$

This implies:

- (1) The subdivision schemes are *interpolatory*.
- (2) The subdivision schemes are *local, using four points*.

Define  $\hat{f}_{2i+1}^{(k+1)}$  as the linear interpolant to the data points  $(x_i^{(k)}, f_i^{(k)})$  and  $(x_{i+1}^{(k)}, f_{i+1}^{(k)})$ , evaluated at the parameter  $x_{2i+1}^{(k+1)}$  determined by scheme (2.5):

$$\begin{aligned} \hat{f}_{2i+1}^{(k+1)} &= \frac{(x_{i+1}^{(k)} - x_{2i+1}^{(k+1)}) f_i^{(k)} + (x_{2i+1}^{(k+1)} - x_i^{(k)}) f_{i+1}^{(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} = \frac{s_{2i+1}^{(k+1)} f_i^{(k)} + s_{2i}^{(k+1)} f_{i+1}^{(k)}}{s_i^{(k)}} \\ &= \frac{1}{2} (f_i^{(k)} + f_{i+1}^{(k)}) + \frac{1}{2} \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})(f_{i+1}^{(k)} - f_i^{(k)}). \end{aligned} \quad (2.11)$$

Then,  $f_{2i+1}^{(k+1)}$  can also be written as

$$f_{2i+1}^{(k+1)} = \hat{f}_{2i+1}^{(k+1)} - \mathcal{F}_2(f_{i-1}^{(k)}, f_i^{(k)}, f_{i+1}^{(k)}, f_{i+2}^{(k)}, x_{i-1}^{(k)}, x_i^{(k)}, x_{i+1}^{(k)}, x_{i+2}^{(k)}).$$

Additional conditions on the correction function  $\mathcal{F}_2$  are determined by the following assumptions on the subdivision schemes:

(3) The subdivision schemes are *invariant under addition of linear functions*, i.e., if the data  $(x_i^{(0)}, f_i^{(0)})$  generate the subdivision points  $(x_i^{(k)}, f_i^{(k)})$ , then the data  $(x_i^{(0)}, f_i^{(0)} + \lambda x_i^{(0)} + \mu)$ , with  $\lambda, \mu \in \mathbb{R}$  generate the subdivision points  $(x_i^{(k)}, f_i^{(k)} + \lambda x_i^{(k)} + \mu)$ .

(4) The subdivision schemes are *invariant under affine transformations of the variables*  $x_i^{(0)}$ , i.e., if the initial data  $(x_i^{(0)}, f_i^{(0)})$  yield subdivision points  $(x_i^{(k)}, f_i^{(k)})$ , then the data  $(\lambda x_i^{(0)} + \mu, f_i^{(0)})$ , with  $\lambda, \mu \in \mathbb{R}$ ,  $\lambda \neq 0$ , yield subdivision points  $(\lambda x_i^{(k)} + \mu, f_i^{(k)})$ .

Condition (4) with  $\lambda = 1$ , combined with condition (3), yields that the scheme can be written as

$$f_{2i+1}^{(k+1)} = \hat{f}_{2i+1}^{(k+1)} - \mathcal{F}_3(d_i^{(k)}, d_{i+1}^{(k)}, s_{i-1}^{(k)}, s_i^{(k)}, s_{i+1}^{(k)}).$$

Condition (4) has no consequence for  $\mathcal{G}$ , as  $r_i^{(k)}$  is invariant under the transformation in (4).

The following assumption deals with homogeneity:

(5) The subdivision schemes are *homogeneous*, i.e., if initial data  $(x_i^{(0)}, f_i^{(0)})$  generate subdivision points  $(x_i^{(k)}, f_i^{(k)})$ , then initial data  $(x_i^{(0)}, \lambda f_i^{(0)})$  yield points  $(x_i^{(k)}, \lambda f_i^{(k)})$ .

A direct consequence of homogeneity of the subdivision scheme is then that the function  $\mathcal{F}_3$  is homogeneous in its first two arguments:

$$\mathcal{F}_3(\lambda x, \lambda y, a, b, c) = \lambda \mathcal{F}_3(x, y, a, b, c), \quad \forall \lambda. \quad (2.12)$$

Assumption (4) with  $\mu = 0$  yields

$$\mathcal{F}_3\left(\frac{1}{\lambda}x, \frac{1}{\lambda}y, \lambda a, \lambda b, \lambda c\right) = \mathcal{F}_3(x, y, a, b, c), \quad \forall \lambda,$$

and this together with (2.12) gives homogeneity in the last three arguments of  $\mathcal{F}_3$ :

$$\overline{\mathcal{F}}_3(x, y, \lambda a, \lambda b, \lambda c) = \lambda \overline{\mathcal{F}}_3(x, y, a, b, c), \quad \forall \lambda.$$

Using this homogeneity, the function  $\mathcal{F}$  is defined by

$$\mathcal{F}_3(d_i^{(k)}, d_{i+1}^{(k)}, s_{i-1}^{(k)}, s_i^{(k)}, s_{i+1}^{(k)}) = s_i^{(k)} \mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)}).$$

The final assumption is:

- (6) The function  $\mathcal{F}$  is  $\text{Lip}_\alpha$ ,  $\alpha > 0$ , in all its arguments.

The class of nonuniform interpolatory subdivision schemes examined in this article becomes

$$\begin{cases} lx_{2i}^{(k+1)} = x_i^{(k)}, \\ x_{2i+1}^{(k+1)} = \frac{1}{2}(x_i^{(k)} + x_{i+1}^{(k)}) + \frac{1}{2}s_i^{(k)}\mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)}), \\ f_{2i}^{(k+1)} = f_i^{(k)}, \\ f_{2i+1}^{(k+1)} = \hat{f}_{2i+1}^{(k+1)} - s_i^{(k)}\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)}), \end{cases} \quad (2.13)$$

where  $\hat{f}_{2i+1}^{(k+1)}$  is the piecewise linear interpolant defined in (2.11).

*Remark 2.2* (Symmetry). The invariance under affine transformations of the variables necessarily yields (take  $\lambda = -1$  and  $\mu = 0$  in assumption (4)) that  $\mathcal{F}$  obeys the following symmetry:

$$\mathcal{F}(x, y, r, R) = \mathcal{F}(y, x, R, r). \quad (2.14)$$

*Remark 2.3* (Reproduction of Linear Functions). The homogeneity of  $\mathcal{F}$ , (2.12), becomes

$$\mathcal{F}(\lambda x, \lambda y, r, R) = \lambda \mathcal{F}(x, y, r, R),$$

and taking  $\lambda = 0$  yields  $\mathcal{F}(0, 0, r, R) = 0$ ; i.e., the subdivision scheme (2.13) reproduces linear functions.

## 2.5. Example: A Nonuniform Linear Four-Point Scheme

As an example of nonuniform subdivision on a grid that becomes locally uniform, a nonuniform linear scheme is constructed in this section.



**DEFINITION 2.4 (Linearity).** We call a subdivision scheme of class (2.13) *linear*, if the function  $\mathcal{F}(x, y, r, R)$  is linear in the variables  $x$  and  $y$ , i.e., we can write  $\mathcal{F}$  as

$$\mathcal{F}(x, y, r, R) = \mathcal{K}^1(r, R) x + \mathcal{K}^2(r, R) y. \quad (2.15)$$

In [Dub86], a linear subdivision scheme based on local equidistant cubic interpolation is constructed. This scheme is extended in [DGL87] by including a tension parameter for shape design, and the smoothness properties and approximation order are investigated.

A *nonuniform* linear subdivision scheme will be constructed as a generalization of the uniform linear four-point scheme [DGL87]. We therefore determine the cubic polynomial that interpolates the four data points  $(x_{i-1}^{(k)}, f_{i-1}^{(k)})$ ,  $(x_i^{(k)}, f_i^{(k)})$ ,  $(x_{i+1}^{(k)}, f_{i+1}^{(k)})$ , and  $(x_{i+2}^{(k)}, f_{i+2}^{(k)})$ . The value of  $f_{2i+1}^{(k+1)}$  is defined as the evaluation of this cubic function at  $x_{2i+1}^{(k+1)}$ .

A straightforward calculation yields that the subdivision scheme is contained in the class (2.13) with  $\mathcal{F}_L$  given by (2.15), where the functions  $\mathcal{K}_L^1$  and  $\mathcal{K}_L^2$  satisfy (subscripts  $L$  refer to linear)

$$\begin{aligned} \mathcal{K}_L^1(r, R) &= \frac{(1 - \mathcal{G}^2(r, R))(1 - \mathcal{G}(r, R) + 2R)}{8(1 + r + R)(1 + r)}, \\ \mathcal{K}_L^2(r, R) &= \frac{(1 - \mathcal{G}^2(r, R))(1 + \mathcal{G}(r, R) + 2r)}{8(1 + r + R)(1 + R)}. \end{aligned} \quad (2.16)$$

Analogous to the linear four-point scheme [DGL87], a class of nonuniform linear interpolatory four-point schemes is

$$\begin{cases} f_{2i}^{(k+1)} = f_i^{(k)}, \\ f_{2i+1}^{(k+1)} = \frac{1}{2}(f_i^{(k)} + f_{i+1}^{(k)}) + \frac{1}{2}\mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})(f_{i+1}^{(k)} - f_i^{(k)}) \\ \quad - 16ws_i^{(k)}(\mathcal{K}_L^1(r_i^{(k)}, R_{i+1}^{(k)})d_i^{(k)} + \mathcal{K}_L^2(r_i^{(k)}, R_{i+1}^{(k)})d_{i+1}^{(k)}), \end{cases} \quad (2.17)$$

where the tension parameter  $w = 1/16$  results in reproduction of cubic polynomials.

