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SINGULAR NONLINEAR H_∞ OPTIMAL CONTROL PROBLEM

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SUMMARY

The theory of nonlinear H_∞ optimal control for affine nonlinear systems is extended to the more general context of singular H_∞ optimal control of nonlinear systems using ideas from the linear H_∞ theory. Our approach yields under certain assumptions a necessary and sufficient condition for solvability of the state feedback singular H_∞ control problem. The resulting state feedback is then used to construct a dynamic compensator solving the nonlinear output feedback H_∞ control problem by applying the certainty equivalence principle.

KEY WORDS nonlinear systems; L_2-gain; Hamilton–Jacobi equations; robust stability

1. INTRODUCTION

In this paper we study a singular nonlinear H_∞ problem. This kind of problem naturally arises when studying certain robustness problems such as parameter uncertainty and multiplicative uncertainty as will be shown in Sections 6 and 7 (generalizing the linear case [7,15]). We consider systems of the form

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u_1 + g_2(x)u_2 + k(x)d_1 \\ y &= c(x) + d_2 \\ z &= \begin{pmatrix} h(x) \\ u_1 \end{pmatrix} \end{aligned} \tag{1}$$

where $u = (u_1\ u_2) \in \mathbb{R}^m$, $u_1 \in \mathbb{R}^{m_1}$, $u_2 \in \mathbb{R}^{m-m_1}$, $d_1 \in \mathbb{R}^q$, $d_2 \in \mathbb{R}^l$, $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^p$. Furthermore $x = (x_1, \ldots, x_n)$ are local coordinates for a smooth state-space manifold M. We assume that f, g_1, g_2, k, c and h are C^r-functions ($r \leqslant 2$). We also assume that there exists an equilibrium $x_0 \in M$, without loss of generality $x_0 = 0$. Hence $f(0) = 0$, furthermore we assume $c(0) = 0$ and $h(0) = 0$.

Our aim is to find conditions under which there exists a feedback such that the L_2-gain of the resulting closed-loop system from disturbance d_1, d_2 to output z is less than (or equal to) a certain bound γ. For the case $m_1 = m$ this is just the regular suboptimal H_∞ control problem studied in References 17, 18, 10 and 1. Most of the results obtained in the present paper are extensions of results about the regular nonlinear H_∞ problem obtained in these papers. For linear systems the singular H_∞ problem has been studied extensively. One approach to the state feedback problem is given by Petersen, Zhou and Khargonekar[13,11,23]. Another approach is

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discussed by Trentelman and Stoorvogel.[14,15] We will extend the first approach to nonlinear systems. After that we will use such a feedback to construct an affine measurement controller of the form

$$\begin{aligned} \dot{\xi} &= k(\xi) + p(\xi)y \\ u &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} q_1(\xi) \\ q_2(\xi) \end{pmatrix} = q(\xi) \end{aligned} \tag{2}$$

with $k(0) = 0$ and $q(0) = 0$, such that the closed-loop system (1), (2) has L_2-gain less than a certain bound γ.

A useful method for finding such a compensator is the worst-case certainty equivalence principle. This principle consists in solving first the state feedback problem and then replacing the state x by the state estimation corresponding to the worst possible disturbance which corresponds to the applied input and the resulting output.

Obtaining convenient conditions for this certainty equivalence principle to hold is a current research topic. In a recent article of Didinsky *et al.*[5] necessary conditions for a certainty equivalence controller to exist are given (see also References 3 and 2). this worst-case certainty equivalence principle will be used for systems of the form (1) by using a singular H_∞ state feedback as obtained in Section 4.

This note is further organized as follows. In the next section we shall briefly recall some results about the L_2-gain of nonlinear systems. In the third section we shall consider the disturbance attenuation approach used to solve the general (singular) H_∞ problem for linear systems. In the fourth section the singular nonlinear H_∞ state feedback problem will be considered. In Section 5 we will use the solution to the state feedback problem to construct a controller of the form (2) for the system (1). In Sections 6 and 7 we will use the obtained results to consider some problems of robust control of nonlinear systems with parameter uncertainty or multiplicative perturbations. A third perturbation model, the numerator–denominator perturbation model or coprime factor uncertainty, is for nonlinear systems considered in Reference 20.

