Discrete Applied Mathematics 35 (1992) 73–79
North-Holland

Note

A greedy reduction algorithm for setup optimization

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Received 2 October 1989
Revised 31 October 1990

Abstract

Faigle, U. and R. Schrader, A greedy reduction algorithm for setup optimization, Discrete Applied Mathematics 35 (1992) 73–79.

A reduction algorithm for setup optimization in general ordered sets is proposed. Moreover, the class of weakly cycle-free orders is introduced. All orders in this class are Dilworth optimal. Cycle-free orders and bipartite Dilworth optimal orders are proper subclasses. The algorithm allows greedy setup optimization in cycle-free orders and coincides with the algorithm of Syslo et al. [5] in the class of bipartite orders.

1. Introduction

The setup problem for a (partial) order P on the ground set E is the following scheduling problem: find a permutation of the elements of E which respects the precedences imposed by P and juxtaposes as few incomparable pairs of elements, relative to P, as possible.

The adjacent incomparable pairs of elements in such a linear extension of P divide the permutation into a collection of chains of P that cover E. Hence the width $w(P)$

* Supported by Sonderforschungsbereich 303 (DFG), University of Bonn.

(minus one) yields a lower bound for the setup number of P. A natural problem, therefore, is to decide, whether P is *Dilworth optimal*, i.e., whether the width bound yields the exact setup number. As Bouchitte and Habib [1] have shown, this problem (and hence the general setup problem!) is NP-hard.

Nevertheless, interesting classes of Dilworth optimal orders exist. Duffus et al. [3] have exhibited the cycle-free orders to form such a class and have derived a polynomial-time algorithm to solve the setup problem for this class. Syslo et al. [5] have provided a greedy-type algorithm which recognizes all bipartite Dilworth optimal orders and solves the associated setup problem. These two classes have a structural property in common: each member in either class possesses a simplicial element. The algorithm of Syslo et al. explicitly makes use of that fact while, interestingly, the algorithm of Duffus et al. does not.

It is the purpose of this note to point out that the notion of a simplicial element may be strengthened in order to achieve a common generalization of cycle-free and bipartite Dilworth optimal orders. In particular, we propose a greedy-type reduction algorithm which may be applied to arbitrary orders and attempts a simplicial decomposition of these. If the decomposition is completely successful, the order is seen to be Dilworth optimal and its setup problem is solved.

The main result is presented in Section 2. Cycle-free orders are discussed in Section 3. Section 4 outlines the case of bipartite Dilworth optimal orders and closes with further remarks and open problems.

2. Basic definitions and properties

Let $P=(E,\leq)$ be a finite (partially) ordered set. For each $x\in E$, we denote by

$$N(x)=\{y\in E: y>x\}\cup\{y\in E: y<x\}$$

its set of *neighbors* (relative to the comparability graph of P). We furthermore write

$$N[x]=\{x\}\cup N(x).$$

The element $x\in E$ is said to be *simplicial* (relative to P) if $N[x]$ is a *chain* in P, i.e., if all elements in $N[x]$ are pairwise comparable.

Recall that a subset $A\subseteq E$ of pairwise incomparable elements in P is an *antichain* and that the *width* of P is defined as the number

$$w(P)=\max\{|A|: A \text{ antichain in } P\}.$$

According to the theorem of Dilworth [2] is $w(P)$ equal to the minimal number of chains needed in order to cover the ground set E. A fundamental observation is formulated in the following lemma.

Lemma 1. *Let $x\in E$ be simplicial in P. Then $w(P\setminus N[x])=w(P)-1$.*

Consider a fixed simplicial element $x \in P$. By x^+ we denote the smallest element $y > x$ such that y has at least two incomparable lower neighbours in P. Similarly, we write x^- for the largest element $y < x$ such that y has at least two incomparable upper neighbours in P. (Note that neither x^+ nor x^- need to exist.)

A *primal twin* of x is an element $z \notin N[x]$ such that $z < x^+$ and for every $y > z$,

$$y \geq x^+.$$

A *dual twin* of x is an element $z \notin N[x]$ such that $z > x^-$ and for every $y < z$,

$$y \leq x^-.$$

We say that $x \in E$ is *p-simplicial* provided x is simplicial in P and has a primal twin unless x^+ does not exist. Dually, $x \in E$ is *d-simplicial* provided x is simplicial in P and has a dual twin unless x^- does not exist.

