

# LIMITING CONDITIONAL DISTRIBUTIONS FOR BIRTH-DEATH PROCESSES

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## Abstract

In a recent paper [16], one of us identified all of the quasi-stationary distributions for a non-explosive, evanescent birth-death process for which absorption is certain, and established conditions for the existence of the corresponding limiting conditional distributions. Our purpose is to extend these results in a number of directions. We shall consider separately two cases depending on whether or not the process is evanescent. In the former case we shall relax the condition that absorption is certain. Furthermore, we shall allow for the possibility that the minimal process might be explosive, so that the transition rates alone will not necessarily determine the birth-death process uniquely. Although we shall be concerned mainly with the minimal process, our most general results hold for any birth-death process whose transition probabilities satisfy both the backward and the forward Kolmogorov differential equations.

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## 1 Introduction

Let  $X = \{X(t), t \geq 0\}$  be a birth-death process taking values in  $\mathcal{N} = \{0, 1, \dots\}$  with birth rates  $\{\lambda_n, n \geq 0\}$  and death rates  $\{\mu_n, n \geq 0\}$ , all strictly positive except  $\mu_0$ , which might be equal to 0. When  $\mu_0 = 0$  the process is irreducible, but when  $\mu_0 > 0$  the process may evanesce by escaping from  $\mathcal{N}$ , via 0, to an absorbing state  $-1$ . We shall allow for the possibility that  $X$  might also escape from  $\mathcal{N}$  by performing infinitely many jumps in a finite time.

In the evanescent case ( $\mu_0 > 0$ ) we shall be concerned with the conditional probabilities

$$(1.1) \quad r_{ij}(t) = \Pr[X(t) = j | X(0) = i, X(t) \in \mathcal{N}], \quad i, j \in \mathcal{N}, \quad t \geq 0.$$

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In particular, we shall derive conditions under which  $\{r_{ij}(t)\}$  converges as  $t \rightarrow \infty$  to a proper *limiting conditional distribution*. Such a distribution is useful in modelling the long-term behaviour of the process before absorption occurs. In order to deal specifically with cases where escape from  $\mathcal{N}$  is not certain, typically, when there is drift away from the absorbing state, we shall also study conditional probabilities of the form

$$(1.2) \quad \bar{r}_{ij}(t) = \Pr[X(t) = j | X(0) = i, X(t) \in \mathcal{N}, X(t+s) = -1 \text{ for some } s > 0],$$

$$i, j \in \mathcal{N}, \quad t \geq 0.$$

Thus, we shall extend the results of van Doorn [16], who studied processes for which absorption at  $-1$  is certain. Our results cannot be deduced from general theory (see, for example, Theorems 1 and 2 of Flaspohler [5] and Theorem 5.2 of Pollett [12]), which usually rests on  $X$  being  $\lambda$ -positive recurrent.

In the irreducible case ( $\mu_0 = 0$ ) we shall be concerned with limits of conditional probabilities of the form

$$(1.3) \quad \bar{r}_{ij}(t) = \Pr[X(t) = j | X(0) = i, X(t+s) = 0 \text{ for some } s > 0],$$

$$i, j \in \mathcal{N}, \quad t \geq 0.$$

When  $X$  is positive recurrent these limits constitute the stationary distribution of the process. On the other hand, when  $X$  is transient and the limits constitute a proper distribution they may be useful in modelling quasi-stationary behaviour of the process before the last exit from state 0, that is, before the drift to infinity has set in.

## 2 Notation and preliminaries

Let  $Q = \{q_{ij}, i, j \in S\}$ , where  $S = \{-1\} \cup \mathcal{N}$ , be the  $q$ -matrix of transition rates, so that  $q_{i,i+1} = \lambda_i$ ,  $q_{i,i-1} = \mu_i$  and  $q_{i,i} = -(\lambda_i + \mu_i)$ , for all  $i \in \mathcal{N}$ , and  $q_{ij} = 0$ , otherwise. Note that  $Q$  will be conservative over  $\mathcal{N}$  when and only when  $\mu_0 = 0$ . But, since  $Q$  is conservative over  $S$ , the transition function  $P(t) = \{p_{ij}(t), i, j \in S\}$ , where

$$p_{ij}(t) = \Pr[X(t) = j | X(0) = i], \quad i, j \in S, \quad t \geq 0,$$

always satisfies the backward differential equations,

$$(2.1) \quad p'_{ij}(t) = \sum_{k \in S} q_{ik} p_{kj}(t), \quad i, j \in S, \quad t \geq 0,$$

but might not satisfy the forward differential equations,

$$(2.2) \quad p'_{ij}(t) = \sum_{k \in S} p_{ik}(t) q_{kj}, \quad i, j \in S, \quad t \geq 0.$$

We shall *assume* that  $P$  satisfies both the backward *and* the forward equations.

We will use Anderson's notation [1]: define  $\pi = \{\pi_n, n \geq 0\}$  by  $\pi_0 = 1$  and, for  $n \geq 1$ ,

$$(2.3) \quad \pi_n = \pi_{n-1} \frac{\lambda_{n-1}}{\mu_n} = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n},$$

and let

$$(2.4) \quad A = \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n}, \quad B = \sum_{n=0}^{\infty} \pi_n,$$

$$(2.5) \quad C = \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=0}^n \pi_i, \quad D = \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=n+1}^{\infty} \pi_i.$$

The series  $C$  and  $D$  determine the behaviour of the process near the boundary point at infinity, for they represent the expected passage times of the process, from 0 to  $\infty$ , and from  $\infty$  to 0, respectively, see, e.g., Pages 263-264 of [1]. The process is non-explosive ( $Q$  is regular) if and only if  $C = \infty$ , in which case the minimal solution,  $F(t) = \{f_{ij}(t), i, j \in S\}$ , to (2.1) is the unique solution, and hence there is a unique birth-death process with transition rates  $Q$ . If  $C < \infty$  and  $D = \infty$ , then  $F$  satisfies (2.2) uniquely, but is dishonest. In these cases our given  $P$  must be the minimal transition function. In the remaining case, when  $C < \infty$  and  $D < \infty$ , there are infinitely many solutions to (2.1) and (2.2), exactly one of which is honest, and, while general theory dictates that the given  $P$  must satisfy the backward equations, it is not possible to determine from the transition rates alone whether  $P$  satisfies the forward equations; it depends on the particular rule (entrance law) for "restarting" the process after an explosion (see Lemma 4.2.1 of [1]). The behaviour of the process near the boundary is summarized in the table on Page 262 of [1].

The series  $A$  and  $B$  have no immediate physical interpretation, but they are related to  $C$  and  $D$  as follows:

$$(2.6) \quad C + D = AB.$$

So, for example,  $A + B = \infty$  if and only if at least one of  $C$  and  $D$  diverges. Note also that  $B = \infty$  implies  $D = \infty$ , while  $A = \infty$  implies  $C = \infty$ . In the evanescent case ( $\mu_0 > 0$ ), the minimal process is eventually absorbed at  $-1$  with probability 1 if and only if  $A = \infty$ , while in the irreducible case ( $\mu_0 = 0$ ), the minimal process is recurrent if and only if  $A = \infty$ , and then positive recurrent if and only if  $B < \infty$ , in which case  $B$  normalizes  $\pi$  to produce the unique stationary distribution.