In the limit  $k \rightarrow \infty$ , the nonuniform linear scheme converges to the uniform linear scheme when  $\mathcal{G}$  is chosen such that (2.6), (2.7), and (2.8) are satisfied. Indeed, in that case the ratios, defined in (2.1), satisfy  $r_i^{(k)} \rightarrow 1$ , and in (2.17)  $\mathcal{G} \rightarrow 0$ . Therefore in (2.16)  $\mathcal{K}_L^j \rightarrow 1/16$ , i.e., the linear four-point scheme for equidistant data becomes

$$\begin{aligned} f_{2i+1}^{(k+1)} &= \frac{1}{2}(f_i^{(k)} + f_{i+1}^{(k)}) - ws_i^{(k)}(d_i^{(k)} + d_{i+1}^{(k)}) \\ &= \frac{1}{2}(f_i^{(k)} + f_{i+1}^{(k)}) - ws_i^{(k)}(g_{i+1}^{(k)} - g_{i-1}^{(k)}) \\ &= -wf_{i-1}^{(k)} + \left(\frac{1}{2} + w\right)f_i^{(k)} + \left(\frac{1}{2} + w\right)f_{i+1}^{(k)} - wf_{i+2}^{(k)}. \end{aligned} \quad (2.18)$$

It is proved in [DGL87], using a double step estimate on jumps of differences  $d_j^{(k)}$ , that it is sufficient for convergence to  $C^1$  limit functions that  $0 < w < 1/8$ , and this range on the tension parameter can also be enlarged using more steps; see, e.g., [DGL91].

Smoothness for the non-uniform case follows from the proof in the uniform case, provided  $\{x_i^{(k)}\}$  become locally uniform, according to (2.8).

In order to prove this, we need the following lemma:

LEMMA 2.5 (Technical Lemma). *Let the function  $\Psi: \mathbb{R}_+^s \rightarrow \mathbb{R}$  satisfy*

$$\Psi \in \text{Lip}_\alpha(\mathbb{R}^s), \quad \alpha > 0, \quad \text{and} \quad \Psi(1, 1, \dots) = 1. \quad (2.19)$$

If

$$|x_j^{(k)} - 1| \leq A_0 \rho_0^k, \quad \forall k, \quad j = 1, \dots, s, \quad \rho_0 < 1,$$

then

$$|\Psi(x_1^{(k)}, \dots, x_s^{(k)})| \leq 1 + A_1 \rho_1^k, \quad \text{with} \quad \rho_1 < 1 \quad \text{and} \quad A_1 < \infty.$$

*Proof.* By (2.19)

$$\forall x, y \in \mathbb{R}^s \quad \exists \alpha > 0, \quad |\Psi(x+y) - \Psi(x)| \leq A_2 \|y\|^\alpha, \quad A_2 < \infty.$$

$$\begin{aligned} |\Psi(x_1^{(k)}, \dots, x_s^{(k)})| &= |\Psi(x_1^{(k)}, \dots, x_s^{(k)}) - \Psi(1, \dots, 1) + 1| \\ &\leq 1 + |\Psi(x_1^{(k)}, \dots, x_s^{(k)}) - \Psi(1, \dots, 1)| \\ &\leq 1 + A_2 \max_j |x_j^{(k)} - 1|^\alpha \leq 1 + A_2 (A_0 \rho_0^k)^\alpha = 1 + A_1 \rho_1^k, \end{aligned}$$

where  $\rho_1 = \rho_0^\alpha < 1$  and  $A_1 = A_2 A_0^\alpha < \infty$ . ■

Next we can prove the following

THEOREM 2.6 (Nonuniform linear subdivision). *Consider the subdivision scheme (2.13), where the function  $\mathcal{G}$  satisfies (2.8). Let  $\mathcal{F}$  be a function that is  $\text{Lip}_\alpha$  in its arguments, and let it be linear according to Definition 2.4.*

*If it can be proved on a uniform grid that*

$$\max_i |d_i^{(k+n)}| \leq \lambda^n \max_i |d_i^{(k)}|, \quad \lambda < 1, \quad n \in \mathbb{N}, \quad 1 \leq n < \infty \quad (2.20)$$

*(and hence the nonuniform scheme on uniform data generates a continuously differentiable limit function), then the nonuniform linear subdivision scheme (2.13) generates  $C^1$  limit functions for any initial nonuniform data.*

*Proof.* The proof is based on the fact that the grid becomes locally uniform. Since  $n$  is finite, and the subdivision scheme is local and linear,  $d_i^{(k+n)}$  is a finite linear combination of  $d_j^{(k)}$ , which contains  $\mathcal{H}^\ell$  in different arguments, i.e., we have (see Lemma 2.5)

$$\begin{aligned} d_i^{(k+n)} &= \sum_j \mathcal{H}_{i,j}(\mathcal{H}^\ell(r, R), \dots) d_j^{(k)} \\ &= \sum_j (\mathcal{H}_{i,j}(\mathcal{H}^\ell(r, R)) - \mathcal{H}_{i,j}(\mathcal{H}^\ell(1, 1)) + \mathcal{H}_{i,j}(\mathcal{H}^\ell(1, 1))) \cdot d_j^{(k)}, \end{aligned}$$

where  $\mathcal{H}_{i,j}$  is multivariate in  $\mathcal{H}^\ell(r, R)$ . By (2.8)

$$|\mathcal{H}^\ell(r, R) - \mathcal{H}^\ell(1, 1)| \leq |\mathcal{H}^\ell(1, 1)| (1 + A_2 \rho_2^k),$$

and hence by Lemma 2.5

$$|d_i^{(k+n)}| \leq \lambda \cdot \max_j |d_j^{(k)}| + C_1 \rho_1^k \max_j |d_j^{(k)}|.$$

It follows from the estimate in the uniform case ( $r = R = 1$ ) that  $\lambda < 1$ .

Hence,  $\max_i |d_i^{(k)}|$  is a Cauchy sequence with limit 0, and hence the limit function generated by the nonuniform scheme is  $C^1$ . ■

*Remark 2.7* (Nonlinear subdivision). This theorem does not apply to nonlinear subdivision schemes, since  $\lambda$  in the proof for uniform nonlinear schemes depends on the data in general.

We now apply Theorem 2.6 to the nonuniform linear scheme (2.17). Since the ratios  $r_j^{(k)}$  converge to 1 as  $k$  increases, this nonuniform scheme converges to the uniform linear four-point scheme. Since the uniform scheme generates  $C^1$  limit functions for  $0 < w < 1/8$  [DGL87], the nonuniform linear four-point scheme (2.17) also generates  $C^1$  functions for this range of the tension parameter (and this range can be extended). The leads to the following:

**COROLLARY 2.8.** *The nonuniform linear four-point scheme (2.17) generates continuously differentiable limit functions if the tension parameter satisfies  $0 < w < 1/8$ .*

*Remark 2.9* (Bivariate subdivision). The linear nonuniform four-point interpolatory subdivision scheme (2.15) can naturally be generalized to a *nonuniform subdivision scheme for rectangular data* in two dimensions for functional data  $\{(x_i, y_j, f_{i,j})\}$  on a rectangular grid. The reader is referred to, e.g., [DGL87] for the equidistant case. A nonuniform algorithm works as follows. First, apply a monotonicity preserving subdivision scheme in the class (2.5) which satisfies (2.8) for the  $x_i$ -data and separately for the

$y_i$ -data, which refines the grid. Then define  $f_{2i,2j}^{(k+1)} = f_{i,j}^{(k)}$  and calculate  $f_{2i+1,2j}^{(k+1)}$  by applying scheme (2.15) in the  $x$ -direction. Finally, the data  $f_{i,2j+1}^{(k+1)}$  are set by application of scheme (2.15) in the  $y$ -direction. It is easily checked that subdivision in the  $x$ -direction commutes with subdivision in the  $y$ -direction.

In the next sections, we discuss *convexity preserving* subdivision schemes and analyse the smoothness properties of the limit function and its approximation order.

### 3. CONVEXITY PRESERVATION

In this section, we examine convexity preservation of the class of nonuniform subdivision schemes (2.13).