2. THE L_2-GAIN OF NONLINEAR SYSTEMS

In this section we will first consider (mainly following Reference 19) the L_2-gain for the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3}$$

Definition 1

Let γ be a fixed nonnegative constant. The system (3) is said to have L_2-gain less than or equal to γ if for all $x \in M$ there exists a constant $K(x)$, $0 \leqslant K(x) < \infty$, with $K(0) = 0$, such that

$$\int_0^T \|y(t)\|^2 \, dt \leqslant \gamma^2 \int_0^T \|u(t)\|^2 \, dt + K(x) \tag{4}$$

for all $T \geqslant 0$ and all $u \in L_2(0, T)$ with $y(t) = h(\varphi(t, 0, x, u))$ denoting the output of (3) resulting from u for initial state $x(0) = x$.

The system has L_2-gain less than γ if there exists some $0 \leqslant \bar{\gamma} < \gamma$ such that the system (3) has L_2-gain less than or equal to $\bar{\gamma}$. The L_2-gain is equal to γ if it has L_2-gain less than or equal to γ and not less than γ.

Without proof we recall some fundamental results originating from Willems;[22] see Van der Schaft.[18,19]

Theorem 2

The system (3) has L_2-gain less than or equal to γ if and only if there exists a solution V: $M \to \mathbb{R}^+$ to the *integral dissipation inequality*

$$V(x(t_1)) - V(x(t_0)) \leqslant \frac{1}{2}\int_{t_0}^{t_1} (\gamma^2 \| u(t) \|^2 - \| y(t) \|^2)\, dt, \qquad V(0) = 0 \tag{5}$$

for all $t_1 \geqslant t_0$ and all $u \in L_2(t_0, t_1)$, where $x(t_1) = \phi(t_1, t_0, x(t_0), u)$.

Further, there exists a nonnegative C^1-solution to the integral dissipation inequality (5) if and only if there exists a nonnegative C^1-solution to the *differential dissipation inequality*

$$V_x(x) f(x) + V_x(x) g(x) u \leqslant \tfrac{1}{2}\gamma^2 \| u \|^2 - \tfrac{1}{2} \| y \|^2, \qquad V(0) = 0 \tag{6}$$

for all $u \in \mathbb{R}^m$, with $y = h(x)$.

And there exists a nonnegative C^1-solution to (6) if and only if there exists a nonnegative C^1-solution to the *Hamilton–Jacobi inequality*

$$V_x(x) f(x) + \frac{1}{2}\frac{1}{\gamma^2} V_x(x) g(x) g^{\mathrm{T}}(x) V_x^{\mathrm{T}}(x) + \frac{1}{2} h^{\mathrm{T}}(x) h(x) \leqslant 0, \qquad V(0) = 0 \tag{7}$$

for all $x \in M$.

Also some kind of stability can be concluded from the solvability of the Hamilton–Jacobi equation.

Definition 3

The system (3) is called *zero-state observable* if for any trajectory such that $u(t) \equiv 0$, $y(t) \equiv 0$ implies $x(t) \equiv 0$.

Theorem 4[6,18]

Assume (3) is zero-state observable. Suppose there exists a smooth solution $V \geqslant 0$ to either (5), (6) or (7). Then $V(x) > 0$, $x \neq 0$, and the free system $\dot{x} = f(x)$ is locally asymptotically stable. Furthermore, assume that V is proper (i.e., for each $c > 0$ the set $\{x \in M \mid 0 \leqslant V(x) \leqslant c\}$ is compact), then $\dot{x} = f(x)$ is globally asymptotically stable.