In the analysis of birth-death processes, a prominent role is played by a sequence of polynomials,  $\{Q_n, n \geq 0\}$ , called *birth-death polynomials*. They are determined uniquely by the recurrence relation

$$(2.7) \quad -xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x), \quad n \geq 0,$$

together with  $Q_{-1}(x) = 0$  and  $Q_0(x) = 1$ . Since the transition function,  $P$ , satisfies both the backward and the forward equations, it can be represented as

$$(2.8) \quad p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) d\psi(x), \quad i, j \in \mathcal{N}, \quad t \geq 0,$$

where  $\psi$  is a positive Borel measure with total mass 1 and with infinite support on the non-negative real axis (Karlin and McGregor [6]);  $\psi$  is called the *spectral measure of  $P$* . The polynomials  $\{Q_n, n \geq 0\}$  are orthogonal with respect to  $\psi$ , since taking  $t = 0$  in (2.8) yields

$$\pi_j \int_0^\infty Q_i(x) Q_j(x) d\psi(x) = \delta_{ij}, \quad i, j \in \mathcal{N},$$

where  $\delta_{ij}$  denotes the Kronecker delta. It is well known that  $Q_n$  has  $n$  positive, simple zeros,  $x_{ni}$  ( $i = 1, \dots, n$ ), which satisfy the “interlacing” property

$$(2.9) \quad 0 < x_{n+1,i} < x_{ni} < x_{n+1,i+1}, \quad i = 1, \dots, n, \quad n \geq 1,$$

from which it follows that the limits

$$(2.10) \quad \xi_i = \lim_{n \rightarrow \infty} x_{ni}, \quad i \geq 1,$$

exist and satisfy  $0 \leq \xi_i \leq \xi_{i+1} < \infty$ .

### 3 The evanescent case

We shall first assume that  $\mu_0 > 0$ . This guarantees that our process is genuinely evanescent since, because  $P$  satisfies the backward equations,  $p_{i0}(t) > 0$  for all  $i \in \mathcal{N}$  and  $t > 0$ , and hence there is a positive probability of escape from  $\mathcal{N}$  to  $-1$ .

#### 3.1 Preliminaries

Following Karlin and McGregor [6, 7], we define the *dual process* to be a birth-death process on  $\mathcal{N}$  with birth rates,  $\{\lambda_n^d, n \geq 0\}$ , and death rates,  $\{\mu_n^d, n \geq 0\}$ , given by

$$\lambda_n^d = \mu_n, \quad \mu_0^d = 0, \quad \mu_{n+1}^d = \lambda_n, \quad n \geq 0.$$

Accordingly, we define  $\pi_0^d = 1$  and, for  $n \geq 1$ ,

$$(3.1) \quad \pi_n^d = \pi_{n-1}^d \frac{\lambda_{n-1}^d}{\mu_n^d} = \frac{\mu_0}{\lambda_{n-1}\pi_{n-1}} = \frac{\mu_0}{\mu_n\pi_n}.$$

Denoting by  $\{Q_n^d, n \geq 0\}$  the corresponding birth-death polynomials, we then have

$$(3.2) \quad Q_0^d(x) = 1, \quad Q_{n+1}^d(x) = 1 - \frac{x}{\mu_0} \sum_{k=0}^n \pi_k Q_k(x), \quad n \geq 0,$$

and

$$(3.3) \quad Q_n(x) = 1 + \sum_{k=0}^{n-1} Q_{k+1}^d(x) \frac{\mu_0}{\lambda_k \pi_k} = \sum_{k=0}^n \pi_k^d Q_k^d(x), \quad n \geq 0,$$

and so

$$(3.4) \quad Q_{n+1}^d(x) = \frac{\lambda_n \pi_n}{\mu_0} \{Q_{n+1}(x) - Q_n(x)\}, \quad n \geq 0;$$

see van Doorn [13] (cf. (2.15) of Karlin and McGregor [6]). Here, and henceforth, the empty sum should be interpreted as zero. Note that (3.2) and (3.3) yield

$$(3.5) \quad Q_n(0) = \sum_{k=0}^n \pi_k^d, \quad Q_n^d(0) = 1, \quad n \geq 0.$$

Since  $\{Q_n^d\}$  constitutes a sequence of birth-death polynomials,  $Q_n^d$  has  $n$  positive, simple zeros,  $x_{ni}^d$  ( $i = 1, \dots, n$ ), which satisfy the interlacing property (2.9). Moreover, Theorem I.7.2 in Chihara [2] may be used to show that

$$0 < x_{ni}^d < x_{ni} < x_{n,i+1}^d < x_{n,i+1}, \quad i = 1, \dots, n-1,$$

and so, denoting  $\xi_i^d = \lim_{n \rightarrow \infty} x_{ni}^d$ , we have that

$$(3.6) \quad 0 \leq \xi_i^d \leq \xi_i \leq \xi_{i+1}^d < \infty, \quad i \geq 1.$$

Recall that under our assumption that  $P$  satisfies both the backward *and* the forward equations, the birth-death process,  $X$ , is uniquely determined by its rates, and is hence the minimal process, if and only if  $A + B = \infty$ , that is, at least one of  $C$  and  $D$  in (2.5) diverges. Correspondingly, the spectral measure  $\psi$  in (2.8) is uniquely determined. In this case, we also have that  $\xi_1 = \gamma$ , where  $\gamma = \gamma(\psi)$  is the infimum of the support of  $\psi$ . If the series  $C$  diverges (and hence  $A + B = \infty$ ), then  $X$  is non-explosive and absorption at  $-1$  occurs with probability 1 if and only if  $A = \infty$ . If the series  $C$  converges while the series  $D$  diverges, so that again  $A + B = \infty$ , the process is explosive and is absorbed either at infinity or at  $-1$ .

If  $A + B < \infty$ , that is, both  $C$  and  $D$  converge, then  $X$  is not determined uniquely by its rates; we say that the rate problem associated with  $X$  is *indeterminate* (see van Doorn [15]). But, remarkably, the rates determine a one-parameter family of spectral measures,  $\psi$ , and, through (2.8), a one-parameter family of transition functions, indexed by  $\gamma(\psi)$  in the range  $[\xi_1^d, \xi_1]$ . If  $\gamma(\psi) = \xi_1$ , then we obtain the *minimal process*; the boundary at infinity is *completely absorbing*. If  $\gamma(\psi) = \xi_1^d$ , then the corresponding process is called the *maximal process* and the boundary is *completely reflecting*; it is then the unique *honest* process. When  $\xi_1^d < \gamma(\psi) < \xi_1$ , the boundary is said to be *mixed*.