**THEOREM 3.1 (Sufficient Convexity Condition).** *Let  $\beta \in \text{Lip}_\alpha(\mathbb{R}_+)$ ,  $\alpha > 0$ , be a function such that for all  $r \in \mathbb{R}_+$*

$$\beta(r) \geq 0 \quad \text{and} \quad \beta(r) + \beta(1/r) \leq 2. \quad (3.1)$$

*Then, the subdivision scheme (2.13) satisfying (2.7) and*

$$0 \leq \mathcal{F}(x, y, r, R) \leq \frac{1}{4} \min\{\beta(r)(1 + \mathcal{G}(r, R)) x, \beta(R)(1 - \mathcal{G}(r, R)) y\}, \quad (3.2)$$

*preserves convexity.*

*Proof.* Consider the data  $\{(x_i^{(k)}, f_i^{(k)}) \in \mathbb{R}^2\}$  generated by the subdivision scheme (2.13) where  $\mathcal{G}$  satisfies (2.7). Convexity preserving properties of (2.13) are analysed by examining the second order divided differences  $d_i^{(k)}$ : the changes in the first order divided differences must be nonnegative.

According to (2.2), (2.5), and (2.13) the first order differences are

$$\begin{aligned} g_{2i}^{(k+1)} &= \frac{f_{2i+1}^{(k+1)} - f_{2i}^{(k+1)}}{x_{2i+1}^{(k+1)} - x_{2i}^{(k+1)}} = \frac{\hat{f}_{2i+1}^{(k+1)} - s_i^{(k)} \mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)}) - f_i^{(k)}}{s_{2i}^{(k+1)}} \\ &= \frac{\frac{1}{2}(f_{i+1}^{(k)} - f_i^{(k)})(1 + \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})) - s_i^{(k)} \mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})}{\frac{1}{2}s_i^{(k)}(1 + \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)}))}. \end{aligned}$$

Thus

$$g_{2i}^{(k+1)} = g_i^{(k)} - 2 \frac{\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})}{1 + \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})}, \quad (3.3)$$

and similarly

$$g_{2i+1}^{(k+1)} = g_i^{(k)} + 2 \frac{\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})}{1 - \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})}. \quad (3.4)$$

This yields that the second order differences  $d_{2i+1}^{(k+1)}$  and  $d_{2i}^{(k+1)}$  become

$$d_{2i+1}^{(k+1)} = g_{2i+1}^{(k+1)} - g_{2i}^{(k+1)} = 4 \frac{\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})}{1 - \mathcal{G}^2(r_i^{(k)}, R_{i+1}^{(k)})}, \quad (3.5)$$

$$d_{2i}^{(k+1)} = d_i^{(k)} - 2 \frac{\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})}{1 + \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})} - 2 \frac{\mathcal{F}(d_{i-1}^{(k)}, d_i^{(k)}, r_{i-1}^{(k)}, R_i^{(k)})}{1 - \mathcal{G}(r_{i-1}^{(k)}, R_i^{(k)})}. \quad (3.6)$$

It has to be proved for convexity preservation that  $d_i^{(k+1)} \geq 0$ , if  $d_i^{(k)} \geq 0$ .

Since  $\mathcal{G}$  satisfies (2.7), the non-negativity of  $d_{2i+1}^{(k+1)}$  is equivalent to the non-negativity of  $\mathcal{F}$  assumed in (3.2).

The non-negativity of  $d_{2i}^{(k+1)}$  is obtained as

$$\begin{aligned} d_{2i}^{(k+1)} &= d_i^{(k)} - \frac{2\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})}{1 + \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})} - \frac{2\mathcal{F}(d_{i-1}^{(k)}, d_i^{(k)}, r_{i-1}^{(k)}, R_i^{(k)})}{1 - \mathcal{G}(r_{i-1}^{(k)}, R_i^{(k)})} \\ &\geq d_i^{(k)} - \frac{2}{1 + \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})} \frac{1}{4} (1 + \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})) \beta(r_i^{(k)}) d_i^{(k)} \\ &\quad - \frac{2}{1 - \mathcal{G}(r_{i-1}^{(k)}, R_i^{(k)})} \frac{1}{4} (1 - \mathcal{G}(r_{i-1}^{(k)}, R_i^{(k)})) \beta(R_i^{(k)}) d_i^{(k)} \\ &= d_i^{(k)} - \frac{1}{2} \beta(r_i^{(k)}) d_i^{(k)} - \frac{1}{2} \beta(R_i^{(k)}) d_i^{(k)} \\ &= d_i^{(k)} \left( 1 - \frac{1}{2} \beta(r_i^{(k)}) - \frac{1}{2} \beta(1/r_i^{(k)}) \right) \geq 0, \end{aligned}$$

which completes the proof.  $\blacksquare$

EXAMPLE 3.2. An example of a function  $\beta$  that satisfies (3.1) is

$$\beta(r) = 2 \frac{1 - \gamma + \gamma r}{1 + r}, \quad \text{for } 0 \leq \gamma \leq 1,$$

as for  $r \in \mathbb{R}_+$ ,  $\beta \in \text{Lip}_1$  and

$$\beta(r) \geq 0, \quad \beta(r) + \beta(1/r) = 2 \frac{1 - \gamma + \gamma r}{1 + r} + 2 \frac{(1 - \gamma) r + \gamma}{1 + r} = 2.$$

The special case  $\beta(r) = 1$  is obtained by the choice  $\gamma = 1/2$ .

Concerning  $C^0$ -convergence, the following theorem can be formulated:

**THEOREM 3.3 (Convergence).** *Given is a convex data set  $\{(x_i^{(0)}, f_i^{(0)}) \in \mathbb{R}^2\}$ , where  $\{x_i^{(0)}\}$  is strictly monotone. Let the  $k$ th level data  $\{(x_i^{(k)}, f_i^{(k)})\}_i$  be defined by a subdivision scheme (2.13) satisfying (2.7), (3.1), and (3.2).*

*Repeated application of such a subdivision scheme to the data set  $\{(x_i^{(0)}, f_i^{(0)})\}$  generates values  $(x_i^{(k)}, f_i^{(k)})$  of a continuous function which is convex and interpolates the initial data points  $(x_i^{(0)}, f_i^{(0)})$ .*

*Proof.* Define the function  $f^{(k)}$  as the piecewise linear interpolant to the data  $(x_i^{(k)}, f_i^{(k)})$ . By construction the functions  $f^{(k)}$  interpolate the initial data and are convex. Convexity of every  $f^{(k)}$  implies convergence to a continuous convex limit function [FM98]. ■

#### 4. CONVERGENCE TO A CONTINUOUSLY DIFFERENTIABLE FUNCTION

In this section, we first present a lemma dealing with sufficient conditions for convergence of subdivision schemes of class (2.13) to continuously differentiable limit functions. Next, we apply this result to convexity preserving subdivision schemes.

**LEMMA 4.1 (Sufficient Smoothness Conditions).** *Given is a data set  $\{(x_i^{(0)}, f_i^{(0)}) \in \mathbb{R}^2\}$ , where  $\{x_i^{(0)}\}$  is strictly monotone. Let the  $k$ th level data  $\{(x_i^{(k)}, f_i^{(k)})\}$  be defined by a subdivision scheme (2.13) where  $\mathcal{G}$  satisfies (2.7).*

*A sufficient condition for convergence of such a subdivision scheme to a continuously differentiable limit function is that the quantities*

$$\max_i |d_i^{(k)}|$$

*form a Cauchy sequence in  $k$  with limit 0.*

*Proof.* The construction follows the lines of the proof of smoothness of the limit curve generated by the linear four-point scheme in [DGL87], as we did in [KvD98a]. For any data set  $\{(x_i^{(k)}, f_i^{(k)})\}$ , the function  $g^{(k)}$  is defined as the linear interpolant of the data points  $(x_{2i+1}^{(k+1)}, g_i^{(k)})$ , where  $g_i^{(k)}$  are first order divided differences, see (2.2).

It is sufficient for the convergence of this sequence  $g^{(k)}$  that there exists a  $C_1 \in \mathbb{R}$  and  $\mu_1 < 1$  (where  $\mu_1$  may depend on the initial data), such that

$$\|g^{(k+1)} - g^{(k)}\|_\infty \leq C_1 \mu_1^k. \quad (4.1)$$

By construction, the maximal distance between the functions the piecewise linear functions  $g^{(k)}$  and  $g^{(k+1)}$  must necessarily occur at a point  $x_{4i-1}^{(k+2)}$  or  $x_{4i+1}^{(k+2)}$  for some  $i$ , i.e., it must hold that

$$\|g^{(k+1)} - g^{(k)}\|_\infty = \max_i \max\{\delta_{2i-1}^{(k+1)}, \delta_{2i}^{(k+1)}\} \leq C_1 \mu_1^k,$$

where the distances  $\delta_j^{(k)}$  are given by

$$\delta_{2i-1}^{(k+1)} = \left| g_{2i-1}^{(k+1)} - \frac{(s_{4i-1}^{(k+2)} + s_{2i}^{(k+1)}) g_{i-1}^{(k)} + s_{4i-2}^{(k+2)} g_i^{(k)}}{s_{2i-1}^{(k+1)} + s_{2i}^{(k+1)}} \right|,$$

$$\delta_{2i}^{(k+1)} = \left| g_{2i}^{(k+1)} - \frac{s_{4i+1}^{(k+2)} g_{i-1}^{(k)} + (s_{2i-1}^{(k+1)} + s_{4i}^{(k+2)}) g_i^{(k)}}{s_{2i-1}^{(k+1)} + s_{2i}^{(k+1)}} \right|.$$