In this note we first consider the singular state feedback H_∞ problem for nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + g_1(x) u_1 + g_2(x) u_2 + k(x) d_1 \\ y &= x \\ z &= \begin{pmatrix} h(x) \\ u_1 \end{pmatrix} \end{aligned} \tag{8}$$

We define this problem as follows.

Definition 5

(Singular nonlinear state feedback H_∞ optimal control problem.) Find, if existing, the smallest value $\gamma^* \geq 0$ such that for any $\gamma > \gamma^*$ there exists a state feedback

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} l_1(x) \\ l_2(x) \end{pmatrix} = l(x); \qquad l(0) = 0 \tag{9}$$

such that the L_2-gain of the closed-loop system (8) and (9) from d_1 to z less than or equal to γ and the origin is local asymptotically stable.

The definition of the measurement feedback H_∞ problem is then

Definition 6

(Singular nonlinear measurement feedback H_∞ optimal control problem). Find, if existing, the smallest value $\gamma^* \geq 0$ such that for any $\gamma > \gamma^*$ there exists a compensator

$$\begin{aligned} \dot{\xi} &= k(\xi) + p(\xi)y & k(0) &= 0 \\ u &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} q_1(\xi) \\ q_2(\xi) \end{pmatrix} = q(\xi) & q(0) &= 0 \end{aligned} \tag{10}$$

such that the L_2-gain of the closed-loop system (1) and (10) from d_1, d_2 to z is less than or equal to γ and the origin is local asymptotically stable.

As in Theorem 4 stability may be deduced from the solvability of the Hamilton–Jacobi equation under an extra assumption made on the system as we shall see in Section 4.

Furthermore we consider the linearization of (8) around the origin, denoted as

$$\begin{aligned} \dot{\bar{x}} &= F\bar{x} + G_1\bar{u}_1 + G_2\bar{u}_2 + K\bar{d}_1 \\ \bar{y} &= x \\ \bar{z} &= \begin{pmatrix} H\bar{x} \\ \bar{u}_1 \end{pmatrix} \end{aligned} \tag{11}$$

where $\bar{u} = (\bar{u}_1, \bar{u}_2) \in \mathbb{R}^m$, $\bar{u}_1 \in \mathbb{R}^{m_1}$, $\bar{u}_1 \in \mathbb{R}^{m-m_1}$, $\bar{x} \in \mathbb{R}^n$, $\bar{d}_1 \in \mathbb{R}^q$, $\bar{z} \in \mathbb{R}^p$ and the matrices F, G_1, G_2, K and H are defined as:

$$F = \frac{\partial f}{\partial x}(0); \qquad G_1 = g_1(0); \qquad G_2 = g_2(0); \qquad K = k(0); \qquad H = \frac{\partial h}{\partial x}(0) \tag{12}$$

We look at the corresponding H_∞ control problem for this system (11). Hence we search for a stabilizing state feedback

$$\bar{u} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = \begin{pmatrix} L_1\bar{x} \\ L_2\bar{x} \end{pmatrix} = L\bar{x} \tag{13}$$

such that the H_∞ norm from $\bar{d}_1$ to $\bar{z}$ is smaller than some value γ.

This problem is a special case of the general linear state feedback H_∞ control problem considered by Zhou and Khargonekar[23] for the system:

$$\begin{aligned} \dot{\bar{x}} &= F\bar{x} + G\bar{u} + K\bar{d}_1 \\ \bar{z} &= C\bar{z} + D_1\bar{u} + D_2\bar{d}_1 \end{aligned} \tag{14}$$

where in our situation $\bar{z} \in \mathbb{R}^{(p+m_1)}$ and

$$G = (G_1 \ \ G_2); \qquad C = \begin{pmatrix} H \\ 0 \end{pmatrix}; \qquad D_1 = \begin{pmatrix} 0 & 0 \\ I_{m_1} & 0 \end{pmatrix}; \qquad D_2 = 0 \tag{15}$$

For the case that (F, G) is stabilizable Petersen, Zhou and Khargonekar[23,13] solved this problem. This solution will be recalled in the next section.

