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On the uniqueness of invariant tori in $D_4 \times S^1$ symmetric systems

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Abstract. The uniqueness of the branch of two-tori in the D_4-equivariant Hopf bifurcation problem is proved in a neighbourhood of a particular limiting case where, after reduction, the Euler equations for the rotation of a free rigid body apply.

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1. Introduction

Motivated by the problem of N oscillators coupled in a ring geometry, Hopf bifurcation in the presence of D_N symmetry is analysed in some detail in Golubitsky, Stewart and Schaeffer [GSS88]. The particular case of $N = 4$ has also found application in a variety of studies of spatially-periodic standing wave patterns in oscillatory convection (see, for example, [SK91], [SRK92], [CK94]). In these studies the standing waves are born in a symmetry-breaking Hopf bifurcation from a spatially-uniform equilibrium solution of the governing hydrodynamic equations. The D_4-symmetry has its origins in spatial reflection and translation symmetries of the physical problem, and is inherited by the normal form of the bifurcation problem in the course of a centre manifold reduction from the partial differential equations. In this context, the phase space variables $\boldsymbol{z} \equiv (z_1, z_2) \in \mathbb{C}^2$ in the normal form of the bifurcation problem, $\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}, \lambda)$, $\boldsymbol{f} : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2$, have simple interpretations. Specifically, z_j is the amplitude of the spatial Fourier mode $\cos(\boldsymbol{k}_j \cdot \boldsymbol{x})$, where $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are linearly independent vectors in $\mathbb{R}^2$ [SK91], [SRK92]. Thus, for example, a time-periodic solution with $z_2 = 0$ corresponds to a one-dimensional standing wave that is periodic in the $\boldsymbol{k}_1$-direction; near the bifurcation point the amplitude of the standing wave is given, approximately, by $|z_1|$.

In their analysis of the D_N-equivariant Hopf bifurcation, Golubitsky, Stewart and Schaeffer [GSS88] determine the general form of a smooth $D_N \times S^1$-equivariant vector field $\boldsymbol{f} : \mathbb{C}^2 \to \mathbb{C}^2$, where the S^1 symmetry is the usual Hopf normal form symmetry. They then compute the stability of those periodic solutions guaranteed to exist by the equivariant Hopf theorem in terms of the coefficients in the Taylor expansion of $\boldsymbol{f}$. The case $N = 4$ differs from the others in that it has an extra cubic equivariant; only in this case is stability determined, generically, by the cubic truncation of the vector field. Swift [Swi88] carries out a fairly complete analysis of the cubic truncation of the $D_4 \times S^1$-equivariant vector field. In particular, he shows, under certain circumstances, that the dynamics are determined by analysing the flow of the so-called associated spherical system. Equilibrium solutions of the associated spherical system correspond to branches of periodic solutions in the original bifurcation problem, and periodic solutions correspond to branches of two-tori. This correspondence allowed Swift to use the Poincaré–Bendixson theorem to prove the generic existence of a branch of two-tori in the D_4-equivariant Hopf bifurcation problem [Swi88].

Our paper addresses the question of uniqueness of the branch of two-tori in the D_4-equivariant Hopf bifurcation problem. The analysis exploits a particular limiting case of the associated spherical flow, where the Euler equations for the rotation of a free rigid body apply. In this limit, the sphere is filled with periodic orbits and isolated heteroclinic orbits. We show, in a neighbourhood of this limiting case, that at most one symmetry-related pair of periodic solutions survives perturbation from the Euler limit, and hence that the branch of two-tori in the corresponding Hopf bifurcation problem is unique. It also follows from our analysis that in a neighbourhood of the Euler system the associated spherical system is structurally stable.

Section 2 provides background to the problem. We introduce the associated spherical system and state the precise relationship between periodic orbits in the associated spherical system and two-tori in the D_4-equivariant Hopf bifurcation problem. Section 3 contains our analysis of the spherical system, in a neighbourhood of the limit where the Euler equations apply. The results are summarized in section 4.

2. Statement of the problem

We follow closely the notation and the results of Golubitsky, Stewart and Schaeffer [GSS88], chapter XVIII and Swift [Swi88]. We assume that elements of $D_4 \times S^1$ act on $(z_1, z_2) \in \mathbb{C}^2$ as follows

$$\gamma(z_1, z_2) = (\mathrm{i}z_1, -\mathrm{i}z_2) \qquad (\gamma \in \mathbb{Z}_4) \tag{2.1a}$$

$$\kappa(z_1, z_2) = (z_2, z_1) \tag{2.1b}$$

$$\theta(z_1, z_2) = (\mathrm{e}^{\mathrm{i}\theta} z_1, \mathrm{e}^{\mathrm{i}\theta} z_2) \qquad (\theta \in S^1). \tag{2.1c}$$

Here the rotation γ and the reflection κ generate the group D_4. A Hilbert basis for the $D_4 \times S^1$ invariant real functions is given by

$$N = |z_1|^2 + |z_2|^2 \tag{2.2a}$$

$$P = |z_1|^2 |z_2|^2 \tag{2.2b}$$

$$S = (z_1\overline{z_2})^2 + (\overline{z_1}z_2)^2 \tag{2.2c}$$

$$T = i(|z_1|^2 - |z_2|^2)((z_1\overline{z_2})^2 - (\overline{z_1}z_2)^2). \tag{2.2d}$$

These invariants satisfy the relation

$$T^2 = (4P - N^2)(S^2 - 4P^2). \tag{2.3}$$

Any smooth $D_4 \times S^1$ equivariant mapping f from $\mathbb{C}^2 \to \mathbb{C}^2$ can be written in the form [GSS88]

$$f(z_1, z_2) = F_1 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + F_2 \begin{bmatrix} {z_1}^2\overline{z_1} \\ {z_2}^2\overline{z_2} \end{bmatrix} + F_3 \begin{bmatrix} \overline{z_1}{z_2}^2 \\ {z_1}^2\overline{z_2} \end{bmatrix} + F_4 \begin{bmatrix} {z_1}^3\overline{z_2}^2 \\ \overline{z_1}^2{z_2}^3 \end{bmatrix} \tag{2.4}$$

where $F_1, \ldots, F_4$ are smooth complex-valued functions of the invariants N, P, S, and T.