Let  $T$  denote the (possibly defective) random variable representing the time at which absorption at  $-1$  occurs. Since, by the forward equations (2.2),

$$\Pr[T \leq t | X(0) = i] = p_{i,-1}(t) = \mu_0 \int_0^t p_{i0}(u) du,$$

we have

$$\Pr[t < T < \infty | X(0) = i] = \mu_0 \int_t^\infty p_{i0}(u) du,$$

so that, from (2.8),

$$(3.7) \quad \Pr[t < T < \infty | X(0) = i] = \mu_0 \int_0^\infty \frac{e^{-xt}}{x} Q_i(x) d\psi(x), \quad i \in \mathcal{N}.$$

In particular, setting  $a_i = \Pr[T < \infty | X(0) = i]$ , we have

$$(3.8) \quad a_i = \mu_0 \int_0^\infty \frac{Q_i(x)}{x} d\psi(x), \quad i \in \mathcal{N},$$

which, together with the recurrence relation (2.7), leads to

$$(3.9) \quad a_i = a_0 - \mu_0(1 - a_0) \sum_{n=0}^{i-1} \frac{1}{\lambda_n \pi_n} = a_0 \sum_{n=0}^i \pi_n^d - \sum_{n=1}^i \pi_n^d, \quad i \in \mathcal{N}.$$

When  $X$  is the minimal process, (3.5) and Lemma 6 on Page 527 of Karlin and McGregor [6] give us

$$(3.10) \quad a_0 = 1 - \lim_{n \rightarrow \infty} \frac{1}{Q_n(0)} = \frac{\sum_{n=1}^{\infty} \pi_n^d}{\sum_{n=0}^{\infty} \pi_n^d},$$

whether or not  $X$  is uniquely determined by its rates. This, together with (3.9), yields

$$(3.11) \quad a_i = \frac{\sum_{n=i+1}^{\infty} \pi_n^d}{\sum_{n=0}^{\infty} \pi_n^d} = \frac{\mu_0 \sum_{n=i}^{\infty} \frac{1}{\lambda_n \pi_n}}{1 + \mu_0 \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n}}, \quad i \in \mathcal{N},$$

with the interpretation that  $a_i = 1$ , for all  $i \in \mathcal{N}$ , whenever  $\sum_{n=0}^{\infty} \pi_n^d = \infty$  (equivalently, in this case,  $A = \infty$ ). The result (3.11) is given in Theorem 10 of Karlin and McGregor [7] under slightly stronger conditions.

We conclude this section with two representations for  $\sum_{k=0}^{\infty} p_{ik}(t)$  which will be used later. Firstly, if  $X$  is non-explosive, or, otherwise, if  $X$  is the maximal process, then, by the honesty of  $P$ , we have that

$$(3.12) \quad \sum_{k=0}^{\infty} p_{ik}(t) = 1 - p_{i,-1}(t), \quad i \in \mathcal{N}, \quad t \geq 0.$$

A representation is given in the next lemma under the condition that the rate problem associated with  $X$  is indeterminate. We shall use the notation  $Q_{\infty}^d(x) = \lim_{n \rightarrow \infty} Q_n^d(x)$  which, as we shall see, defines an entire function,  $Q_{\infty}^d$ , under the condition at hand.

**Lemma 3.1** Let  $P$  be the transition function of an evanescent birth-death process on  $\mathcal{N}$  satisfying  $A + B < \infty$ , where  $A$  and  $B$  are given in (2.4), and let  $\psi$  be its spectral measure.

Then,

$$(3.13) \quad \sum_{k=0}^{\infty} p_{ik}(t) = \mu_0 \int_0^{\infty} e^{-xt} Q_i(x) \left( \frac{1 - Q_{\infty}^d(x)}{x} \right) d\psi(x), \quad i \in \mathcal{N}, \quad t \geq 0.$$

**Proof:** By (2.8) and (3.2), we have

$$\begin{aligned} \sum_{k=0}^n p_{ik}(t) &= \int_0^{\infty} e^{-xt} Q_i(x) \sum_{k=0}^n \pi_k Q_k(x) d\psi(x) \\ &= \mu_0 \int_0^{\infty} e^{-xt} Q_i(x) \left( \frac{1 - Q_{n+1}^d(x)}{x} \right) d\psi(x). \end{aligned}$$

The result follows on letting  $n \rightarrow \infty$  and interchanging the limit and integral, which can be justified by dominated convergence as in Pages 535-536 of Karlin and McGregor [6].  $\square$

If  $X$  is explosive but uniquely determined by its rates (equivalently,  $C < \infty$  and  $D = \infty$ ), then  $Q_n^d$  does not converge to an entire function as  $n \rightarrow \infty$  and, correspondingly, a convenient representation such as (3.13) does not seem to exist.

### 3.2 Asymptotics for birth-death polynomials

By making the identifications  $\gamma_{2n+2} = \mu_n, \gamma_{2n+3} = \lambda_n$  (and, correspondingly,  $R_n^*(x) = Q_n(x), R_n(x) = Q_n^d(x)$ , etc.) we obtain from Kijima and van Doorn [9] a number of asymptotic results which are collected in this section and will be used later in evaluating limiting conditional distributions. We discern four cases determined by the convergence or divergence of the series  $C$  and  $D$  in (2.5). The quantities  $a_i = Pr[T < \infty | X(0) = i]$  correspond to the minimal process and, hence, satisfy (3.11).

*Case 1:  $C + D < \infty$ .* Both  $Q_n$  and  $Q_n^d$  converge uniformly on bounded sets to entire functions,  $Q_\infty$  and  $Q_\infty^d$ , respectively, whose zeros are simple (multiplicity one) and are precisely the points  $\xi_i$  ( $i \geq 1$ ) and  $\xi_i^d$  ( $i \geq 1$ ), respectively; moreover,

$$0 < \xi_i^d < \xi_i < \xi_{i+1}^d, \quad i \geq 1.$$

Hence, (3.2) leads to

$$(3.14) \quad \frac{x}{\mu_0} \sum_{n=0}^{\infty} \pi_n Q_n(x) = 1 - Q_\infty^d(x),$$

while substitution of (3.2) into (3.3) and subsequent use of (3.11) gives us

$$(3.15) \quad \frac{x}{\mu_0} \sum_{n=0}^{\infty} a_n \pi_n Q_n(x) = 1 - \frac{Q_\infty(x)}{\sum_{n=0}^{\infty} \pi_n^d}.$$

Recall that  $C + D < \infty$  if and only if  $A + B < \infty$ , and note that  $A < \infty$  if and only if  $\sum_{n=0}^{\infty} \pi_n^d < \infty$ .

*Case 2:  $C = \infty$  and  $D < \infty$ .* As in Case 1,  $Q_n^d$  converges to an entire function,  $Q_\infty^d$ , but now

$$0 < \xi_i^d = \xi_i < \xi_{i+1}^d, \quad i \geq 1.$$

Hence, from (3.2), we arrive at (3.14) again. Note that  $a_i = 1$  for all  $i \in \mathcal{N}$ , since, in this case,  $A = \infty$  and  $B < \infty$ .