3. LINEAR DISTURBANCE ATTENUATION

We consider linear systems of the form (11).

This system is said to satisfy the *ARE* (*Algebraic Riccati Equation*) *with constant* γ if, for arbitrary $Q > 0$, there exists an $\varepsilon > 0$ such that the Riccati equation ($G = (G_1 \ \ G_2)$)

$$F^{\mathrm{T}}P + PF + \frac{1}{\gamma^2}PKK^{\mathrm{T}}P - PG\begin{pmatrix} I_{m_1} & 0 \\ 0 & 0 \end{pmatrix}G^{\mathrm{T}}P - \frac{1}{\varepsilon}PG\begin{pmatrix} 0 & 0 \\ 0 & I_{m-m_1} \end{pmatrix}G^{\mathrm{T}}P + H^{\mathrm{T}}H + \varepsilon Q = 0 \tag{16}$$

has a positive-definite solution P.

The following lemma shows that the existence of a positive-definite solution P of ARE (16) does not depend on the choice of Q.

Lemma 7[13]

Suppose there exists a positive-definite matrix $Q \in \mathbb{R}^{n\times n}$ and a constant $\varepsilon > 0$ such that the algebraic Riccati equation (16) has a positive-definite solution. Then given any positive-definite $\tilde{Q} \in \mathbb{R}^{n\times n}$ there exists a constant $\varepsilon^* > 0$ such that the ARE (16) with Q replaced by $\tilde{Q}$ has a positive-definite solution for all $\varepsilon \in (0, \varepsilon^*]$.

Now we have the following connection with the H_∞ control problem (see Zhou and Khargonekar.[23])

Theorem 8

Consider the system (11). Let $\gamma > 0$. Then there exists a linear feedback of the form (13) such that $F + GL$ is stable and

$$\left\| \begin{pmatrix} H \\ L_1 \end{pmatrix} (sI - F - GL)^{-1}K \right\|_\infty < \gamma \tag{17}$$

if and only if for any $Q > 0$ there exists an $\varepsilon > 0$ such that the algebraic Riccati equation (16) has a positive-definite solution P.

Moreover if $P>0$ is a solution of the ARE (16) for some $Q>0$ and constant $\varepsilon>0$ then if we choose

$$L=-\begin{pmatrix} I_{m_1} & 0 \\ 0 & \dfrac{1}{2\varepsilon} I_{m-m_1} \end{pmatrix} G^{\mathrm{T}}P \tag{18}$$

the closed-loop system $F+GL$ is stable and (17) holds.

Furthermore if there exists a positive-definite solution P of (16), then there also exists a stabilizing solution of (16) (see Reference 11).

Theorem 9

Suppose for $Q>0$ there exists an $\varepsilon^*>0$ such that (16) has a positive-definite solution P_ε for every $\varepsilon \in (0, \varepsilon^*]$. Then for every $\varepsilon \in (0, \varepsilon^*)$ there also exists a stabilizing solution $\tilde{P}_\varepsilon>0$ for (16), i.e., there exists a solution $\tilde{P}_\varepsilon>0$ for which also holds that

$$F+\frac{1}{\gamma^2}KK^{\mathrm{T}}\tilde{P}_\varepsilon - G\begin{pmatrix} I_{m_1} & 0 \\ 0 & \dfrac{1}{\varepsilon} I_{m-m_1} \end{pmatrix} G^{\mathrm{T}}\tilde{P}_\varepsilon \tag{19}$$

is asymptotically stable.