With each p-simplicial element x we associate a chain

$$C_p(x) = \begin{cases} \{y \in N[x] : y < x^+\}, & \text{if } x^+ \text{ exists,} \\ N[x], & \text{otherwise.} \end{cases}$$

Dually, we associate with each d-simplicial element x the chain

$$C_d(x) = \begin{cases} \{y \in N[x] : y > x^-\}, & \text{if } x^- \text{ exists,} \\ N[x], & \text{otherwise.} \end{cases}$$

Our next result is the analogue of Lemma 1.

Lemma 2. (a) *If $x \in P$ is p-simplicial, then* $w(P \setminus C_p(x)) = w(P) - 1$.
(b) *If $x \in P$ is d-simplicial, then* $w(P \setminus C_d(x)) = w(P) - 1$.

Proof. It suffices to prove (a). W.l.o.g., we assume that x^+ exists and, hence, that x has a primal twin z.

Suppose the lemma is false and $P \setminus C_p(x)$ contains an antichain A of size $|A| = w(P)$. Because $N[x]$ is a chain and $w(P \setminus N[x]) = w(P) - 1$, there must be exactly one element $a \in A$ with the property $a \geq x^+$. The crucial observation now is the definition of a primal twin: every $y > z$ must be comparable with this element a. Hence the set $A' = (A \setminus \{a\}) \cup \{z\}$ is also an antichain in $P \setminus C_d(x)$. Thus we have found an antichain

$$A'' = A' \cup \{x\}$$

of size $|A''| = w(P) + 1$ in P, a contradiction, which proves the lemma. □

Based on Lemma 2, we obtain a greedy-type reduction algorithm for obtaining a chain cover of P with $w(P)$ chains by successively identifying primal- or dual-simplicial elements and removing the associated chains:

S-algorithm.

Input: Ordered set P;

Output: A suborder Q of P and a number w such that Q has no p- or d-simplicial element and $w(P) = w + w(Q)$.

(1) $Q \leftarrow P$;
$w \leftarrow 0$;
(2) **if** Q has a p-simplicial element x
then $Q \leftarrow (Q \setminus C_p(x)) w \leftarrow (w+1)$ **goto** (2)
endif;
(3) **if** Q has a d-simplicial element x
then $Q \leftarrow (Q \setminus C_d(x)) w \leftarrow (w+1)$ **goto** (2)
endif;
(4) **stop;**

Recall that a *linear extension* of P is a permutation $L = x_1 x_2 \dots x_n$ of the elements in E such that $x_i < x_j$ in P implies $i < j$. The *setup problem* for P consists in identifying a linear extension with the smallest possible number of incomparable adjacent pairs of elements.

Lemma 3. (a) *If $x \in E$ is p-simplicial and L is a linear extension of $P \setminus C_p(x)$, then $C_p(x) \oplus L$ is a linear extension of P.*

(b) *If $x \in E$ is d-simplicial and L is a linear extension of $P \setminus C_d(x)$, then $L \oplus C_d(x)$ is a linear extension of P.*

(Notation: $U \oplus V$ is the concatenation of U and V.)

Let us call an order P *weakly cycle-free* if the S-algorithm may be carried out in such a way that it produces the suborder $Q = \emptyset$ (the terminology is motivated in the next section).

Proposition 4. *If P is weakly cycle-free and $E \neq \emptyset$, then the S-algorithm may be carried out in such a way that a linear extension L of P is produced with at most $w(P) - 1$ incomparable adjacent pairs.*

Note that the linear extension in Proposition 4 must be optimal because each linear extension of an order P contains at least $w(P) - 1$ incomparable adjacent pairs.

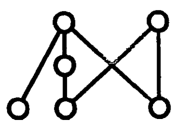

Fig. 1.

Hence the class of weakly cycle-free orders provides examples of orders where the "setup number" equals the "Dilworth number" (minus one). In this sense, weakly cycle-free orders are ***Dilworth optimal***. The converse, however, does not hold (see, e.g., Fig. 2).

3. Cycle-free orders

We will now exhibit the class of cycle-free orders as a proper subclass of weakly cycle-free orders. Here we call the order P ***cycle-free*** if its comparability graph $G(P)$ is ***chordal***, i.e., if $G(P)$ contains no cycle of length 4 or more as a vertex-induced subgraph.