A straightforward computation (details can be found in [KvD97a]) yields that

$$\delta_{2i}^{(k+1)} \leq |d_i^{(k)}| + \frac{2}{1 - \mu_g} |\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})|.$$

As it is required that  $\mathcal{F}$  is  $\text{Lip}_\alpha$  in its arguments,  $\alpha > 0$  (see Condition (6) in Section 2.4), and hence continuous, and the fact that  $r_i^{(k)}, R_i^{(k)}$  are assumed to be bounded, the homogeneity of  $\mathcal{F}$  yields that, for  $d_{i+1}^{(k)}/d_i^{(k)} \leq 1$ ,

$$\begin{aligned} |\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})| &= |d_i^{(k)}| \cdot |\mathcal{F}(1, d_{i+1}^{(k)}/d_i^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})| \\ &\leq C_2 \cdot |d_i^{(k)}|, \quad C_2 < \infty, \end{aligned}$$

and by similarly examining the case  $d_{i+1}^{(k)}/d_i^{(k)} \geq 1$ , it is finally obtained

$$|\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})| \leq C_2 \max\{|d_i^{(k)}|, |d_{i+1}^{(k)}|\}, \quad C_2 < \infty.$$

The estimate of  $\delta_{2i}^{(k+1)}$  is completed with

$$\delta_{2i}^{(k+1)} \leq |d_i^{(k)}| + \frac{2}{1 - \mu_g} C_2 \max\{|d_i^{(k)}|, |d_{i+1}^{(k)}|\} \leq \left(1 + \frac{2C_2}{1 - \mu_g}\right) \max_j |d_j^{(k)}|,$$

and a similar result can be derived for  $\delta_{2i-1}^{(k+1)}$ .

The conclusion is that it is sufficient for convergence of the (continuous) functions  $g^{(k)}$  that  $\max_i |d_i^{(k)}|$  is a Cauchy sequence in  $k$  with limit 0. The proof that also  $g^{(\infty)} = f^{(\infty)'}$  can be given using the uniform convergence of Bernstein polynomials; see [DGL87]. ■

Lemma 4.1 holds for all subdivision schemes in class (2.13). We continue this section with examination of  $C^1$ -smoothness of convexity preserving subdivision schemes.

**THEOREM 4.2 (Strict Convexity and Smoothness).** *Let  $\{x_i^{(0)}\}$  be strictly monotone and let the data set  $\{(x_i^{(0)}, f_i^{(0)})\}$  be strictly convex. Consider the class of subdivision schemes (2.13) where  $\mathcal{G}$  satisfies (2.6), (2.7), and (2.8), and  $\beta$  satisfies (3.1) and*

$$\beta \in \text{Lip}_\alpha(\mathbb{R}_+), \quad \beta(1) = 1, \quad \text{and} \quad \forall \bar{r} > 0 \quad \exists \bar{\beta} > 0 : r > \bar{r} \rightarrow \beta(r) \geq \bar{\beta}. \quad (4.2)$$

*Assume further that there exist  $\nu, \mu$ ,  $0 < \nu \leq \mu < 1$  such that  $\forall x, y$ ,  $0 < x, y < \infty$ :*

$$\mathcal{F}(x, y, r, R) \geq \frac{1}{4} \nu \max\{\beta(r)(1 + \mathcal{G}(r, R)) x, \beta(R)(1 - \mathcal{G}(r, R)) y\}, \quad (4.3)$$

$$\mathcal{F}(x, y, r, R) \leq \frac{1}{4} \mu \min\{\beta(r)(1 + \mathcal{G}(r, R)) x, \beta(R)(1 - \mathcal{G}(r, R)) y\}. \quad (4.4)$$

*Repeated application of such a subdivision scheme leads to a continuously differentiable function which is convex and interpolates the initial data  $(x_i^{(0)}, f_i^{(0)})$ .*

*Proof.* It is sufficient to prove that  $\max_i d_i^{(k)}$  is a geometric sequence in  $k$  with limit 0. Using (3.5) one easily shows (see [KvD97a] for the technical details) that

$$d_{2i+1}^{(k+1)} \leq \mu(1 + C_1 \rho_1^k) \max_j d_j^{(k)},$$

where Lemma 2.5 is applied for the function  $\beta$ . As  $\mu < 1$ , there exists a  $k^* < \infty$  such that  $\mu(1 + C_1 \rho_1^k) < 1$ ,  $\forall k \geq k^*$ . Similarly, it is obtained for  $d_{2i}^{(k+1)}$  from (3.6) that

$$d_{2i}^{(k+1)} \leq (1 - \frac{1}{2} \nu \beta(r_i^{(k)}) - \frac{1}{2} \nu \beta(R_i^{(k)})) d_i^{(k)}.$$

As the ratios are assumed to satisfy  $r_i^{(k)} \geq \bar{r}$ , see (2.8), it follows from (4.2) that  $\beta(r_i^{(k)}) \geq \bar{\beta} > 0$ . Hence

$$d_{2i}^{(k+1)} \leq (1 - \nu \bar{\beta}) d_i^{(k)}.$$

In addition, as also  $\nu > 0$  we obtain  $1 - \nu \bar{\beta} < 1$ . ■

Strict convexity is required since in general the limit function cannot be  $C^1$  if the data are convex but not strictly convex. For example, this is the case if data are drawn from  $f(x) = |x|$  (including the point  $(0, 0)$ ).

The conditions (4.3) and (4.4) in Theorem 4.2 are natural since we have to require that the data are strictly convex. As in the equidistant case, see [KvD98a], these conditions are only a little more restrictive than the convexity condition (3.2).



**THEOREM 4.3 (A Class of Smooth Convex Schemes).** *Let  $\{x_i^{(0)}\}$  be strictly monotone and let the data set  $\{(x_i^{(0)}, f_i^{(0)})\}$  be strictly convex. Consider the class of subdivision schemes (2.13) with  $\mathcal{F}$  given by*

$$\mathcal{F}(x, y, r, R) = \frac{1}{4} \frac{1}{\frac{1}{\beta(r)(1+\mathcal{G}(r, R))x} + \frac{1}{\beta(R)(1-\mathcal{G}(r, R))y}}, \quad (4.5)$$

where the function  $\beta$  satisfies (4.2) and

$$\beta(r) + \beta(1/r) = 2, \quad \forall r, \quad (4.6)$$

and where  $\mathcal{G}$  satisfies conditions (2.6), (2.7), and (2.8).

Repeated application of such a subdivision scheme to the data  $\{(x_i^{(0)}, f_i^{(0)})\}$  leads to a continuously differentiable function which is convex and interpolates the initial data points  $(x_i^{(0)}, f_i^{(0)})$ .

*Proof.* Define  $q_i^{(k)}$  as

$$q_i^{(k)} = \frac{\beta(r_i^{(k)})(1+\mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})) d_i^{(k)}}{\beta(R_{i+1}^{(k)})(1-\mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})) d_{i+1}^{(k)}}, \quad (4.7)$$

and the sequence  $q^{(k)}$  as

$$q^{(k)} = \max_i \max\{q_i^{(k)}, 1/q_i^{(k)}\}.$$

First we show that  $q^{(k)}$  is a bounded sequence, i.e.,

$$\exists q^* < \infty \quad \text{such that} \quad q^{(k)} \leq q^*, \quad \forall k.$$

Next, we show that  $\mathcal{F}$  satisfies conditions (4.3) and (4.4).

It is obtained using (3.5) and (3.6) that

$$\begin{aligned} \frac{d_{2i}^{(k+1)}}{d_{2i+1}^{(k+1)}} &= \frac{1-\mathcal{G}^2(r_i^{(k)}, R_{i+1}^{(k)})}{\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})} \left( \frac{1}{4} d_i^{(k)} - \frac{1}{2} \frac{\mathcal{F}(d_{i-1}^{(k)}, d_i^{(k)}, r_{i-1}^{(k)}, R_i^{(k)})}{1-\mathcal{G}(r_{i-1}^{(k)}, R_i^{(k)})} \right) \\ &\quad - \frac{1}{2} (1-\mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})). \end{aligned} \quad (4.8)$$

For the special choice (4.5), it is easily shown that

$$\begin{aligned} \mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)}) &= \frac{1}{4} \beta(r_i^{(k)})(1+\mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})) d_i^{(k)} \frac{1}{1+q_i^{(k)}}, \\ \mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)}) &= \frac{1}{4} \beta(R_{i+1}^{(k)})(1-\mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})) d_{i+1}^{(k)} \frac{q_i^{(k)}}{1+q_i^{(k)}}, \end{aligned}$$

which is substituted in (4.8) to obtain

$$\frac{d_{2i}^{(k+1)}}{d_{2i+1}^{(k+1)}} \leq \frac{1 - \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})}{\beta(r_i^{(k)})} q^{(k)}, \quad (4.9)$$

since  $\beta$  satisfies (4.6). Therefore

$$q_{2i}^{(k+1)} \leq \frac{\beta(r_{2i}^{(k+1)}) \cdot (1 + \mathcal{G}(r_{2i}^{(k+1)}, R_{2i+1}^{(k+1)})) \cdot (1 - \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)}))}{\beta(R_{2i+1}^{(k+1)}) \cdot \beta(r_i^{(k)}) \cdot (1 - \mathcal{G}(r_{2i}^{(k+1)}, R_{2i+1}^{(k+1)}))} q^{(k)}$$

This estimate is the crucial step for giving the bounds on  $q_{2i}^{(k+1)}$ . (A similar estimate can be given for  $q_{2i+1}^{(k+1)}$ .)