*Case 3:  $C < \infty$  and  $D = \infty$ .* In this case,  $A < \infty$  and  $B = \infty$ , so that  $a_i < 1$  for all  $i \in \mathcal{N}$ . As in Case 1,  $Q_n$  converges to an entire function,  $Q_\infty$ , but now

$$0 = \xi_1^d < \xi_i = \xi_{i+1}^d < \xi_{i+1}, \quad i \geq 1.$$



Also, (3.15) holds true again. On the other hand, we have that

$$(3.16) \quad \sum_{n=0}^{\infty} \pi_n Q_n(x) = \infty, \quad 0 < x < \xi_1,$$

but it is not known whether this series diverges at the point  $x = \xi_1$ .

*Case 4:*  $C = \infty$  and  $D = \infty$ . We readily see from (3.5) that

$$(3.17) \quad \sum_{n=0}^{\infty} \pi_n Q_n(0) = \infty$$

since  $D = \infty$ , and from (3.11) that

$$(3.18) \quad \sum_{n=0}^{\infty} a_n \pi_n Q_n(0) = \infty \quad \text{if } A < \infty,$$

since  $C = \infty$ , which, together, gives us sufficient information when  $\xi_1 = 0$ . It is shown in [9] that in the opposite case  $\xi_1 > 0$  there are only two possibilities:

*Case 4.1:*  $A = \infty$ ,  $B < \infty$  and  $\xi_1 > 0$ . In this case, we have  $a_i = 1$  for all  $i \in \mathcal{N}$  and

$$0 < \xi_i^d = \xi_i \leq \xi_{i+1}^d, \quad i \geq 1,$$

while

$$(3.19) \quad \sum_{n=0}^{\infty} \pi_n Q_n(x) = \frac{\mu_0}{x}, \quad 0 < x \leq \xi_1^d = \xi_1.$$

*Case 4.2:*  $A < \infty$ ,  $B = \infty$  and  $\xi_1 > 0$ . In this case,  $a_i < 1$  for all  $i \in \mathcal{N}$  and

$$0 = \xi_1^d < \xi_i = \xi_{i+1}^d \leq \xi_{i+1}, \quad i \geq 1.$$

Moreover, we have that

$$(3.20) \quad \sum_{n=0}^{\infty} \pi_n Q_n(x) = \infty, \quad 0 < x \leq \xi_1,$$

but

$$(3.21) \quad \sum_{n=0}^{\infty} a_n \pi_n Q_n(x) = \frac{\mu_0}{x}, \quad 0 < x \leq \xi_1.$$

### 3.3 Limiting conditional distributions

Obviously, the conditional probabilities  $r_{ij}(t)$  of (1.1) can be written as

$$(3.22) \quad r_{ij}(t) = \frac{p_{ij}(t)}{\sum_{k=0}^{\infty} p_{ik}(t)}, \quad t \geq 0.$$

We shall begin by showing that when  $X$  is non-explosive their limits as  $t \rightarrow \infty$  can be evaluated for all  $i, j \in \mathcal{N}$  as

$$(3.23) \quad \lim_{t \rightarrow \infty} r_{ij}(t) = \frac{\pi_j Q_j(\gamma)}{\sum_{k=0}^{\infty} \pi_k Q_k(\gamma)}, \quad \gamma = \gamma(\psi),$$

where  $\psi$  is the spectral measure of  $X$ , with the interpretation that the limit is 0 whenever the sum  $\sum_{k=0}^{\infty} \pi_k Q_k(\gamma)$  diverges.

So, suppose that  $C = \infty$ . By (3.12) and (3.22) we may write

$$r_{ij}(t) = \frac{p_{ij}(t)}{1 - p_{i,-1}(t)}, \quad i, j \in \mathcal{N}.$$

Hence, when  $A = \infty$ , we can use the results of van Doorn [16], to conclude that

$$(3.24) \quad \lim_{t \rightarrow \infty} r_{ij}(t) = \frac{\xi_1 \pi_j Q_j(\xi_1)}{\mu_0} > 0,$$

if  $\xi_1 > 0$ , while  $r_{ij}(t) \rightarrow 0$  otherwise. Moreover, when  $A < \infty$ , we evidently have  $r_{ij}(t) \rightarrow 0$ , since absorption at  $-1$  is not certain. Since  $\gamma(\psi) = \xi_1$ , we have arrived at (3.23) by virtue of the appropriate results of Section 3.2: (3.14) in Case 2, (3.19) in Case 4.1, (3.20) in Case 4.2 and (3.17) in Case 4 when  $\xi_1 = 0$ .

The proof of (3.24) in van Doorn [16] depends crucially on the existence of a convenient integral representation for  $\sum_{k=0}^{\infty} p_{ik}(t)$ , which, under the conditions at hand, equals  $1 - p_{i,-1}(t)$ . By Lemma 3.1, however, such an integral representation also exists when the rate problem associated with  $X$  is indeterminate. In this case we can use arguments similar to those of van Doorn [16], to conclude that

$$(3.25) \quad \lim_{t \rightarrow \infty} r_{ij}(t) = \frac{\gamma \pi_j Q_j(\gamma)}{\mu_0 (1 - Q_{\infty}^d(\gamma))}, \quad \gamma = \gamma(\psi).$$

Thus, in view of (3.14) in Case 1, (3.23) is valid. We have proved the following result.

**Theorem 3.3** Let  $X$  be an evanescent birth-death process on  $\mathcal{N}$  and let  $\psi$  be its spectral measure. If  $C = \infty$  or if  $C + D < \infty$  (equivalently,  $A + B < \infty$ ), then (3.23) holds for all  $i, j \in \mathcal{N}$ , with the interpretation that the limit is zero whenever the sum in the denominator diverges, that is, whenever  $\gamma(\psi) = 0$  or  $B = \infty$ .

The problem of determining  $\lim_{t \rightarrow \infty} r_{ij}(t)$  when  $C < \infty$  and  $D = \infty$ , that is, when  $X$  is explosive, but uniquely determined by the birth and death rates, remains unsolved. We conjecture that (3.23) is valid under any circumstance.