A ***trampoline*** in a graph G is an induced subgraph on two equicardinal sets $U=\{u_1,\dots,u_n\}$ and $W=\{w_1,\dots,w_n\}$ of vertices such that the restriction of G to U is a complete graph, W is an independent set and, for each i and j, w_i is adjacent to u_j if and only if $i=j$ or $i=j+1$ (mod n).

It is easily verified that a comparability graph of an ordered set cannot contain any trampoline. Hence, by the characterization of Farber [4], the comparability graph $G(P)$ of the cycle-free order P is ***strongly chordal*** and, in particular, contains a *simple* vertex x, which, by definition, has the property that for all $u,v\in N(x)$,

$$N[u]\subseteq N[v] \quad \text{or} \quad N[v]\subseteq N[u].$$

Lemma 5. *Let P be an arbitrary order on E and $x\in E$ a simple element relative to the comparability graph $G(P)$ of P. Then x is p-simplicial or d-simplicial in P.*

Proof. Apparently, each simple element in P must also be simplicial. W.l.o.g., we now assume that both x^+ and x^- exist and that $N[x^-]\subseteq N[x^+]$. We claim that x has a primal twin.

Indeed, choose any element $z\notin N[x]$ such that $x^-<z<x^+$ and for all $y\geq z$, $y=z$ or $y\geq x^+$. Then each upper neighbour u of z satisfies $u>x^-$ and therefore $u\in N[x^-]\subseteq N[x^+]$. Because $u<x^+$ is impossible by the choice of z, $u\geq x^+$ must hold. □

In view of Lemma 5, each cycle-free order is, in particular, weakly cycle-free. Hence the S-algorithm of the previous section will produce an optimal linear exten-

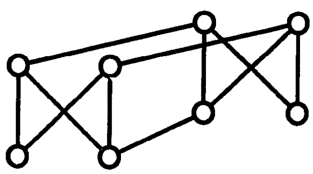

Fig. 2.

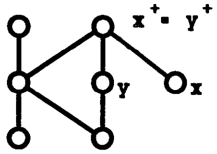

Fig. 3.

sion when applied to a cycle-free order (Proposition 4). Note that, in this case, the choice of p- or d-simplicial elements in the S-algorithm may be carried out "greedily" as every suborder of a cycle-free order is again cycle-free.

We remark that a different approach for the computation of optimal linear extensions of cycle-free orders was taken by Duffus et al. [3]. In contrast to our algorithm, their method first constructs a covering of the ground set E with $w(P)$ chains relative to the cycle-free order P. From such a chain covering then an optimal linear extension is extracted in an iterative procedure. Moreover, the correctness proof of Duffus et al. relies on P being cycle-free, while our algorithm may also be successful in the presence of cycles (see Fig. 1).

4. General remarks

Recall that the order P is said to be Dilworth optimal if it admits a linear extension with exactly $w(P)-1$ incomparable adjacent pairs of elements. It was observed by Syslo et al. [5] that each bipartite Dilworth optimal order contains a simplicial element. Since, by definition, bipartite orders contain no chain with three elements, each simplicial element of a bipartite order is, in particular, p- or d-simplicial. In fact, it is straightforward to verify that a bipartite order is Dilworth optimal if and only if it is weakly cycle-free. Moreover, the S-algorithm may be applied in a greedy fashion to recognize such bipartite orders. In this sense, our S-algorithm may be viewed as a proper extension of the algorithm of Syslo et al. [5].

The problem of recognizing Dilworth optimal orders that are not bipartite is NP-complete [1]. The example in Fig. 2 shows that such orders need not have simplicial elements at all.

An interesting open problem concerns the complexity status of recognizing general weakly cycle-free orders. The difficulty there lies in the fact that the S-algorithm may get struck if the p- or d-simplicial elements are chosen in the "wrong" order (see Fig. 3). The S-algorithm successfully decomposes P if it begins with y but NOT if it selects x first.

We mention in closing that the class of weakly cycle-free orders cannot be characterized by forbidden (induced) suborders. Indeed, if this were the case, each suborder of a weakly cycle-free order would possess a simplicial element. As is well known, however, the latter property implies cycle-freeness in the strict sense.

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