It follows that the general cubic vector field is given by

$$\begin{aligned}\dot{z_1} &= z_1(\lambda + i\omega + A(|z_1|^2 + |z_2|^2) + B|z_1|^2) + C\overline{z_1}z_2^2 \\ \dot{z_2} &= z_2(\lambda + i\omega + A(|z_1|^2 + |z_2|^2) + B|z_2|^2) + C\overline{z_2}z_1^2\end{aligned} \tag{2.5}$$

where

$$A = A_R + iA_I \qquad B = B_R + iB_I \qquad C = C_R + iC_I \tag{2.6}$$

with A_R, A_I, B_R, B_I, C_R, $C_I \in \mathbb{R}$. While the invariants could be used to divide out the S^1 action, from a computational point of view it seems better to view z_1, z_2 as the Cayley–Klein parameters of $SU(2)$, (see for instance [Mil72]) and to transform (2.5) to the Euler angles. Let

$$u + iv = r\sin(\theta)e^{i\phi} = 2z_1\overline{z_2} \tag{2.7a}$$

$$w = r\cos(\theta) = |z_1|^2 - |z_2|^2 \tag{2.7b}$$

$$e^{i\psi} = \frac{z_1 z_2}{|z_1 z_2|} \tag{2.7c}$$

where (r, θ, ϕ) are the usual spherical coordinates on $\mathbb{R}^3$. There are two inverses of this transformation, reflecting the fact that $SU(2)$ is the twofold universal covering of $SO(3)$, given by the two choices for the square root:

$$z_1 = r^{1/2}\cos(\theta/2)e^{i\frac{(\phi+\psi)}{2}} \tag{2.8a}$$

$$z_2 = r^{1/2}\sin(\theta/2)e^{i\frac{(-\phi+\psi)}{2}}\ . \tag{2.8b}$$

In terms of these coordinates (2.5) is given by

$$\begin{aligned}\dot{u} &= 2\lambda u + B_R ru - B_I vw + 2A_R ru + C_R ru - C_I vw \\ \dot{v} &= 2\lambda v - C_R rv - C_I uw + B_I uw + B_R rv + 2A_R rv \\ \dot{w} &= 2\lambda w + 2C_I vu + 2B_R rw + 2A_R rw \\ \dot{\psi} &= 2\omega + O(r)\end{aligned} \tag{2.9}$$

where $r \equiv \sqrt{u^2 + v^2 + w^2}$. As ψ does not occur in the first three equations we obtain a reduced system in $\mathbb{R}^3$. This reduction is more efficient than the reduction with the Hilbert invariants which would yield an equation on a variety in $\mathbb{R}^4$. Note that an equilibrium solution of the reduced $(\dot{u}, \dot{v}, \dot{w})$-equations corresponds to a periodic solution of the full system (2.9). Similarly, a periodic solution of the reduced equations generates a flow on the torus once we incorporate the ψ-dynamics.

A special property of equations (2.9) for u, v, and w is most easily seen in spherical coordinates:

$$\begin{aligned}\dot{r} &= r\left(2\lambda + r\left(2A_R + C_R\sin^2(\theta)\cos(2\phi) + B_R\left(1 + \cos^2(\theta)\right)\right)\right) \\ \dot{\theta} &= -r\sin(\theta)\left(C_I\sin(2\phi) - C_R\cos(\theta)\cos(2\phi) + B_R\cos(\theta)\right) \\ \dot{\phi} &= -r\left(C_R\sin(2\phi) - B_I\cos(\theta) + C_I\cos(\theta)\cos(2\phi)\right).\end{aligned} \tag{2.10}$$

Following Swift [Swi88], we define the associated spherical system to be

$$\begin{aligned}\frac{d\theta}{d\tau} &= -\sin(\theta)\left(2C_I\cos(\phi)\sin(\phi) - C_R\cos(\theta)\cos(2\phi) + B_R\cos(\theta)\right) \\ \frac{d\phi}{d\tau} &= -\left(2C_R\sin(\phi)\cos(\phi) - B_I\cos(\theta) + C_I\cos(\theta)\cos(2\phi)\right).\end{aligned} \tag{2.11}$$

The precise relation between periodic solutions of (2.10) and (2.11) is given in the next lemma. This is the *associated spherical system lemma* in [Swi88], slightly differently formulated and proved. First we define

$$f(\theta, \phi) = 2A_R + C_R\sin^2(\theta)\cos(2\phi) + B_R\left(1 + \cos^2(\theta)\right). \tag{2.12}$$

Lemma 2.1. *If $(\theta^*(\tau), \phi^*(\tau))$ is a T-periodic solution of (2.11) and*

$$\lambda A(T) = \lambda \int_0^T a(\tau)\,\mathrm{d}\tau \stackrel{\text{def}}{=} \lambda \int_0^T f(\theta^*(\tau), \phi^*(\tau))\,\mathrm{d}\tau < 0$$

then there exists a unique $r(0) > 0$ such that (2.10) has a periodic solution with initial condition $\theta(0) = \theta^*(0)$ and $\phi(0) = \phi^*(0)$.

Conversely, if $(r^*(t), \theta^*(t), \phi^*(t))$ is a periodic solution of (2.10), then the solution of (2.11) with initial condition $\theta(0) = \theta^*(0)$ and $\phi(0) = \phi^*(0)$ is periodic.