4. SINGULAR NONLINEAR STATE FEEDBACK H_∞ CONTROL

Now consider the singular state feedback H_∞ optimal control problem for an affine nonlinear systems of the form

$$\Sigma: \begin{cases} \dot{x}=f(x)+g_1(x)u_1+g_2(x)u_2+k(x)d_1 \\ y=x \\ z=\begin{pmatrix} h(x) \\ u_1 \end{pmatrix} \end{cases} \tag{20}$$

for which we seek a nonlinear static feedback

$$u=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} l_1(x) \\ l_2(x) \end{pmatrix}=l(x) \qquad l(0)=0 \tag{21}$$

such that the closed-loop system (20), (21) has L_2-gain less than (or equal to) γ, i.e., cf. (4), for every $x \in M$ there exists a constant $K(x)$, $0 \leqslant K(x)<\infty$, with $K(0)=0$, such that

$$\int_0^T (\| l_1(x(t))\|^2+\|h(x(t))\|^2)\,\mathrm{d}t \leqslant \gamma^2\int_0^T \|d_1(t)\|^2\,\mathrm{d}t+K(x) \tag{22}$$

for all $d_1 \in L_2[0,T]$ and all $T \geqslant 0$, with $x(t)$ denoting the response of (20), (21) for initial condition $x(0)=x$.

We start by making the following assumption.

Assumption 1

The L_2-gain from d_1 to l_2 is finite, i.e., there exists a constant $N>0$ such that for all $x \in M$ there exists a constant $\tilde{K}(x)$, $0 \leq \tilde{K}(x) < \infty$, with $\tilde{K}(0)=0$ such that

$$\int_0^T \| l_2(x(t)) \|^2 \, dt \leq N \int_0^T \| d_1(t) \|^2 \, dt + \tilde{K}(x) \tag{23}$$

for all $T>0$ and all $d_1 \in L_2[0,T]$, where $x(t)$ is the solution of the state equation of the closed loop system Σ, (21).

We will prove that the feedback (21) also leads to a closed loop system with L_2-gain less than γ when applied to the system:

$$\bar{\Sigma}: \begin{cases} \dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 + k(x)d_1 \\ y = x \\ \bar{z} = \begin{pmatrix} h(x) \\ u_1 \\ \sqrt{\varepsilon}u_2 \end{pmatrix} \end{cases} \tag{24}$$

for ε sufficiently small and vice versa.

Theorem 10

Assume we have a feedback of the form (21). Then (ii) implies (i) and under Assumption 1 also equivalence of (i) and (ii) holds:

(i) The closed loop system Σ with static state feedback (21) has L_2-gain less than γ.
(ii) For ε sufficiently small the closed-loop system $\bar{\Sigma}$ with static state feedback (21) has L_2-gain less than γ.

Proof. (i)$\Rightarrow$(ii) By Assumption 1 there exists a constant $N>0$ such that for all $x \in M$ there exists a constant $\tilde{K}(x)$, $0 \leq \tilde{K}(x) < \infty$, with $\tilde{K}(0)=0$ such that

$$\int_0^T \| l_2(x(t)) \|^2 \, dt \leq N \int_0^T \| d_1(t) \|^2 \, dt + \tilde{K}(x) \tag{25}$$

for all $T>0$ and all $d_1 \in L_2[0,T]$.

From the Definition 5 we know that there exists a constant $\delta>0$ such that for every $x \in M$ there exists a constant $K(x)$, $0 \leq K(x) < \infty$, with $K(0)=0$, such that

$$\int_0^T (\| l_1(x(t)) \|^2 + \| h(x(t)) \|^2) \, dt \leq (\gamma^2 - \delta) \int_0^T \| d_1(t) \|^2 \, dt + K(x) \tag{26}$$

for all $d_1 \in L_2[0,T]$ and all $T \geq 0$.