As  $\beta$  and  $\mathcal{G}$  are assumed to be Lipschitz continuous, the ratios  $r_i^{(k)}$  satisfy (2.8), and because of the fact that  $\beta(1) = 1$ ,  $\mathcal{G}(1, 1) = 0$ , Lemma 2.5 implies that there exists a  $k^* < \infty$ , which may depend on the initial data, such that

$$q^{(k+1)} \leq (1 + A_1 \rho_1^k) q^{(k)}, \quad \forall k \geq k^*, \quad \rho_1 < 1, \quad A_1 < \infty. \quad (4.10)$$

Since

$$\frac{\beta(r_i^{(k)})(1 + \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)}))}{\beta(R_{i+1}^{(k)})(1 - \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)}))} \leq \frac{2 \cdot 2}{\bar{\beta}(1 - \mu_{\mathcal{G}})},$$

the following bound is directly obtained from (4.8) and (4.9)

$$\begin{aligned} q^{(k+1)} &\leq \frac{2 \cdot 2}{\bar{\beta}(1 - \mu_{\mathcal{G}})} \frac{1 - \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})}{\beta(r_i^{(k)})} q^{(k)} \leq \frac{2 \cdot 2}{\bar{\beta}(1 - \mu_{\mathcal{G}})} \cdot \frac{2}{\bar{\beta}} q^{(k)} \\ &\leq \frac{8}{\bar{\beta}^2(1 - \mu_{\mathcal{G}})} q^{(k)}, \quad \forall k. \end{aligned} \quad (4.11)$$

The combination of (4.11) for the first  $k^*$  iterations with (4.10) for the subdivision levels above  $k^*$ , yields that there exists an  $A_2 < \infty$  such that

$$q^{(k+1)} \leq (1 + A_2 \rho_1^k) q^{(k)}, \quad \forall k,$$

and therefore

$$q^{(k)} \leq q^{(0)} \prod_{\ell=0}^{k-1} (1 + A_2 \rho_1^\ell).$$

Since  $1 + x \leq e^x$ , we obtain

$$\begin{aligned} \prod_{\ell=0}^{k-1} (1 + A_2 \rho_1^\ell) &\leq \prod_{\ell=0}^{k-1} \exp(A_2 \rho_1^\ell) = \exp\left(A_2 \sum_{\ell=0}^{k-1} \rho_1^\ell\right) \\ &= \exp\left(A_2 \frac{1 - \rho_1^k}{1 - \rho_1}\right) \leq \exp\left(\frac{A_2}{1 - \rho_1}\right) = A_3 < \infty, \end{aligned}$$

and hence the sequence  $q^{(k)}$  is bounded (in  $k$ ),

$$q^{(k)} \leq A_3 q^{(0)}, \quad \forall k.$$

Now, the proof of the theorem can be completed with  $q^* = \max(1, A_3 q^{(0)})$ . Take

$$\nu = \frac{1}{1+q^*} > 0 \quad \text{and} \quad \mu = \frac{q^*}{1+q^*} < 1,$$

and application of (4.9) results in

$$\begin{aligned} \mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)}) &= \frac{1}{4} \beta(r_i^{(k)})(1 + \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})) d_i^{(k)} \frac{1}{1+q_i^{(k)}} \\ &\geq \frac{1}{4} \beta(r_i^{(k)})(1 + \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})) d_i^{(k)} \frac{1}{1+q^*} \\ &= \frac{1}{4} \nu \beta(r_i^{(k)})(1 + \mathcal{G}(r_i^{(k)}, R_{i+1}^{(k)})) d_i^{(k)}. \end{aligned}$$

The other lower bound and two upper bounds can be estimated similarly, which shows that  $\mathcal{F}$  satisfies (4.3) and (4.4). ■

Note that theorem 4.3 shows that there exists a class of  $C^1$  convexity preserving subdivision schemes. First,  $\mathcal{G}$  has to satisfy (2.6), (2.7), and (2.8) which has a solution; see, e.g., (2.9). Second, the choice of  $\beta$  in Example 3.2 satisfies (4.6).

The class of schemes is not unique, however, as we show in Section 5.

## 5. APPROXIMATION ORDER

In the previous sections, subdivision schemes in the class (2.13) satisfying convexity condition (3.2) have been constructed. The approximation properties of these schemes are examined in this section. Section 5.1 deals with a sufficient condition for approximation order of a certain degree. The approximation order of nonuniform convexity preserving subdivision schemes is investigated in Section 5.2. The conditions for approximation order four, discussed in Section 5.3, yield that convexity is preserved only if the data are equidistant. Nevertheless, the resulting scheme in this section leads to a relation between nonuniform convexity preserving subdivision schemes and rational interpolation, which is shown in Section 5.4.

### 5.1. General Properties

In this section, we examine the approximation properties of nonuniform convexity preserving subdivision schemes (2.13) with (3.2).

**DEFINITION 5.1.** Given a sufficiently smooth function  $g: I \rightarrow \mathbb{R}$ , where  $I \subset \mathbb{R}$  is a bounded interval. Data are drawn from  $g$  at  $\{(x_i^{(0)})\}$  ( $x_i^{(0)} < x_{i+1}^{(0)}, \forall i$ ), i.e.,

$$f_i^{(0)} = g(x_i^{(0)}).$$

Denote

$$h = \max_i |x_{i+1}^{(0)} - x_i^{(0)}|.$$

Then the approximation order is defined as the maximal  $p$  for which it holds that the limit function  $f_h^{(\infty)}$  of the subdivision scheme applied to the data  $\{(x_i^{(0)}, f_i^{(0)})\}$  satisfies

$$\|f_h^{(\infty)} - g\|_I \leq Ch^p, \quad C < \infty.$$

To be able to formulate explicit statements on the approximation order of convexity preserving subdivision schemes, we need the notion of stability.

**DEFINITION 5.2 (Stability of Subdivision Schemes).** A subdivision scheme is called *stable* if for perturbed data  $\tilde{f}_i^{(0)}$  to  $f_i^{(0)}$ :

$$|\tilde{f}_i^{(0)} - f_i^{(0)}| \leq \delta, \quad \forall i \Rightarrow \|\tilde{f}^{(\infty)} - f^{(\infty)}\|_\infty \leq C\delta, \quad C < \infty.$$

*Remark 5.3.* Linear convergent subdivision schemes are necessarily stable.

The next lemma provides a sufficient condition for approximation order of a certain degree  $p$ . The lemma applies to nonuniform convexity preserving subdivision as well as nonuniform linear subdivision.

**LEMMA 5.4 (Sufficient Conditions).** *Let subdivision scheme (2.13) be stable and let it reproduce polynomials of degree  $p-1$ , with  $p \geq 1$ . Then, the subdivision scheme has approximation order  $p$ .*

*Proof.* Without loss of generality, consider the interval  $I_i = [x_i^{(0)}, x_{i+1}^{(0)}]$ . It is necessary and sufficient for approximation order  $p$  that on each  $I_i$ ,  $i = 0, \dots, N-1$

$$\|f_h^{(\infty)} - g\|_{I_i, \infty} \leq Ch^p, \quad C < \infty,$$

with

$$h = \max_i |x_{i+1}^{(0)} - x_i^{(0)}|.$$

This can be achieved by defining  $\tilde{g}$  as the  $(p-1)$ st degree Taylor polynomial of  $g$  at  $x = x_{2i+1}^{(1)}$ , which obviously satisfies

$$\|g - \tilde{g}\|_{I_i, \infty} \leq C_1 h^p, \quad C_1 < \infty.$$

Now apply subdivision scheme (2.13) to the (perturbed) data  $\tilde{f}_i^{(0)}$  drawn from  $\tilde{g}$  at the same values  $x_i^{(0)}$ . As a direct result of the stability, (see Definition 5.2), the limit function  $\tilde{f}_h^{(\infty)}$  then satisfies

$$\|f_h^{(\infty)} - \tilde{f}_h^{(\infty)}\|_{I_i, \infty} \leq C_2 h^p, \quad C_2 < \infty,$$

and since the subdivision scheme reproduces polynomials of degree  $p$ , it also holds that

$$\|\tilde{f}_h^{(\infty)} - \tilde{g}\|_{I_i, \infty} = 0.$$

This yields

$$\begin{aligned} \|f_h^{(\infty)} - g\|_{I_i, \infty} &= \|f_h^{(\infty)} - \tilde{f}_h^{(\infty)} + \tilde{f}_h^{(\infty)} - \tilde{g} + \tilde{g} - g\|_{I_i, \infty} \\ &\leq \|f_h^{(\infty)} - \tilde{f}_h^{(\infty)}\|_{I_i, \infty} + \|\tilde{f}_h^{(\infty)} - \tilde{g}\|_{I_i, \infty} + \|\tilde{g} - g\|_{I_i, \infty} \\ &\leq C_2 h^p + 0 + C_1 h^p = Ch^p, \end{aligned}$$

which is valid for all  $i$ . ■

## 5.2. Approximation Order of Convexity Preserving Subdivision Schemes

First, we observe that any subdivision scheme of the form (2.13) which preserves convexity has approximation order two:

**THEOREM 5.5 (Approximation Order Two).** *Nonuniform convexity preserving subdivision schemes of class (2.13) which satisfy the conditions of Theorem 3.3 have at least approximation order two.*

*Proof.* Let the data  $f_i^{(0)}$  be drawn from a function  $g \in C^2([0, 1])$ , and let  $g$  be convex. Without loss of generality, consider the interval  $I_i = [x_i^{(0)}, x_{i+1}^{(0)}]$ . The upper envelope  $f_{U,i}^{(0)}$  is defined as the linear interpolant to the data points  $(x_i^{(0)}, f_i^{(0)})$  and  $(x_{i+1}^{(0)}, f_{i+1}^{(0)})$ , and the lower envelope  $f_{L,i}^{(0)}$  is defined by the maximum of the interpolating lines, one through  $(x_{i-1}^{(0)}, f_{i-1}^{(0)})$  and  $(x_i^{(0)}, f_i^{(0)})$ , and the second through  $(x_{i+1}^{(0)}, f_{i+1}^{(0)})$  and

$(x_{i+2}^{(0)}, f_{i+2}^{(0)})$ . It is easily checked that the intersection point  $x_{L,i}^{(0)}$  of the lower envelope lines is at

$$x_{L,i}^{(0)} = \frac{1}{2} (x_i^{(0)} + x_{i+1}^{(0)}) + \frac{1}{2} s_i^{(0)} \frac{d_{i+1}^{(0)} - d_i^{(0)}}{d_i^{(0)} + d_{i+1}^{(0)}}.$$

As the function  $g$  is  $C^2$  and convex,  $g$  is located between the upper envelope  $f_{U,i}^{(0)}$  and the lower envelope  $f_{L,i}^{(0)}$  of the initial data, and as the subdivision scheme preserves convexity, the same holds for all linear interpolants  $f_h^{(k)}$  and also for their limit function  $f_h^{(\infty)}$ . The distance between  $g$  and  $f_h^{(\infty)}$  is therefore bounded by the difference between  $f_{L,i}^{(0)}$  and  $f_{U,i}^{(0)}$  at  $x_{L,i}^{(0)}$ :

$$\|f_h^{(\infty)} - g\|_{L_i, \infty} \leq s_i^{(0)} \frac{1}{\frac{1}{d_i^{(0)}} + \frac{1}{d_{i+1}^{(0)}}} \leq \max_i s_i^{(0)} \cdot \max_i \max\{d_i^{(0)}, d_{i+1}^{(0)}\} = \mathcal{O}(h^2),$$

as both  $s_i^{(0)}$  and  $d_i^{(0)}$  are  $\mathcal{O}(h)$ . In fact, it has been used in the proof of Theorem 3.3 that  $\{f_{L,i}^{(0)}\}$  is a lower bound of the monotone decreasing sequence of piecewise linear functions  $f_h^{(k)}$ . ■

In order to prove that the approximation order of the convexity preserving schemes (2.13) with (4.5) equals three, we want to apply Lemma 5.4. As a consequence, we have to examine the stability properties of this class of convexity preserving subdivision schemes. The proof of the stability of nonuniform convexity preserving subdivision schemes is very involved, and is only briefly sketched here—see [KvD98b] for a detailed discussion. It uses the fact that the grid becomes locally uniform, and that the subdivision scheme converges to the uniform scheme.

The proof uses induction in the level of subdivision, and the following inequality is easily shown to be valid:

$$\begin{aligned} \|\tilde{f}^{(k+1)} - f^{(k+1)}\|_\infty &= \|\tilde{f}^{(k+1)} - \tilde{f}^{(k)} + \tilde{f}^{(k)} - f^{(k)} + f^{(k)} - f^{(k+1)}\|_\infty \\ &\leq \|\tilde{f}^{(k)} - f^{(k)}\|_\infty + \|\tilde{f}^{(k+1)} - \tilde{f}^{(k)} + f^{(k)} - f^{(k+1)}\|_\infty \\ &\leq \|\tilde{f}^{(k)} - f^{(k)}\|_\infty + \max_i s_i^{(0)} \\ &\quad \cdot \max_i |\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)}) - \mathcal{F}(\tilde{d}_i^{(k)}, \tilde{d}_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)})|. \end{aligned}$$

For the proof we make use of a Taylor series in the first two arguments of  $\mathcal{F}$  in (4.5). Moreover, we use the facts that the data at all levels are strictly convex and the ratios of second order differences are bounded (by  $q^*$ ), see the proof of Theorem 4.3. In this way, we can prove:

**PROPOSITION 5.6.** *The nonuniform convexity preserving subdivision scheme (2.13) with (4.5) is stable.*

As any convexity preserving scheme must necessarily reproduce linear functions, see Remark 2.3, the stability of subdivision scheme (2.13) with (4.5) is sufficient for an approximation order two. In the next theorem we show that this subdivision scheme can have even approximation order three for a suitable choice of  $\beta$ .

**THEOREM 5.7 (Approximation Order Three).** *Convexity preserving subdivision schemes in the class (2.13) with (4.5) have approximation order 3 if and only if  $\beta$  satisfies*

$$\beta(r) = \frac{2}{1+r}. \quad (5.1)$$

*Proof.* A scheme (2.13) with (4.5) has the property of stability, by Proposition 5.6. According to Lemma 5.4, reproduction of quadratic polynomials is sufficient for approximation order 3. A necessary and sufficient condition for quadratic reproduction is

$$\Delta^2 f_i^{(k)} = D, \quad \forall i \Rightarrow \Delta^2 f_i^{(k+1)} = D, \quad \forall i.$$

It can be easily shown that the subdivision scheme reproduces quadratic polynomials if and only if (5.1) holds. This proves the sufficient part.

Necessary conditions for approximation order 3 are obtained by comparing the nonuniform convexity preserving scheme (2.13) with (4.5) to the nonuniform linear scheme (2.15) with (2.16) applied to the initial data  $\{(x_i^{(0)}, f_i^{(0)})\}$ .

As the nonuniform linear four-point scheme (2.15) has approximation order four, it is necessary for the approximation order to be  $p$  ( $p \leq 4$ ) that

$$\max_i |s_i^{(k)}(\mathcal{F}(d_i^{(k)}, d_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)}) - \mathcal{F}_L(\bar{d}_i^{(k)}, \bar{d}_{i+1}^{(k)}, r_i^{(k)}, R_{i+1}^{(k)}))| \leq Ch^p,$$

$$C < \infty,$$

where the differences  $\bar{d}_i^{(k)}$  are defined by the nonuniform linear scheme. This condition must then also hold for the initial data which satisfy  $\bar{d}_i^{(0)} = d_i^{(0)}$ . As  $s_i^{(0)} \leq h$ , it is then necessary that  $\mathcal{F}$  satisfies

$$\max_i |\mathcal{F}(d_i^{(0)}, d_{i+1}^{(0)}, r_i^{(0)}, R_{i+1}^{(0)}) - \mathcal{F}_L(d_i^{(0)}, d_{i+1}^{(0)}, r_i^{(0)}, R_{i+1}^{(0)})| \leq Ch^{p-1},$$

$$C < \infty.$$

As we consider only initial data, the superscripts are omitted. An approximation order three is examined, so the function  $g$  is assumed to satisfy  $g \in C^3(I)$ , and it is therefore necessary for examination of the approximation order that we introduce differences  $\Delta^2 f_i$ ,  $\Delta^2 f_{i+1}$  and  $\Delta^3 f_i$ . For  $h \rightarrow 0$ , these differences satisfy  $\Delta^2 f_i^{(0)} \rightarrow g''(x_i^{(0)})$  and  $\Delta^3 f_i^{(0)} \rightarrow g'''(x_i^{(0)})$ , and are therefore bounded.