Proof. First introduce the new time $\tau(t)$, where $\dot{\tau} = r(t)$, $\tau(0) = 0$, to (2.10). The first equation transforms to

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \lambda + rf(\theta, \phi). \tag{2.13}$$

Suppose that $(\theta^*(\tau), \phi^*(\tau))$ is a T-periodic solution of (2.11), then substituting $a(\tau) = f(\theta^*(\tau), \phi^*(\tau))$ in (2.13), we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \lambda + ra(\tau)$$

which can be integrated. Let $A(\tau) = \int_0^\tau a(s)\mathrm{d}s$. The solution is T-periodic if and only if

$$r(0) = \frac{\lambda}{\mathrm{e}^{-A(T)} - 1} \int_0^T \mathrm{e}^{-A(s)}\mathrm{d}s$$

in which case the solution is given by

$$r(\tau) = \frac{\lambda \mathrm{e}^{A(\tau)}}{\mathrm{e}^{-A(T)} - 1} \left(\int_\tau^T \mathrm{e}^{-A(s)}\mathrm{d}s + \mathrm{e}^{-A(T)} \int_0^\tau \mathrm{e}^{-A(s)}\mathrm{d}s \right).$$

Note that the radius r is a positive function of τ whenever $A(T)$ and λ have opposite signs. From this explicit equation the first part of the lemma follows. The second part is trivial.□

The foregoing lemma enables us to restrict our analysis to (2.11), provided $\lambda A(T) < 0$. Apologizing for yet another transformation we write (2.11) in Cartesian coordinates on the unit sphere (adding the equation $\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0$). By a time rescaling we put

$$C_I = 1 \quad B_I = b \quad C_R = \mu \quad B_R = \nu. \tag{2.14}$$

This gives the system of equations

$$\begin{aligned} \dot{x} &= -(b+1)yz + \mu\left(-x^3 + xy^2 + x\right) + \nu\left(x^3 + xy^2 - x\right) \\ \dot{y} &= (b-1)xz + \mu\left(-yx^2 - y + y^3\right) + \nu\left(yx^2 + y^3 - y\right) \\ \dot{z} &= 2xy + \mu\left(zy^2 - zx^2\right) + \nu\left(x^2 z + y^2 z\right) \end{aligned} \tag{2.15}$$

where $x^2 + y^2 + z^2 = 1$. We refer to (2.15) with $\mu = \nu = 0$ as the 'unperturbed system'. Note that the unperturbed equations are identical to the Euler equations for the angular momentum of a free rigid body; they possess a first integral

$$H = \frac{1}{2}(b-1)x^2 + \frac{1}{2}(b+1)y^2. \tag{2.16}$$

In this case, the spherical system lemma applies provided A_R in (2.12) does not have the same sign as λ. Excluding the degenerate cases $b = \pm 1$, the unperturbed system possesses precisely three symmetry-related pairs of fixed-points at $(x, y, z) =$

$(\pm1, 0, 0)$, $(0, \pm1, 0)$, $(0, 0, \pm1)$; one pair is hyperbolic and the others are elliptic. Which of the three pairs of fixed-points is hyperbolic depends on whether $b > 1$, $1 > b > -1$, or $b < -1$. Without loss of generality we take $b > 1$, in which case the hyperbolic fixed-points are on the x-axis; the results for the other cases can be obtained by a simple coordinate transformation. The pair of hyperbolic points are connected by heteroclinic orbits; all the other orbits are periodic. See figure 1.

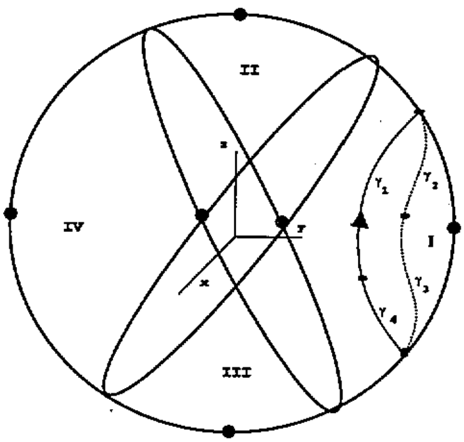

Figure 1. The Euler flow on the sphere, $b > 1$.

Note that there are four regions filled with families of periodic solutions. The heteroclinic orbits that separate the regions I–IV are obtained by intersecting the unit sphere with the surface $H = \frac{1}{2}(b-1)$, where H is given by (2.16). Thus for example, region I is defined by $0 < y \leqslant 1$, $z^2 < \frac{2y^2}{(b-1)}$, with $x^2 = 1 - y^2 - z^2$. The family of periodic orbits surrounding the fixed-point $(0, 1, 0)$ is related to the family surrounding the fixed-point $(0, -1, 0)$ by the symmetry $(x, y, z) \to (x, -y, -z)$ of (2.15); similarly the periodic orbits surrounding the $(0, 0, 1)$ are related by symmetry to those surrounding $(0, 0, -1)$. It is our main result that the perturbed system, with μ, ν sufficiently small, possesses at most one symmetry-related pair of limit cycles.