When  $A < \infty$ , so that absorption at  $-1$  of the minimal process is not certain, then, as we have seen, the limits of the conditional probabilities  $r_{ij}(t)$  can be identically zero. However, by considering the conditional probabilities  $\bar{r}_{ij}(t)$ , given in (1.2), we shall now obtain a non-trivial limiting conditional distribution. Using (2.8), (3.7), and the Markov property, these probabilities can be evaluated as follows. Suppose that  $X$  is the minimal process. Then,

$$\begin{aligned} \bar{r}_{ij}(t) &= \Pr[X(t) = j | t < T < \infty, X(0) = i] \\ &= \frac{\Pr[X(t) = j, t < T < \infty | X(0) = i]}{\Pr[t < T < \infty | X(0) = i]} \\ &= \frac{a_j p_{ij}(t)}{\Pr[t < T < \infty | X(0) = i]} \\ &= \frac{a_j \pi_j}{\mu_0} \frac{\int_0^\infty e^{-xt} Q_i(x) Q_j(x) d\psi(x)}{\int_0^\infty e^{-xt} x^{-1} Q_i(x) d\psi(x)}, \end{aligned}$$

where  $a_i = \Pr[T < \infty | X(0) = i]$  is given by (3.10) and (3.11). Karlin and McGregor [7] show in the proof of their Theorem 11 that for any continuous function  $f$

$$(3.26) \quad \lim_{t \rightarrow \infty} \frac{\int_0^\infty e^{-xt} f(x) d\psi(x)}{\int_0^\infty e^{-xt} d\psi(x)} = f(\gamma), \quad \gamma = \gamma(\psi).$$

On the basis of this result it is easy to see that for all  $i, j \in \mathcal{N}$ ,

$$(3.27) \quad \lim_{t \rightarrow \infty} \bar{r}_{ij}(t) = \frac{\gamma a_j \pi_j Q_j(\gamma)}{\mu_0} > 0,$$

if  $\gamma = \gamma(\psi) > 0$ , while  $\bar{r}_{ij}(t) \rightarrow 0$  otherwise. Since  $\gamma(\psi) = \xi_1$  and in view of the asymptotic results (3.15) in Cases 1 and 3, (3.21) in Case 4.2, and (3.18) in Case 4 when  $\xi_1 = 0$ , as well as Theorem 3.3, we have deduced that, for all  $i, j \in \mathcal{N}$ , and under any circumstance

$$(3.28) \quad \lim_{t \rightarrow \infty} \bar{r}_{ij}(t) = \frac{a_j \pi_j Q_j(\gamma)}{\sum_{k=0}^\infty a_k \pi_k Q_k(\gamma)}, \quad \gamma = \gamma(\psi).$$

The only point in the analysis leading to (3.27) where  $X$  is required to be minimal process is in the derivation of (3.10). When the rate problem associated with  $X$  is indeterminate we can use (3.8), (3.9) and the results on Pages 529–530 of Karlin and McGregor [6], to show that

$$(3.29) \quad a_j = \frac{Q_\infty^d(\gamma) \mu_0 \sum_{n=j}^\infty \frac{1}{\lambda_n \pi_n} - Q_\infty(\gamma)}{Q_\infty^d(\gamma) \{1 + \mu_0 \sum_{n=0}^\infty \frac{1}{\lambda_n \pi_n}\} - Q_\infty(\gamma)}, \quad j \in \mathcal{N}, \quad \gamma = \gamma(\psi).$$

Note that when  $X$  is the minimal process, and hence  $\gamma(\psi) = \xi_1$ , (3.11) and (3.29) coincide. Thus, (3.27) is also valid with the prescription (3.29). Moreover, arithmetical manipulations

involving (3.5) and (3.15) show that (3.28) is also valid when the rate problem associated with  $X$  is indeterminate.

Thus, we have proved the following elegant result.

**Theorem 3.4** Let  $X$  be an evanescent birth-death process on  $\mathcal{N}$  and let  $\psi$  be its spectral measure. Then, (3.28) holds for all  $i, j \in \mathcal{N}$ , with the interpretation that the limit is zero whenever the sum in the denominator diverges, that is, whenever  $\gamma(\psi) = 0$ .

### 3.4 Quasi-stationary distributions

It is well known (see, for example, Vere-Jones [17]) that if, for an absorbing Markov chain, the limit of (1.1) exists and defines a proper distribution, then it is a quasi-stationary distribution. Recall (van Doorn [16]) that a proper distribution,  $m = \{m_i, i \in \mathcal{N}\}$ , over  $\mathcal{N}$  is called a *quasi-stationary distribution (for  $P$ )* if

$$\frac{p_i(t)}{(1 - p_{-1}(t))} = m_i, \quad i \in \mathcal{N}, t \geq 0,$$

where  $p_i(t) = \Pr[X(t) = i]$ , whenever  $m$  is the initial distribution, that is,  $p_i(0) = m_i, i \in \mathcal{N}$ , and  $p_{-1}(0) = 0$ . This is equivalent to the condition that, for some  $\mu > 0$ ,  $m$  is a  $\mu$ -invariant measure on  $\mathcal{N}$  for  $P$ , that is,

$$\sum_{i \in \mathcal{N}} m_i p_{ij}(t) = e^{-\mu t} m_j, \quad j \in \mathcal{N}, t \geq 0;$$

see Proposition 1.1 of Nair and Pollett [10]. Since  $p'_{ij}(0+) = q_{ij}$ , where  $Q = \{q_{ij}\}$  is the  $q$ -matrix, an obvious formal argument suggests that a quasi-stationary distribution,  $m$ , should also satisfy

$$(3.30) \quad \sum_{i \in \mathcal{N}} m_i q_{ij} = -\mu m_j, \quad j \in \mathcal{N}.$$

Accordingly, we call a collection,  $m = \{m_i, i \in \mathcal{N}\}$ , of positive numbers which satisfies (3.30) a  $\mu$ -invariant measure on  $\mathcal{N}$  for  $Q$ . This argument is valid when (and only when)  $P$  satisfies the forward equations over  $\mathcal{N}$  (see Theorem 3.1 of Nair and Pollett [10]), and so, in the present context, general theory dictates that *every* quasi-stationary distribution,  $m$ , and, in particular, the limiting conditional distributions of Theorem 3.3, must satisfy

$$\lambda_{i-1} m_{i-1} - (\lambda_i + \mu_i) m_i + \mu_{i+1} m_{i+1} = -\mu m_i, \quad i \in \mathcal{N},$$

for some  $\mu > 0$ . Indeed, it is easy to prove that, for every  $x$  in the range  $0 \leq x \leq \xi_1$ ,  $\{\pi_i Q_i(x)\}$  (cf. (3.23)) determines an  $x$ -invariant measure for  $Q$ , which is unique up to constant multiples (see Lemma 1 of Kijima and Seneta [8] or Theorem 4.1(b) of Pollett [12]).

Analogous statements, both general, and specific to the present context, hold for the limiting conditional distributions determined by (1.2); see Elmes, Pollett and Walker [3].

What is striking about birth-death processes with certain absorption is that  $\mu$ -invariant probability measures on  $\mathcal{N}$  for  $Q$  exist if and only if  $\xi_1 > 0$ , and are *always* quasi-stationary distributions (see, for example, Theorem 3.1 of van Doorn [16]). Whether this holds true for general Markov chains is not known. The best available general results, which avoid the “classical” premise of  $\lambda$ -positivity, are given by Ferrari, Kesten, Martinez and Picco [4] (see, also, Pakes [11]).

Our next result follows immediately from the results in Section 3.2. The quantities  $a_i = Pr[T < \infty | X(0) = i]$  correspond to the minimal process and, hence, satisfy (3.11.)