So we substitute the differences  $d_i$  according to (2.4) and

$$\begin{aligned} d_{i+1} &= \frac{1}{2} s_{i+1} (1 + r_{i+1}) \Delta^2 f_{i+1} = \frac{1}{2} s_i R_{i+1} (1 + r_{i+1}) \Delta^2 f_{i+1} \\ &= \frac{1}{2} s_i (1 + R_{i+1}) \Delta^2 f_{i+1}, \end{aligned}$$

and hence

$$\frac{d_i}{1 + r_i} = \frac{1}{2} s_i \Delta^2 f_i = \mathcal{O}(h) \quad \text{and} \quad \frac{d_{i+1}}{1 + R_{i+1}} = \frac{1}{2} s_i \Delta^2 f_{i+1} = \mathcal{O}(h).$$

Similarly third order differences can be introduced, and this finally yields after straightforward computations that

$$\begin{aligned} &|\mathcal{F}(d_i, d_{i+1}, r_i, R_{i+1}) - \mathcal{F}_L(d_i, d_{i+1}, r_i, R_{i+1})| \\ &\leq \frac{1}{C_0 \min g''(\xi) h} \cdot (\max_i |C_1(r_i, R_{i+1})| \cdot (\max_{\xi \in I} g''(\xi))^2 h^2 \\ &\quad + \max_i |C_2(r_i, R_{i+1})| \cdot \max_{\xi \in I} g''(\xi) \max_{\xi \in I} |g'''(\xi)| h^3) + \mathcal{O}(h^3), \end{aligned} \quad (5.2)$$

where  $C_0 > 0$ . The functions  $\mathcal{K}_L^1$  and  $\mathcal{K}_L^2$  have been substituted according to (2.16), and the functions  $C_j(r, R)$ ,  $j = 1, 2$ , can be easily computed and depend on  $r, R, \beta$  and  $\mathcal{G}$ ,

It is necessary for approximation order 3 that  $C_1(r, R) = 0$ ,  $\forall r, R$ , which finally results in

$$\frac{1 - \mathcal{G}(r, R)}{\beta(r)(1+r)} + \frac{1 + \mathcal{G}(r, R)}{\beta(R)(1+R)} = 1, \quad \forall r, R.$$

Taking  $R = 1$  yields that necessarily (5.1) must hold as it is required that the scheme has approximation order three for all choices of  $\mathcal{G}$ . ■

Summarising, Theorem 5.7 shows that any subdivision scheme (2.13) with  $\mathcal{F}$  given by

$$\mathcal{F}(x, y, r, R) = \frac{1}{2} \frac{1}{\frac{1+r}{(1+\mathcal{G}(r, R))x} + \frac{1+R}{(1-\mathcal{G}(r, R))y}}, \quad (5.3)$$



preserves convexity, generates continuously differentiable limit functions, and it has approximation order 3.

*Remark 5.8.* Reproduction of polynomials of degree  $p$  is *not necessary* for approximation order  $p+1$ . The reader is referred to e.g., [KvD98a], where a (uniform) rational convexity preserving subdivision scheme is constructed that does not reproduce *cubic* polynomials, but that has approximation order *four*.

### 5.3. Convexity Preservation and Approximation Order Four?

In this section, we further restrict the class of schemes to have approximation order 4. In addition to (5.1), it is necessary that  $C_2(r, R) = 0$ ,  $\forall r, R$  in (5.2). Hence, after straightforward algebra,  $\mathcal{G}$  must satisfy

$$\mathcal{G}(r, R) = \frac{r-R}{r+R}, \quad (5.4)$$

i.e., the subdivision scheme is uniquely determined.

The resulting subdivision scheme preserves convexity, but although this scheme for subdivision of the  $x$ -data is monotonicity preserving, the grid does not become locally uniform. Therefore, theorem 4.2 cannot be applied, and it is not clear whether the limit function is  $C^1$  or not. Numerical experiments however, show that the grid becomes dense and that the scheme has approximation order four and generates  $C^1$  limit functions.

So far, we constructed a subdivision scheme that preserves convexity and that satisfies necessary conditions for approximation order four, but we did not prove convergence properties. Now we examine a more general class of schemes than (4.5), namely

$$\mathcal{F}(x, y, r, R) = \frac{1}{4} \frac{1}{\frac{1}{\beta_1(r, R)(1+\mathcal{G}(r, R))x} + \frac{1}{\beta_2(r, R)(1-\mathcal{G}(r, R))y}}, \quad (5.5)$$

where by condition (2.14)  $\beta_2(r, R) = \beta_1(R, r)$ .

By going through the proof of Theorem 3.1, it is easily seen that

$$0 \leq \beta_j(r, R) \leq 1 \quad \text{and} \quad \beta_j(1, 1) = 1, \quad j = 1, 2, \quad (5.6)$$

are sufficient conditions for convexity preservation.

Again, by analogous arguments to those in Section 5, it can be shown that a necessary condition for approximation order 3 is

$$\frac{1+\mathcal{G}(r, R)}{\beta_2(r, R)(1+R)} + \frac{1-\mathcal{G}(r, R)}{\beta_1(r, R)(1+r)} = 1. \quad (5.7)$$

By considering the special case  $r = R$ , it is obtained that

$$\beta_1(r, r) = \beta_2(r, r) = \frac{2}{1+r},$$

i.e.,  $\beta_1$  and  $\beta_2$  do not satisfy the sufficient convexity condition (5.6), and it can also be derived that this scheme *does not* preserve convexity in general. We conclude that a subdivision scheme of class (2.13) with (5.5) and (5.7) that has at least approximation order three is not convexity preserving for all possible initial strictly convex data.

Still, it is an interesting question what kind of subdivision schemes are obtained if we demand that the schemes satisfy the necessary conditions for approximation order 4, even if they are not convexity preserving. The necessary condition for approximation order four yields that  $\beta_1$  is given by

$$\beta_1(r, R) = 2 \frac{(1 - \mathcal{G}(r, R))(1 + r + R)}{(1 + r)(1 - \mathcal{G}(r, R) + 2R)}, \quad (5.8)$$

and  $\beta_2(r, R) = \beta_1(R, r)$ . Numerical experiments show that the approximation order of this nonuniform rational subdivision scheme is indeed four. Next, we show a relation with rational interpolation.

#### 5.4. Connection With Fourth Order Rational Interpolation

The class of rational subdivision schemes, i.e., subdivision schemes of the form (2.13) with (5.5) and (5.8) have a connection with *rational interpolation*. As is pointed out in [FM98], the *uniform* convexity preserving subdivision scheme in [KvD98a] reproduces the following class of rational polynomials:

$$S(\xi) = \frac{a_0 + a_1\xi + a_2\xi^2}{1 + a_3\xi}. \quad (5.9)$$

In [Sch73], the rational function (5.9) is examined as a basis function in a class of rational splines. Although that class of interpolating splines is  $C^2$ -continuous, the equations in the spline coefficients are nonlinear and therefore difficult to solve.

For equidistant data, the function of the form (5.9) that interpolates the data  $\{(x_i, f_i)\}_{i=-1}^2$  with  $x_i = i$ , can be written as

$$S_i(\xi) = (1 - \xi) f_i + \xi f_{i+1} - \frac{1}{2} \frac{3\xi(1 - \xi)}{2 - \xi} \frac{\xi + 1}{d_i + d_{i+1}}, \quad \xi = \frac{x - x_i}{x_{i+1} - x_i}, \quad (5.10)$$

where  $d_i$  and  $d_{i+1}$  are second order differences, as defined in (2.3). If the data are convex, the spline  $S_i(\xi)$  is convex in the interval  $[x_i, x_{i+1}]$ , but the spline function  $\{S_i(\xi)\}_i$  is not globally convex for any convex data, however.

Evaluation of (5.10) in  $\xi = 1/2$  defines the uniform convexity preserving subdivision scheme that is examined in [KvD98a]:

$$f_{2i+1} = S_i(\xi = 1/2) = \frac{1}{2} (f_i + f_{i+1}) - \frac{1}{4} \frac{1}{\frac{1}{d_i} + \frac{1}{d_{i+1}}}.$$

This approach of making a rational fit and then evaluating it at the point  $\xi = 1/2$  cannot be extended to nonuniform data: it is easily checked that evaluating at, e.g.,  $x_{2i+1}^{(k+1)}$  given by (2.5) results in the scheme (5.5) with (5.8), which is the subdivision scheme that satisfies necessary conditions for approximation order four, but this scheme does not preserve convexity in general, as shown above.

It can be shown that this scheme preserves convexity if  $\beta$  satisfies (5.1) and  $\mathcal{G}$  satisfies (5.4), but then it is not clear whether the subdivision scheme is  $C^1$  or not, as is observed in Section 5.3.

## 6. MIDPOINT SUBDIVISION

In this section, we briefly examine midpoint convexity preserving subdivision schemes, i.e., the class of subdivision schemes (2.13) with (4.5) and  $\mathcal{G} = 0$ .