3. The perturbed Euler flow

Analysis in region I ($b > 1$)

We rewrite (2.15) in the form

$$\begin{aligned} \dot{x} &= -z\frac{\partial H}{\partial y} + P_1 \\ \dot{y} &= z\frac{\partial H}{\partial x} + P_2 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} P_1 &= \mu\left(-x^3 + xy^2 + x\right) + \nu\left(x^3 + xy^2 - x\right) \\ P_2 &= \mu\left(-yx^2 - y + y^3\right) + \nu\left(yx^2 + y^3 - y\right). \end{aligned} \tag{3.2}$$

The function $\eta = \eta(h, \mu, \nu)$ is defined in the following standard way. Let

$$S = \{(x, \sqrt{1 - x^2}, 0), 0 \leqslant x \leqslant 1\} \tag{3.3}$$

$q \in H^{-1}(h) \cap S$ and let $\tilde{\gamma}$ be the trajectory of (3.1) in forward time from q to the next intersection with S at the point p. We let

$$\eta(h, \mu, \nu) = H(p) - H(q). \tag{3.4}$$

The perturbed system has a periodic orbit in region I (see figure 1) iff

$$\frac{1}{2}(b-1) < h < \frac{1}{2}(b+1) \tag{3.5}$$

and

$$0 = H(p) - H(q). \tag{3.6}$$

It is a consequence of (3.1) that $\eta(h, \mu, \nu)$ satisfies

$$\eta(h, \mu, \nu) = \oint_{\gamma(h)} \frac{P_1}{z}\,\mathrm{d}y - \oint_{\gamma(h)} \frac{P_2}{z}\,\mathrm{d}x + \mathcal{O}((|\mu| + |\nu|)^2) \tag{3.7}$$

where $\gamma(h)$ is the periodic orbit of the unperturbed system at the level $H = h$. Let

$$\oint_{\gamma(h)} \frac{P_1}{z}\,\mathrm{d}y - \oint_{\gamma(h)} \frac{P_2}{z}\,\mathrm{d}x = \mu I_1(h) + \nu I_2(h). \tag{3.8}$$

We apply the implicit function theorem to the equation $\frac{I_2(h)}{I_1(h)} + \frac{\mu}{\nu} = 0$ to show that the perturbed system, with μ, ν sufficiently small, has a surviving periodic orbit at the level $H = h(\mu/\nu)$. The main result of this section is that $\frac{I_2(h)}{I_1(h)}$ is a strictly decreasing function of h for $h \in ((b-1)/2, (b+1)/2)$.

Lemma 3.1. *In region I, the quotient of the integrals I_2 and I_1 satisfies the relation*

$$\frac{I_2(s)}{I_1(s)} = \frac{E(s) - K(s) + K(s)s^2}{K(s) - E(s) - K(s)s^2 + 2\,E(s)s^2} \tag{3.9}$$

where $K(s)$ and $E(s)$ are the complete elliptic integrals of the first and the second kind, and

$$s = \frac{\sqrt{b+1-2h}\sqrt{b-1}}{2\sqrt{h}} \tag{3.10}$$

$s \in (0, 1)$ *for* $h \in ((b-1)/2, (b+1)/2)$.

Proof. See appendix.

The elliptic functions satisfy the linear differential equation [BF71]

$$\begin{aligned} s\dot{E}(s) &= E(s) - K(s) \\ s\left(1-s^2\right)\dot{K}(s) &= E(s) - \left(1-s^2\right)K(s). \end{aligned} \tag{3.11}$$

We introduce the quotient

$$Q(s) \equiv \frac{I_2(s)}{I_1(s)}. \tag{3.12}$$

In the next lemma we give an *a priori* estimate for the quotient $Q(s)$.

Lemma 3.2. *On the interval* $[0, 1]$, $\frac{1}{3} \leqslant Q(s) \leqslant 1$, *with* $Q(0) = \frac{1}{3}$ *and* $Q(1) = 1$.

Proof. See appendix.

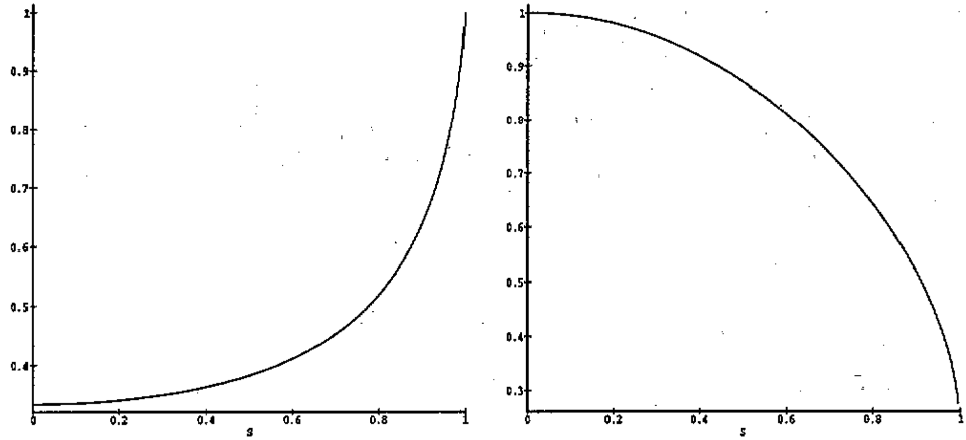

Figure 2. (*a*) $I_2(s)/I_1(s)$; (*b*) $E(s)/K(s)$.

We introduce the quotient of the complete elliptic integrals:

$$R(s) = \frac{E(s)}{K(s)} \tag{3.13}$$

Lemma 3.3. *For* $s \in (0, 1) : R(s) > 1 - s$.

Proof. See appendix.

Lemma 3.4. *The mapping* $s \mapsto Q(s)$ *is strictly monotonic increasing on the interval* (0, 1).