**Theorem 3.5** Consider an evanescent birth-death process on  $\mathcal{N}$  with  $q$ -matrix  $Q$  such that  $\xi_1 > 0$ . Let  $\mu > 0$  and let  $m = \{m_i\}$  be the essentially unique  $\mu$ -invariant measure on  $\mathcal{N}$  for  $Q$ , that is,  $m_i = \pi_i Q_i(\mu)$  and  $\mu \leq \xi_1$ .

(i) Suppose that  $A = \infty$ , so that absorption at  $-1$  occurs with probability 1. Then,  $\sum_{i=0}^{\infty} m_i < \infty$ ; moreover, if either

(a)  $D = \infty$ , or

(b)  $D < \infty$  and  $\mu = \xi_1$ ,

then

$$(3.31) \quad \sum_{i=0}^{\infty} m_i = \frac{\mu_0}{\mu}.$$

(ii) Suppose  $A < \infty$ , so that  $a_i < 1$  for all  $i \in \mathcal{N}$ . Then,  $\sum_{i=0}^{\infty} a_i m_i < \infty$ ; moreover, if either

(a)  $C = \infty$ , or

(b)  $C < \infty$  and  $\mu = \xi_1$ ,

then

$$(3.32) \quad \sum_{i=0}^{\infty} a_i m_i = \frac{\mu_0}{\mu}.$$

**Proof:** The equality (3.31) follows from (3.14) in Case 2 and (3.19) in Case 4.1, while (3.32) follows from (3.15) in Cases 1 and 3, and (3.21) in Case 4.2.  $\square$

## 4 The irreducible case

We shall now study the irreducible case,  $\mu_0 = 0$ .

### 4.1 Preliminaries

The definition of the dual process mirrors that given in Section 3.1. Only minor differences arise in the definition and properties of the dual birth-death polynomials, because dual birth and death rates,  $\{\lambda_n^d, n \geq 0\}$  and  $\{\mu_n^d, n \geq 0\}$ , are now given by

$$\lambda_n^d = \mu_{n+1}; \quad \mu_n^d = \lambda_n, \quad n \geq 0.$$

Analogous to (3.1) – (3.5) we have

$$(4.1) \quad \pi_0^d = 1; \quad \pi_n^d = \pi_{n-1}^d \frac{\lambda_{n-1}^d}{\mu_n^d} = \frac{\lambda_0}{\lambda_n \pi_n} = \frac{\lambda_0}{\mu_{n+1} \pi_{n+1}}, \quad n \geq 1,$$

$$(4.2) \quad Q_n^d(x) = \sum_{k=0}^n \pi_k Q_k(x), \quad n \geq 0,$$

and

$$(4.3) \quad Q_n(x) = 1 - x \sum_{k=0}^{n-1} Q_k^d(x) \frac{1}{\lambda_k \pi_k} = 1 - \frac{x}{\lambda_0} \sum_{k=0}^{n-1} \pi_k^d Q_k^d(x), \quad n \geq 0,$$

giving

$$(4.4) \quad Q_n^d(x) = \frac{\lambda_n \pi_n}{-x} \{Q_{n+1}(x) - Q_n(x)\}, \quad n \geq 0,$$

and

$$(4.5) \quad Q_n(0) = 1; \quad Q_n^d(0) = \sum_{k=0}^n \pi_k, \quad n \geq 0.$$

As before,  $Q_n^d(x)$  has  $n$  positive, simple zeros  $x_{ni}^d$  ( $i = 1, \dots, n$ ), which satisfy the interlacing property (2.9), as well as

$$0 < x_{ni} < x_{ni}^d < x_{n,i+1} < x_{n,i+1}^d, \quad i = 1, \dots, n-1,$$

and

$$(4.6) \quad 0 \leq \xi_i \leq \xi_i^d \leq \xi_{i+1} < \infty, \quad i \geq 1.$$

where  $\xi_i^d = \lim_{n \rightarrow \infty} x_{ni}^d$ .

As before, the condition  $A+B = \infty$  is necessary and sufficient for the birth and death rates to determine a unique process  $X$  satisfying both the backward and the forward equations, in which case  $\xi_1 = \gamma(\psi)$ . If  $C < \infty$  and  $D = \infty$ , so that  $A+B = \infty$ , then  $X$  is explosive and reaches infinity in a finite time *with probability 1*. If  $A+B < \infty$  then, as before,  $X$  is explosive and the rate problem is indeterminate: the rates determine a one-parameter family of spectral measures,  $\psi$ , and a one-parameter family of transition functions, but now indexed by  $\gamma(\psi)$  in the interval  $[0, \xi_1]$ . If  $\gamma(\psi) = \xi_1$  we obtain the *minimal process* for which the boundary at infinity is completely absorbing. If  $\gamma(\psi) = 0$  then the corresponding process is called the *maximal process* and the boundary at infinity is completely reflecting. All probability mass eventually disappears at infinity unless  $\gamma(\psi) = 0$ .

The next lemma plays a central role in determining limiting conditional distributions. We define

$$G_{ij}(t) = \Pr[X(t+s) = j \text{ for some } s > 0 | X(0) = i], \quad i, j \in \mathcal{N}, \quad t \geq 0.$$

**Lemma 4.1** Suppose that the process  $X$  is transient. Then,

$$G_{ij}(t) = \frac{\int_t^\infty p_{ij}(u) du}{\int_0^\infty p_{jj}(u) du}, \quad i, j \in \mathcal{N}, \quad t \geq 0.$$

**Proof:** Let  $V_j(t)$  denote the total time spent in state  $j$  during the interval  $(t, \infty)$ . Then, evidently,

$$E[V_j(t) | X(0) = i] = E\left(\int_t^\infty \mathbf{1}\{X(u) = j\} du | X(0) = i\right) = \int_t^\infty p_{ij}(u) du < \infty.$$

Since we also have, because of the Markov property,

$$E[V_j(t) | X(0) = i] = G_{ij}(t) E[V_j(0) | X(0) = j],$$

the result follows.  $\square$

From (2.8) and Lemma 4.1, we have

$$(4.7) \quad G_{i0}(t) = \frac{\int_0^\infty e^{-xt} x^{-1} Q_i(x) d\psi(x)}{\int_0^\infty x^{-1} d\psi(x)},$$

provided  $X$  is transient. In particular, setting  $a_i = G_{i0}(0)$ , we have

$$(4.8) \quad a_i = \frac{\int_0^\infty x^{-1} Q_i(x) d\psi(x)}{\int_0^\infty x^{-1} d\psi(x)}, \quad i \in \mathcal{N}.$$

On the other hand, as in (3.9), the recurrence relation (2.7) leads to

$$(4.9) \quad \int_0^\infty \frac{Q_i(x)}{x} d\psi(x) = \int_0^\infty \frac{d\psi(x)}{x} - \sum_{n=0}^{i-1} \frac{1}{\lambda_n \pi_n}, \quad i \in \mathcal{N}.$$

When  $X$  is the minimal process, (4.5) and Lemma 6 on Page 527 of Karlin and McGregor [6] tell us that

$$(4.10) \quad \int_0^\infty \frac{d\psi(x)}{x} = \sum_{n=0}^\infty \frac{1}{\lambda_n \pi_n} = \frac{1}{\lambda_0} \sum_{n=0}^\infty \pi_n^d,$$

and, hence, that

$$(4.11) \quad a_i = \frac{\sum_{n=i}^\infty \frac{1}{\lambda_n \pi_n}}{\sum_{n=0}^\infty \frac{1}{\lambda_n \pi_n}} = \frac{\sum_{n=i}^\infty \pi_n^d}{\sum_{n=0}^\infty \pi_n^d}, \quad i \in \mathcal{N},$$

provided  $A < \infty$ . The minimal process is transient if and only if  $A < \infty$  (see Karlin and McGregor [7]), and so, if we interpret the right-hand side of (4.11) as unity when  $A = \infty$ , then (4.11) holds whether or not the process is transient.