**THEOREM 6.1 (Convexity Preserving Midpoint Subdivision).** *The non-uniform convexity preserving subdivision scheme of the form (2.13) with (4.5),  $\mathcal{G}(r, R) = 0$ , and  $\beta(r) = 1$  generates continuously differentiable limit functions.*

*Proof.* Convexity is preserved, because (4.5) satisfies the conditions (3.1) and (3.2). As a result, the scheme converges to continuous limit functions. For  $C^1$ -convergence, without loss of generality consider the nonuniform grid

$$s_i^{(k)} = s_R 2^{-k}, \quad i \geq 0, \quad \text{and} \quad s_i^{(k)} = s_L 2^{-k}, \quad i < 0.$$

Then, we arrive at

$$f_1^{(k+1)} = \frac{1}{2} (f_0^{(k)} + f_1^{(k)}) - \frac{1}{4} s_R 2^{-k} \frac{1}{\frac{1}{\beta(r_0^{(k)}) d_0^{(k)}} + \frac{1}{d_1^{(k)}}},$$

and

$$f_{-1}^{(k+1)} = \frac{1}{2} (f_{-1}^{(k)} + f_0^{(k)}) - \frac{1}{4} s_L 2^{-k} \frac{1}{\frac{1}{d_{-1}^{(k)}} + \frac{1}{\beta(R_0^{(k)}) d_0^{(k)}}},$$

It is simply shown that the following properties hold if  $\beta(r) = 1$ ,

$$q^{(k+1)} \leq q^{(k)}, \quad \forall k \quad \text{and} \quad q_i^{(k)} \leq q^{(k)}, \quad \forall i,$$

where  $q_i^{(k)}$  are defined in (4.7), and as a result

$$\max_i d_i^{(k)} \leq \left( \frac{q^{(k)}}{1+q^{(k)}} \right)^k \max_i d_i^{(0)} \leq \left( \frac{q^{(0)}}{1+q^{(0)}} \right)^k \max_i d_i^{(0)},$$

i.e., the jumps in the differences are a Cauchy sequence in  $k$  with limit 0, which is sufficient for convergence to a  $C^1$  limit function by Lemma 4.1. ■

*Remark 6.2 (Approximation Order Three?).* The proof for  $C^1$ -convergence of Theorem 6.1 is simple using single-step estimates like  $q^{(k+1)} \leq q^{(k)}$ , because of the choice  $\beta(r) = 1$ . However, this choice of  $\beta$  yields that the subdivision scheme does not reproduce quadratic polynomials, as  $\beta$  does not satisfy condition (5.1) in Theorem 5.7. In addition, numerical experiments show that the scheme has only approximation order two in general.

Also in the case of midpoint subdivision, there is a relation with convexity preserving rational splines: In [DG85], a class of  $C^1$  rational splines is introduced that interpolate function values  $f_i$  and derivatives  $\theta_i$ . This spline is defined on the interval  $[x_i, x_{i+1}]$  as

$$S_i(\xi) = \frac{\left( (1-\xi)^3 f_i + \xi(1-\xi)^2 (w_i f_i + s_i \theta_i) + \xi^2(1-\xi)(w_i f_{i+1} - s_i \theta_{i+1}) + \xi^3 f_{i+1} \right)}{1 + (w_i - 3) \xi(1-\xi)},$$

where the local parameter  $\xi$  is defined as  $\xi = (x - x_i)/(x_{i+1} - x_i)$ , and  $w_i$  is a (local) tension parameter.

Define  $\hat{S}_i(\xi)$  as the linear interpolant to  $(x_i, f_i)$  and  $(x_{i+1}, f_{i+1})$ . Then,  $S_i(\xi)$  can be rewritten as  $S_i(\xi) = \hat{S}_i(\xi) - \tilde{S}_i(\xi)$ . The correction term  $\tilde{S}_i(\xi)$  satisfies

$$\tilde{S}_i(\xi) = \frac{s_i \xi(1-\xi)}{1 + (w_i - 3) \xi(1-\xi)} (\xi(\theta_i - g_i) + \xi(\theta_{i+1} - g_i) + (g_i - \theta_i)),$$

where the  $g_i$  are divided differences, defined in (2.2). This class of rational splines is  $C^1$  and interpolates function values  $f_i$  and derivatives  $\theta_i$ .

TABLE I  
Initial Highly Nonuniform Grid

$i$	-2	-1	0	1	2	3	4	5	6
$x_i^{(0)}$	-2	-1	0	1	$1\frac{1}{2}$	7	8	10	11

We choose the derivatives using the Butland-slopes [But80], and take the tension parameter (see [DG85]) to be stationary,

$$\theta_i = \frac{2g_{i-1}g_i}{g_{i-1} + g_i}, \quad w_i = 1 + \frac{m_i}{M_i} + \frac{M_i}{m_i},$$

where  $m_i$  and  $M_i$  are the minimum and the maximum of  $\{\theta_{i+1} - g_i, g_i - \theta_i\}$ , respectively. It is shown in [DG85] that the rational spline interpolant  $S_i(\xi)$  is strictly convex (without straight line segments) for strictly convex data, and that the correction  $\tilde{S}_i(\xi)$  can be written as

$$\tilde{S}_i(\xi) = \frac{s_i \xi(1-\xi)}{\frac{2\xi-1}{g_i} + 2\left(\frac{1-\xi}{d_i} + \frac{\xi}{d_{i+1}}\right)}.$$

Since any other choice than  $\xi = 1/2$  yields that the resulting subdivision scheme is not in the class (2.13), we evaluate  $\tilde{S}_i$  at  $\xi = 1/2$ . This is called midpoint subdivision: the two-point scheme (2.5) with  $\mathcal{G}(r, R) = 0$ . With this choice

$$\tilde{S}_i(1/2) = s_i \mathcal{F}_{GD}(d_i, d_{i+1}, r_i, R_{i+1}) \quad \text{with}$$

$$\mathcal{F}_{GD}(d_i, d_{i+1}, r_i, R_{i+1}) = \frac{1}{4} \frac{1}{\frac{1}{d_i} + \frac{1}{d_{i+1}}}. \quad (6.1)$$

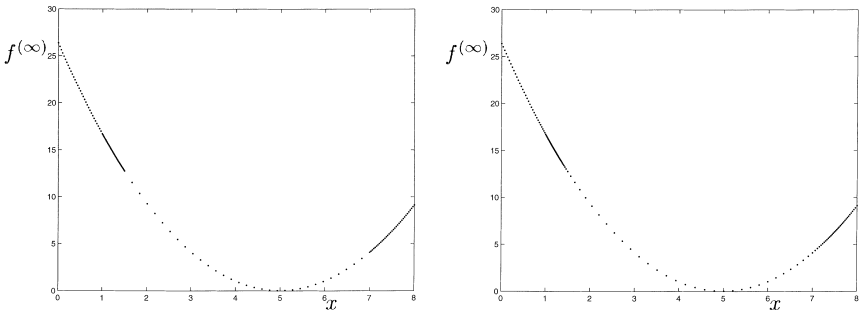


FIG. 1. Nonuniform convex subdivision. Midpoint subdivision on the left and locally uniform subdivision on the right, both for the data from Table I.

TABLE II  
Initial Convex Data Set

$x_i^{(0)}$	-2	-1	0	1	4	5	7	9	11
$f_i^{(0)}$	8	$5\frac{1}{2}$	4	3	$1/4$	1	3	$5\frac{1}{2}$	8

Although the subdivision scheme (6.1) has only approximation order two, it preserves convexity and it generates continuously differentiable limit functions according to Theorem 6.1.

## 7. NUMERICAL EXAMPLES

In this section, nonuniform subdivision schemes are graphically illustrated. We show application of the convexity preserving subdivision scheme (2.13) with  $\mathcal{F}$  given by (5.3). The first example deals with subdivision of data drawn from the function  $x^2 + x^4/1000$  where the parameter values  $x_i^{(0)}$  are given in Table I.

In Fig. 1, we respectively take  $\mathcal{G} = 0$ , and  $\mathcal{G}$  as in (2.9) where  $\ell_1 = 2$ ,  $\ell_2 = 1$  and  $\ell_3 = 2$ , and the limit function is plotted on the interval  $[0, 8]$ .

The second example deals with the data defined in Table II.

The function  $\mathcal{G}$  is taken to be as in (2.9), with  $\ell_1 = 2$ ,  $\ell_2 = 1$  and  $\ell_3 = 2$ , and  $\beta(r)$  as in (5.1). The limit function and its derivative are displayed on the interval  $[0, 7]$  in Fig. 2. It is clearly seen for this example that the derivative is continuous, i.e., the limit function is  $C^1$ , as has been proved in Section 4.

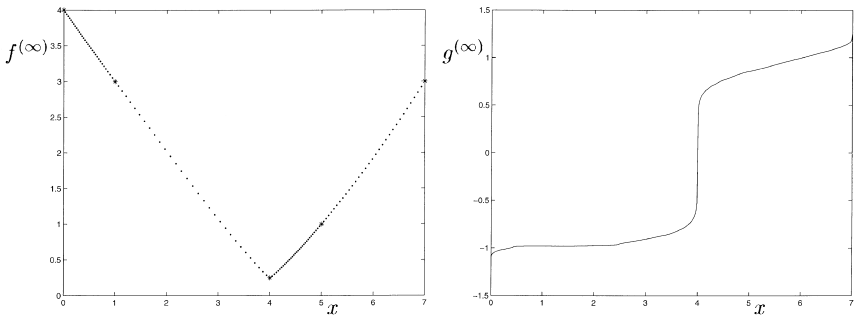


FIG. 2. The limit function  $f^{(\infty)}$  and its derivative  $g^{(\infty)}$  obtained by the nonuniform convexity preserving subdivision scheme (see text) for the data from Table I.

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