Proof. We prove that $Q(s)$ is a strictly monotonic function of s in the interval $(0, 1)$ by showing that $\dot{Q}(s) \neq 0$ in this interval, where

$$\dot{Q}(s) = -\frac{\left(4\,K(s)E(s)s^2 - 2\,K(s)^2s^2 - 8\,K(s)E(s) + 2\,K(s)^2 + 6\,E(s)^2\right)s}{\left(K(s) - E(s) - K(s)s^2 + 2\,E(s)s^2\right)^2} \tag{3.14}$$

follows from (3.11). We conclude that if there is a point s where the first derivative of Q vanishes, then it would satisfy $R(s) = R_1(s)$ or $R(s) = R_2(s)$, where

$$\begin{aligned} R_1(s) &= \tfrac{2}{3} - \tfrac{s^2}{3} - \tfrac{\sqrt{1-s^2+s^4}}{3} \\ R_2(s) &= \tfrac{2}{3} - \tfrac{s^2}{3} + \tfrac{\sqrt{1-s^2+s^4}}{3}. \end{aligned} \tag{3.15}$$

It is a consequence of the previous lemma that the root $R_1(s)$ cannot occur. To show that the second root cannot occur, we examine the second derivative of Q at a point where the first derivative vanishes.

$$\ddot{Q}\rfloor_{R_2(s)} = \frac{4}{\left(s^2 - 2 - \sqrt{1 - s^2 + s^4}\right)\left(s^2 - \sqrt{1 - s^2 + s^4}\right)} \tag{3.16}$$

which is strictly positive on the interval [0, 1]. Combined with the *a priori* bound for Q, it follows that Q must be strictly monotonic increasing. □

Note that the independent variable s in (3.10) is a decreasing function of h. It follows from lemma 3.4 that $\frac{I_2(h)}{I_1(h)}$ is a strictly decreasing function of h for $h \in ((b-1)/2, (b+1)/2)$.

Analysis in region II ($b > 1$)

Here we omit the proofs because they are very similar to the proofs of the corresponding results in region I. We begin by rotating the system so that region II is taken into region I. Specifically, we apply the coordinate transformation and time rescaling

$$\begin{aligned} x &= Z \\ y &= X \\ z &= Y \\ t &= -\frac{2}{b+1}\tau \end{aligned} \tag{3.17}$$

and also introduce new values for b, μ, ν, which we denote by B, M, N:

$$\begin{aligned} B &= \frac{b-3}{b+1} \\ M &= \frac{\mu+\nu}{b+1} \\ N &= \frac{-3\mu+\nu}{b+1}. \end{aligned} \tag{3.18}$$

With this identification, the $(\dot{X}, \dot{Y}, \dot{Z})$-equations have the exact same form as (2.15); as in (3.1) the perturbation of the Euler equations is given by

$$\begin{aligned} \tilde{P}_1 &= M\left(-X^3 + XY^2 + X\right) + N\left(X^3 + XY^2 - X\right) \\ \tilde{P}_2 &= M\left(-YX^2 - Y + Y^3\right) + N\left(YX^2 + Y^3 - Y\right). \end{aligned} \tag{3.19}$$

Note that $B \in (-1, 1)$ when $b > 1$. In this case the hyperbolic fixed points are on the z axis, and h takes values in the range $(0, (B+1)/2)$. Let

$$\oint_{\gamma(h)} \frac{\tilde{P}_1}{Z}\,\mathrm{d}Y - \oint_{\gamma(h)} \frac{\tilde{P}_2}{Z}\,\mathrm{d}X = M\tilde{I}_1(h) + N\tilde{I}_2(h) \tag{3.20}$$

Lemma 3.5. *In region II, the quotient of the integrals* $\tilde{I}_2$ *and* $\tilde{I}_1$ *satisfies the relation*

$$\frac{\tilde{I}_2(s)}{\tilde{I}_1(s)} = \frac{(1-s)(1+s)(K(s)-E(s))}{E(s)-K(s)+K(s)s^2+E(s)s^2} \tag{3.21}$$

with

$$s = \sqrt{\frac{(1-B)(B-2h+1)}{(B+1)(1-B+2h)}}. \tag{3.22}$$

We introduce the quotient

$$\tilde{Q}(s) \equiv \frac{\tilde{I}_2(s)}{\tilde{I}_1(s)}. \tag{3.23}$$

Lemma 3.6.

(i) On the interval $[0, 1]$, $\frac{1}{3} \geqslant \tilde{Q}(s) \geqslant 0$, *with* $\tilde{Q}(0) = \frac{1}{3}$ *and* $\tilde{Q}(1) = 0$.

(ii) The mapping $s \mapsto \tilde{Q}(s)$ *is strictly monotonic decreasing on the interval* $(0, 1)$.

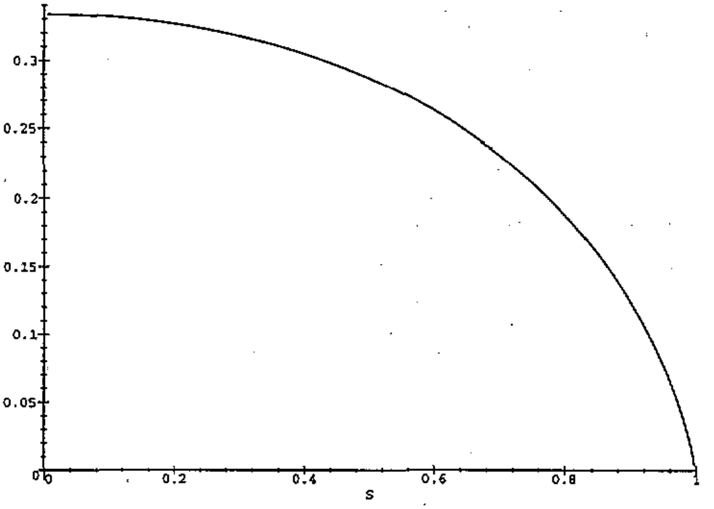

Figure 3. $\tilde{I}_2(s)/\tilde{I}_1(s)$, $s \in [0, 1]$.