When the rate problem associated with  $X$  is indeterminate, we can determine a representation for  $\sum_{k=0}^\infty p_{ik}(t)$  in terms of the entire function,  $Q_\infty^d$ , given by  $Q_\infty^d(x) = \lim_{n \rightarrow \infty} Q_n^d(x)$ .

**Lemma 4.2** Let  $P$  be the transition function of an irreducible birth-death process on  $\mathcal{N}$  with spectral measure  $\psi$  and birth and death rates satisfying  $A + B < \infty$ . Then,

$$(4.12) \quad \sum_{k=0}^\infty p_{ik}(t) = \int_0^\infty e^{-xt} Q_i(x) Q_\infty^d(x) d\psi(x), \quad i \in \mathcal{N}, \quad t \geq 0.$$

**Proof:** By (2.8) and (4.2), we have

$$\sum_{k=0}^n p_{ik}(t) = \int_0^\infty e^{-xt} Q_i(x) \sum_{k=0}^n \pi_k Q_k(x) d\psi(x) = \int_0^\infty e^{-xt} Q_i(x) Q_n^d(x) d\psi(x).$$

The result follows on letting  $n \rightarrow \infty$  and interchanging the limit and integral, which, as shown in Pages 535-536 of Karlin and McGregor [6], is justified by dominated convergence.  $\square$

If  $X$  is explosive but uniquely determined by its rates (equivalently,  $C < \infty$  and  $D = \infty$ ), then  $Q_n^d$  does not converge to an entire function as  $n \rightarrow \infty$  and, accordingly, a convenient representation such as (4.12) does not seem to exist. We finally note that if  $X$  is non-explosive, or, otherwise, if  $X$  is the maximal process, then  $P$  is honest.



## 4.2 Asymptotics for birth-death polynomials

By making the appropriate identifications we obtain the following asymptotic results from Kijima and van Doorn [9], where  $a_i = Pr[X(s) = 0 \text{ for some } s > 0 | X(0) = i]$  corresponds to the minimal process and, hence, satisfies (4.11).

*Case 1:  $C + D < \infty$ .* As in Case 1 of Section 3.2, both  $Q_n$  and  $Q_n^d$  converge uniformly on bounded sets to entire functions,  $Q_\infty$  and  $Q_\infty^d$ , respectively, whose zeros are simple and are precisely the points  $\xi_i$  and  $\xi_i^d$ ; the latter quantities satisfy

$$0 < \xi_i < \xi_i^d < \xi_{i+1}, \quad i \geq 1.$$

Hence, (4.2) leads to

$$(4.13) \quad \sum_{n=0}^{\infty} \pi_n Q_n(x) = Q_\infty^d(x),$$

and combining (4.2), (4.3) and (4.11) gives us

$$(4.14) \quad \frac{x}{\lambda_0} \sum_{n=0}^{\infty} a_n \pi_n Q_n(x) = \frac{1 - Q_\infty(x)}{\sum_{n=0}^{\infty} \pi_n^d}.$$

Note that  $B < \infty$ ,  $A = \sum_{n=0}^{\infty} \pi_n^d / \lambda_0 < \infty$ , and hence  $a_n < 1$  for all  $n \in \mathcal{N}$ .

*Case 2:  $C = \infty$  and  $D < \infty$ .*  $Q_n^d$  converges to an entire function  $Q_\infty^d$ , and

$$0 = \xi_1 < \xi_i^d = \xi_{i+1} < \xi_{i+1}^d, \quad i \geq 1.$$

Hence, from (4.2), we have (4.13) again. In this case  $B < \infty$ ,  $A = \infty$ , and hence  $a_n = 1$  for all  $n \in \mathcal{N}$ .

*Case 3:  $C < \infty$  and  $D = \infty$ .*  $Q_n$  converges to an entire function  $Q_\infty$ , and

$$0 < \xi_i = \xi_i^d < \xi_{i+1}, \quad i \geq 1.$$

In addition, we have (4.14) again. On the other hand,

$$(4.15) \quad \sum_{n=0}^{\infty} \pi_n Q_n(x) = \infty, \quad 0 < x < \xi_1;$$

but it is not known whether this series diverges at the point  $x = \xi_1$ . Note that  $B = \infty$ ,  $A < \infty$ , and hence  $a_n < 1$  for all  $n \in \mathcal{N}$ .

*Case 4:  $C = \infty$  and  $D = \infty$ .* From (4.5) we readily obtain

$$(4.16) \quad \sum_{n=0}^{\infty} \pi_n Q_n(0) = B$$

and

$$(4.17) \quad \sum_{n=0}^{\infty} a_n \pi_n Q_n(0) = \infty \quad \text{if } A < \infty.$$

These results serve our purposes when  $\xi_1 = 0$ . It is shown in [9] that in the opposite case  $\xi_1 > 0$  we must be in the following situation.

*Case 4.1:*  $A < \infty$ ,  $B = \infty$  and  $\xi_1 > 0$ . In this case, we have  $a_n < 1$  for all  $n \in \mathcal{N}$  and

$$0 < \xi_i = \xi_i^d \leq \xi_{i+1}, \quad i \geq 1.$$

Moreover,

$$(4.18) \quad \sum_{n=0}^{\infty} \pi_n Q_n(x) = \infty, \quad 0 < x \leq \xi_1,$$

but

$$(4.19) \quad \frac{x}{\lambda_0} \sum_{n=0}^{\infty} a_n \pi_n Q_n(x) = \frac{1}{\sum_{n=0}^{\infty} \pi_n^d}, \quad 0 < x \leq \xi_1.$$

### 4.3 Limiting conditional distributions

Obviously, the conditional probabilities  $r_{ij}(t)$  of (1.1) satisfy (3.22). So, when  $X$  is non-explosive, we have  $r_{ij}(t) = p_{ij}(t)$  and it is clear that

$$(4.20) \quad \lim_{t \rightarrow \infty} p_{ij}(t) = \frac{\pi_j}{\sum_{k=0}^{\infty} \pi_k}, \quad j \in \mathcal{N},$$

which should be interpreted as zero if the sum in the denominator diverges. In view of (4.13) in Case 2, (4.18) in Case 4.1, and (4.16) in Case 4 when  $\xi_1 = 0$ , the analogue of (3.23) holds. When the rate problem associated with  $X$  is indeterminate, the integral representation (4.12) allows us to obtain an analogue of (3.25), namely

$$\lim_{t \rightarrow \infty} r_{ij}(t) = \frac{\pi_j Q_j(\gamma)}{Q_{\infty}^d(\gamma)}, \quad \gamma = \gamma(\psi).$$

This, together with (4.13), provides us with our analogue of Theorem 3.3.