4. Conclusions and numerical results

It is a consequence of the monotonicity of the two quotients of integrals that in each of the regions I–IV, leaving out an arbitrarily small neighbourhood of the saddle connections, there can be at most one symmetry-related pair of limit cycles when we are close to the Euler flow. This claim is true for *all* values of b provided the nondegeneracy condition $b \neq \pm 1$ holds. We restricted our analysis to the case $b > 1$; the results for other values of b follow using the 'parameter symmetries' of Swift [Swi88] (an example of such a transformation is given by equations 3.17 and 3.18). We have investigated regions I and II; the results for regions III and IV follow from symmetry.

The region of existence of a limit cycle in region I, in the limit as $\mu, \nu \to 0$, is determined by setting $\frac{\mu}{\nu} = -Q$ and noting that $Q \in [\frac{1}{3}, 1]$ (see lemma 3.2). Similarly, there is a limit cycle in region II whenever $\frac{M}{N} = -\tilde{Q} \in [-\frac{1}{3}, 0]$, where $\frac{M}{N} = \frac{\mu+\nu}{\nu-3\mu}$. Figure 4 indicates the region of existence of the branch of tori, near $\mu = \nu = 0$, as well as the bifurcation phenomena. Following the arrow, we encounter first a Hopf bifurcation at the intersection with the line $3\mu + \nu = 0$. A symmetry-related pair of limit cycles grow from the equilibria $(x, y, z) = (0, \pm 1, 0)$. When crossing the line $\mu + \nu = 0$, the limit cycles disappear in the saddle connections and then reappear in regions II and III. These limit cycles then shrink until they die at the Hopf bifurcation line $\nu = 0$. A numerically computed example of the transition from region I to region II is presented in figures 5 and 7. In figure 6, $\mu = 0.1$ and $\nu = -0.11$; we conclude from the two projections, one on the x–y plane and the other one on the x–z plane that the limit cycle lies in region I. Similarly, when $\mu = 0.1$ and $\nu = -0.09$ the limit cycle is in region II.

5. Appendix

Proof of lemma 3.1. From the symmetries of P_1, P_2, and the symmetry of the periodic

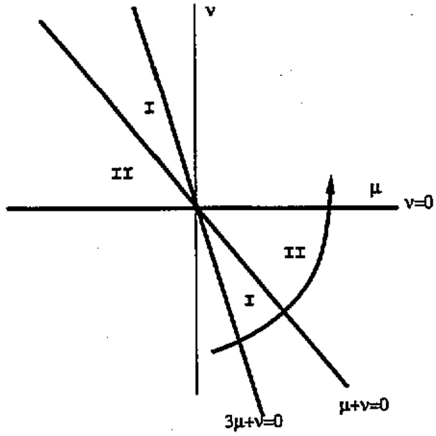

Figure 4. Bifurcation diagram for $b > 1$, and μ, ν sufficiently small. A unique limit cycle exists in region I for values of (μ, ν) between the lines $3\mu + \nu = 0$ and $\mu + \nu = 0$. The limit cycle moves into region II for values of (μ, ν) between the lines $\mu + \nu = 0$ and $\nu = 0$.

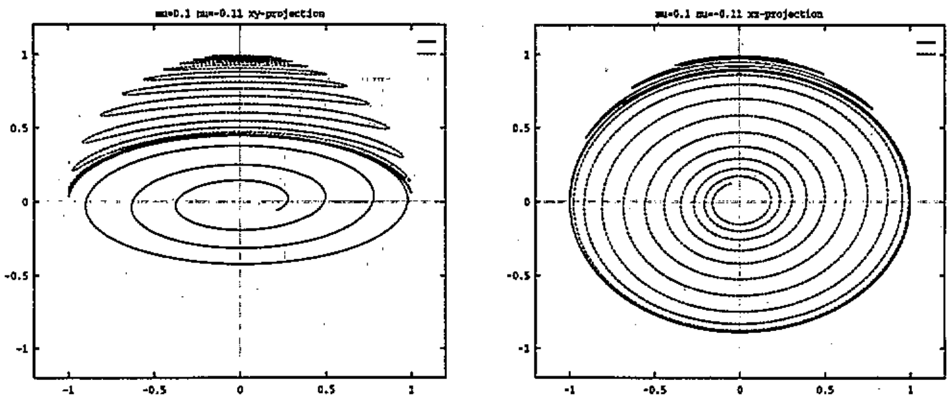

Figure 5. Phase space projections on the x–y plane and the x–z plane for $\mu = 0.1$, $\nu = -0.11$. The limit cycle surrounds the y-axis, i.e., it is in region I.

orbit $\gamma(h)$ of the unperturbed system, it follows that

$$\frac{1}{4}\left(\oint_{\gamma(h)} \frac{P_1}{z}\,\mathrm{d}y - \oint_{\gamma(h)} \frac{P_2}{z}\,\mathrm{d}x\right) = \int_{\gamma_1(h)} \frac{P_1}{z}\,\mathrm{d}y - \int_{\gamma_1(h)} \frac{P_2}{z}\,\mathrm{d}x \tag{5.1}$$

where $\gamma_1(h)$ is the first quarter of the unperturbed periodic orbit $\gamma(h)$ (see figure 1). To evaluate $\int_{\gamma_1(h)} \frac{P_1}{z}\,\mathrm{d}y$ we change variables and parameters as follows

$$h = \frac{(b+1)\,k}{2}$$
$$y = \sqrt{k}\,u$$
$$k = \frac{-b+1}{2\,c^2 - b - 1}.$$

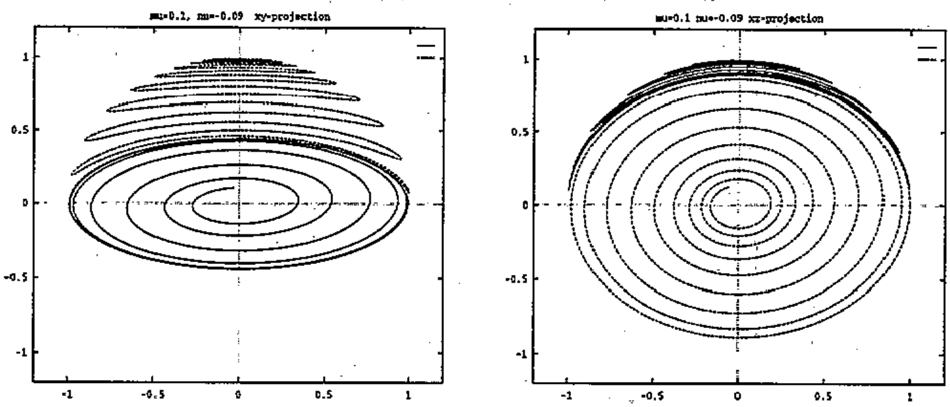

Figure 6. Phase space projections for $\mu = 0.1$, $\nu = -0.09$. The limit cycle surrounds the z-axis, i.e., it is in region II.

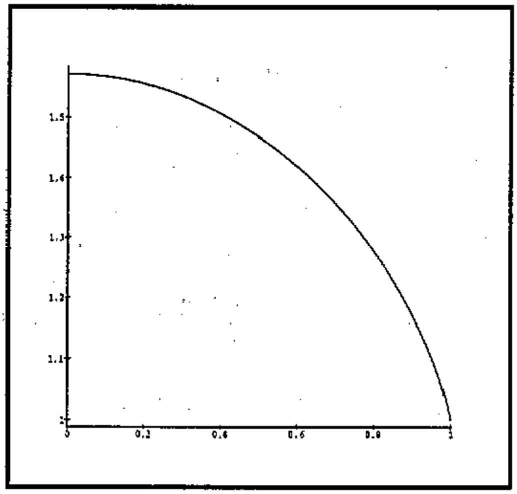

Figure 7. $E(s)$, $s \in [0, 1]$.

The integration is from c to 1, where $0 < c < 1$.

$$\int_{\gamma_1(h)} \frac{P_1}{z}\,\mathrm{d}y = -\nu \int_c^1 \frac{\sqrt{b-1}\sqrt{b+1}\sqrt{2}\sqrt{1-u^2}\sqrt{u^2-c^2}}{\left(-2\,c^2+b+1\right)^{3/2}}\,\mathrm{d}u$$

$$+\,\mu \int_c^1 \frac{\sqrt{1-u^2}\sqrt{2}\sqrt{b+1}\left(b^2-b\right)\sqrt{u^2-c^2}}{\sqrt{b-1}\left(-2\,c^2+b+1\right)^{3/2}}\,\mathrm{d}u$$

$$+\,\mu \int_c^1 \frac{\sqrt{1-u^2}\sqrt{2}\sqrt{b+1}c^2\,(b-1)^{3/2}}{\sqrt{u^2-c^2}\left(-2\,c^2+b+1\right)^{3/2}}\,\mathrm{d}u.$$

The first two integrals can be expressed in terms of complete Elliptic functions using [BF71 formulae 218.11, 361.01]† , the third one using [BF71, formulae 218.09, 310.02]. Writing

† Formula 361.01 contains a misprint in the first edition that has been corrected in the second edition.

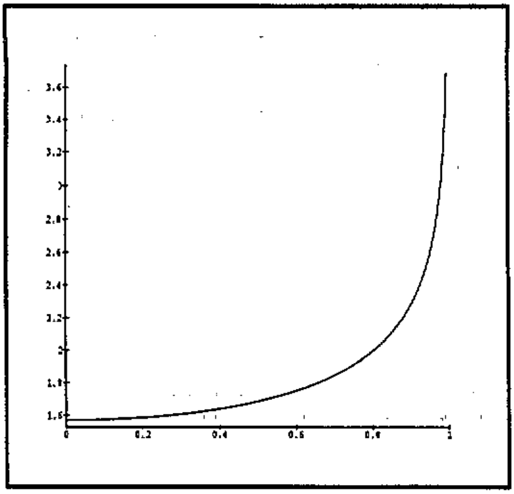

Figure 8. $K(s)$, $s \in [0, 1]$.

$c^2 = 1 - s^2$, which is equivalent to (3.10), this gives

$$\int_{\gamma_1(h)} \frac{P_1}{z}\,\mathrm{d}y = -\nu \frac{\sqrt{b-1}\sqrt{b+1}\sqrt{2}\left(\left(2-s^2\right)E(s) - \left(2-2s^2\right)K(s)\right)}{3\left(2s^2-1+b\right)^{3/2}} + \mu \frac{\sqrt{2}\sqrt{b+1}\left(b^2-b\right)\left(\left(2-s^2\right)E(s)-\left(2-2s^2\right)K(s)\right)}{3\sqrt{b-1}\left(2s^2-1+b\right)^{3/2}} - \mu \frac{\sqrt{2}\sqrt{b+1}\left(1-s^2\right)(b-1)^{3/2}(-K(s)+E(s))}{\left(2s^2-1+b\right)^{3/2}}.$$