**Theorem 4.3** Let  $X$  be an irreducible birth-death process on  $\mathcal{N}$  with spectral measure  $\psi$  and birth and death rates satisfying either  $C = \infty$ , or  $C + D < \infty$  (equivalently,  $A + B < \infty$ ). Then, for all  $i, j \in \mathcal{N}$ ,

$$(4.21) \quad \lim_{t \rightarrow \infty} r_{ij}(t) = \frac{\pi_j Q_j(\gamma)}{\sum_{k=0}^{\infty} \pi_k Q_k(\gamma)}, \quad \gamma = \gamma(\psi),$$

with the interpretation that the limit is zero whenever the sum in the denominator diverges, that is, whenever  $B = \infty$ .

As before, the problem of determining  $\lim_{t \rightarrow \infty} r_{ij}(t)$  when  $C < \infty$  and  $D = \infty$  remains unsolved, but we conjecture that (4.21) is valid under any circumstance.

Suppose that  $X$  is the minimal process. Then, the conditional probabilities  $\bar{r}_{ij}(t)$  of (1.3) satisfy

$$\begin{aligned} \bar{r}_{ij}(t) &= \Pr[X(t) = j | X(0) = i, X(t+s) = 0 \text{ for some } s > 0] \\ &= \frac{a_j p_{ij}(t)}{G_{i0}(t)} \\ &= a_j \pi_j \int_0^\infty \frac{d\psi(x)}{x} \frac{\int_0^\infty e^{-xt} Q_i(x) Q_j(x) d\psi(x)}{\int_0^\infty e^{-xt} x^{-1} Q_i(x) d\psi(x)}, \end{aligned}$$

where we have used (2.8), (4.7) and the Markov property. By using the result of Karlin and McGregor [7] stated in (3.26), we subsequently obtain, for all  $i, j \in \mathcal{N}$ ,

$$(4.22) \quad \lim_{t \rightarrow \infty} \bar{r}_{ij}(t) = \gamma a_j \pi_j Q_j(\gamma) \int_0^\infty \frac{d\psi(x)}{x}, \quad \gamma = \gamma(\psi) = \xi_1,$$

provided that the process is transient, that is,  $A < \infty$ . If the process is recurrent then  $\bar{r}_{ij}(t) = r_{ij}(t)$  and  $a_j = 1$  for all  $i, j \in \mathcal{N}$ . So, in view of (4.10), the asymptotic results (4.14) in Cases 1 and 3, (4.19) in Case 4.1, and (4.17) in Case 4 when  $\xi_1 = 0$ , as well as Theorem 4.3, we arrive at the analogue of Theorem 3.4 in the case when  $X$  is the minimal process. In particular we have, for all  $i, j \in \mathcal{N}$ , that

$$(4.23) \quad \lim_{t \rightarrow \infty} \bar{r}_{ij}(t) = \frac{a_j \pi_j Q_j(\gamma)}{\sum_{k=0}^\infty a_k \pi_k Q_k(\gamma)}, \quad \gamma = \gamma(\psi),$$

where  $a_j$  is given by (4.11).

As in the  $\mu_0 > 0$  case, it is possible to extend this result to non-minimal processes, provided that  $a_j$  is suitably reinterpreted. Indeed, it can be shown that if the rate problem associated with  $X$  is indeterminate, then

$$(4.24) \quad \int_0^\infty \frac{d\psi(x)}{x} = \sum_{n=0}^\infty \frac{1}{\lambda_n \pi_n} + \frac{Q_\infty(\gamma)}{\gamma Q_\infty^d(\gamma)}, \quad \gamma = \gamma(\psi),$$

so that, by (4.8) and (4.9),

$$(4.25) \quad a_j = \frac{Q_\infty(\gamma) + \gamma Q_\infty^d(\gamma) \sum_{n=j}^\infty \frac{1}{\lambda_n \pi_n}}{Q_\infty(\gamma) + \gamma Q_\infty^d(\gamma) \sum_{n=0}^\infty \frac{1}{\lambda_n \pi_n}}, \quad j \in \mathcal{N}, \quad \gamma = \gamma(\psi).$$

It subsequently follows, with (4.13), (4.14) and (4.22), that (4.23) is also valid for non-minimal processes.

In summary, we have the following result.

**Theorem 4.4** Let  $X$  be an irreducible birth-death process on  $\mathcal{N}$  with spectral measure  $\psi$ . Then, (4.23) holds for all  $i, j \in \mathcal{N}$ , with the interpretation that the limit is zero whenever the sum in the denominator diverges, that is, whenever  $\gamma(\psi) = 0$  and  $B = \infty$ .

## 4.4 Quasi-stationary distributions

As for the evanescent case, we have, for every  $x$  in the range  $0 \leq x \leq \xi_1$ , that  $\{\pi_i Q_i(x)\}$  determines an essentially unique  $x$ -invariant measure for  $Q$ . Using the results of Section 4.2, we can prove a result, analogous to Theorem 3.5. The quantities  $a_i = Pr[X(s) = 0 \text{ for some } s > 0 | X(0) = i]$  correspond the minimal process and, hence, satisfy (4.11).

**Theorem 4.5** Consider an irreducible birth-death process on  $\mathcal{N}$  with  $q$ -matrix  $Q$ . Let  $\mu \geq 0$  and let  $m = \{m_i\}$  be the essentially unique  $\mu$ -invariant measure on  $\mathcal{N}$  for  $Q$ , that is,  $m_i = \pi_i Q_i(\mu)$  and  $\mu \leq \xi_1$ .

(i) Suppose that the process is recurrent, that is,  $A = \infty$ . Then  $\xi_1 = \mu = 0$ , so that  $m_i = \pi_i$  for all  $i \in \mathcal{N}$  and

$$(4.26) \quad \sum_{i=0}^{\infty} m_i = \sum_{i=0}^{\infty} \pi_i = B.$$

(ii) Suppose that the process is transient, that is,  $A < \infty$ . Then,  $\sum_{i=0}^{\infty} a_i m_i < \infty$  unless  $C = \infty$  and  $\mu = 0$ . Moreover, if either

(a)  $C = \infty$  and  $0 < \mu \leq \xi_1$ , or

(b)  $C < \infty$  and  $\mu = \xi_1$ ,

then

$$(4.27) \quad \sum_{i=0}^{\infty} a_i m_i = \frac{\lambda_0}{\mu \sum_{i=0}^{\infty} \pi_i^d} = \frac{1}{\mu A}.$$

**Proof:** The equality (4.27) follows from (4.14) in Cases 1 and 3 and (4.19) in Case 4.1.  $\square$

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