To evaluate $-\int_{\gamma_1(h)} \frac{P_2}{z}\,\mathrm{d}x$ we change variables and parameters as follows

$$h = \frac{(b+1)k}{2}$$
$$x = \frac{\sqrt{2}\sqrt{(b+1)(1-k)}}{2}\,v$$
$$k = \frac{-a^2+a^2b}{2-a^2+a^2b}$$

where $a > 1$. Then

$$-\int_{\gamma_1(h)} \frac{P_2}{z}\,\mathrm{d}x = -\nu \int_0^1 \frac{\sqrt{b-1}\sqrt{b+1}\sqrt{2}\sqrt{1-v^2}\sqrt{a^2-v^2}}{\left(2-a^2+a^2b\right)^{3/2}}\,\mathrm{d}v + \mu \int_0^1 \frac{\sqrt{2}\sqrt{b+1}b\sqrt{1-v^2}\sqrt{b-1}\sqrt{a^2-v^2}}{\left(2-a^2+a^2b\right)^{3/2}}\,\mathrm{d}v - \mu \int_0^1 \frac{\sqrt{2}(b+1)^{3/2}\sqrt{b-1}\sqrt{a^2-v^2}}{\sqrt{1-v^2}\left(2-a^2+a^2b\right)^{3/2}}\,\mathrm{d}v.$$

We use [BF71, formulae 219.11, 361.03] to evaluate the first two integrals, and [BF71, formulae 219.01] to evaluate the last one. Writing $a = s^{-1}$, which is equivalent to (3.10), we obtain

$$-\int_{\gamma_1(h)} \frac{P_2}{z}\,\mathrm{d}x = -\nu\,\frac{\sqrt{b-1}\sqrt{b+1}\sqrt{2}\left(E(s)+E(s)s^2-K(s)+K(s)s^2\right)}{3\left(2s^2-1+b\right)^{3/2}} + \mu\,\frac{\sqrt{2}\sqrt{b+1}b\sqrt{b-1}\left(E(s)+E(s)s^2-K(s)+K(s)s^2\right)}{3\left(2s^2-1+b\right)^{3/2}} - \mu\,\frac{\sqrt{2}\,(b+1)^{3/2}\sqrt{b-1}s^2E(s)}{\left(2s^2-1+b\right)^{3/2}}.$$

It follows that

$$I_1(s) = 4\,\frac{\sqrt{2}\sqrt{b+1}\sqrt{b-1}\left(-2\,E(s)s^2+K(s)s^2-K(s)+E(s)\right)}{\left(2s^2-1+b\right)^{3/2}}$$

$$I_2(s) = -4\,\frac{\sqrt{2}\sqrt{b+1}\sqrt{b-1}\left(E(s)-K(s)+K(s)s^2\right)}{\left(2s^2-1+b\right)^{3/2}}.$$

From this (3.9) follows. □

Proof of lemma 3.2 From the expansion

$$K(s) = \frac{\pi}{2}\left(1+\frac{s^2}{4}+\frac{9s^4}{64}+\mathcal{O}(s^6)\right)$$

$$E(s) = \frac{\pi}{2}\left(1-\frac{s^2}{4}-\frac{3s^4}{64}+\mathcal{O}(s^6)\right)$$

see [BF71, formulae 900.00, 900.07], it follows that $Q(s) = \frac{1}{3}\left(1+\frac{s^2}{2}+\mathcal{O}(s^4)\right)$. As $K(s)$ has a logarithmic singularity at $s=1$ (see [BF71, 900.06]) and $E(1)=1$, it follows that $Q(1)=1$.

We prove that $Q(s) > \frac{1}{3}$ for $s \in (0,1)$ by showing that there does not exist a point $s_1 \in (0,1)$ with the property that $Q(s_1) = \frac{1}{3}$ and $Q(s) > \frac{1}{3}$ for $s \in (0, s_1)$. If such a point s_1 did exist then it follows from (3.11) that at $s = s_1$

$$\frac{\mathrm{d}Q}{\mathrm{d}s}(s_1) = -\frac{2s_1}{-3+3{s_1}^2}$$

but this is positive on the interval (0, 1). Hence s_1 does not exist and $Q \geqslant \frac{1}{3}$ on the interval [0, 1]. Similarly, let $(0, s_2)$ be an interval such that $Q(s) < 1$ on $(0, s_2)$ and $Q(s_2) = 1$. From (3.11) it follows that

$$\frac{\mathrm{d}Q}{\mathrm{d}s}(s_2) = -\frac{2}{s_2}$$

which is negative for $s_2 \in (0,1)$, showing that $Q \leqslant 1$ on the interval [0, 1]. □

Proof of lemma 3.3. We let $f(s) = R(s) + s - 1$. Then $f(0) = f(1) = 0$ and

$$f(s) = s + \mathcal{O}(s^2)$$

follows from [BF71, 900.00, 900.07]. Let $s_1 \in (0,1)$ be such that $f(s) > 0$ on the interval $(0, s_1)$ and $f(s_1) = 0$. From (3.11) we derive that

$$\frac{\mathrm{d}f}{\mathrm{d}s}(s_1) = \frac{1-s_1}{1+s_1}$$

which is positive for $s_1 \in (0,1)$, contradicting the choice of s_1. This completes the proof. □

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References

[BF71] Byrd P F and Friedman M D 1971 *Handbook of Elliptic integrals* 2nd edn revised (Berlin: Springer)

[CK94] Clune T and Knobloch E 1994 Pattern selection in three-dimensionnal magnetoconvection *Physica* **74D** 151–76

[GSS88] Golubitsky M, Stewart I and Scraeffer D S 1988 *Singularities and Groups in Bifurcation Theory* (New York: Springer)

[Mil72] W Miller 1972 Symmetry groups and their application *Pure and Applied Mathematics* vol 2 (New York: Academic)

[SK91] Silber M and Knobloch E 1991 Hopf bifurcation on asquare lattice *Nonlinearity* 4 1063–107

[SRK92] Silber M, Riecke H and Kramer L 1992 Symmetry-breaking Hopf bifurcation in anisotropic systems *Physica* **61D** 260–78

[Swi88] Swift J W 1988 Hopf bifurcation with the symmetry of the square *Nonlinearity* **1** 333–77