Now take $\varepsilon>0$ such that $\varepsilon N < \delta$. Then some $\mu>0$ can be found such that with $\bar{K}(x)$ $(=K(x)+\varepsilon\tilde{K}(x))$

$$\int_0^T (\| h(x(t)) \|^2 + \| l_1(x(t)) \|^2 + \varepsilon \| l_2(x(t)) \|^2) \, dt \leq (\gamma^2 - \mu) \int_0^T \| d_1(t) \|^2 \, dt + \bar{K}(x) \tag{27}$$

for all $T>0$ and all $d_1 \in L_2[0,T]$. Hence $u=l(x)$ combined with $\bar{\Sigma}$ leads to a closed-loop system which has L_2-gain from d_1 to z less than γ.

For proving (ii) ⇒ (i) we note that if $u=l(x)$ solves the suboptimal problem for the system $\bar{\Sigma}$ then it also solves the suboptimal problem for Σ because

$$\int_0^T \|\bar{z}(t)\|^2 \, dt \geq \int_0^T \|z(t)\|^2 \, dt \tag{28}$$

□

Remark 11

For linear systems Assumption 1 necessarily holds for every stabilizing feedback $u=l(x)$ (see Reference 4).

Hence can search for a state feedback, within the set of feedbacks satisfying Assumption 1, which makes the L_2-gain for the system $\bar{\Sigma}$ less than γ. Since $\bar{\Sigma}$ is a regular system we can find a min-max solution for this H_∞ problem.

The prehamiltonian K_γ: $T^*M \times \mathbb{R}^q \times \mathbb{R}^m$ corresponding to this problem is

$$K_\gamma(x,p,d_1,u) = p^T(f(x) + g_1(x)u_1 + g_2(x)u_2 + k(x)d_1) - \tfrac{1}{2}\gamma^2\|d_1\|^2 + \tfrac{1}{2}h^T(x)h(x) + \tfrac{1}{2}\|u_1\|^2 + \tfrac{1}{2}\varepsilon\|u_2\|^2$$

which has saddle point solution:

$$K_\gamma(x,p,d_1,u^*) \leq K_\gamma(x,p,d_1^*,u^*) \leq K_\gamma(x,p,d_1^*,u) \tag{29}$$

for every d_1, u, x and p when we choose

$$d_1^* = \frac{1}{\gamma^2}k^T(x)p; \qquad u_1^* = -g_1^T(x)p; \qquad u_2^* = -\frac{1}{\varepsilon}g_2^T(x)p \tag{30}$$

This leads to the Hamiltonian $H_\gamma(x,p) = K_\gamma(x,p,d_1^*(x,p),u^*(x,p))$ given as

$$H_\gamma(x,p) = p^Tf(x) + \tfrac{1}{2}p^T\left[\frac{1}{\gamma^2}k(x)k^T(x) - g_1(x)g_1^T(x) - \frac{1}{\varepsilon}g_2(x)g_2^T(x)\right]p + \tfrac{1}{2}h^T(x)h(x) \tag{31}$$

and using the theory of differential games we find that (see References 2, 3, and 19)

Theorem 12

Consider the nonlinear system Σ. Let $\gamma>0$. Suppose there exists an C^r $(k \geq r \geq 1)$ solution $V \geq 0$ to the Hamilton–Jacobi inequality

$$V_x(x)f(x) + \tfrac{1}{2}V_x(x)\left[\frac{1}{\gamma^2}k(x)k^T(x) - g_1(x)g_1^T(x) - \frac{1}{\varepsilon}g_2(x)g_2^T(x)\right]V_x^T(x) + \tfrac{1}{2}h^T(x)h(x) \leq 0 \tag{32}$$

$$V(0) = 0$$

then the C^{r-1} state feedback

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\begin{pmatrix} g_1^T(x) \\ \frac{1}{\varepsilon}g_2^T(x) \end{pmatrix} V_x^T(x) \tag{33}$$

leads to a closed-loop system which has L_2-gain less than or equal to γ.

Proof. The closed-loop system $\bar{\Sigma}$ and (33) is

$$\dot{x} = f(x) - g_1(x)g_1^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) - \frac{1}{\varepsilon}\, g_2(x)g_2^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) + k(x)d_1$$

$$z = \begin{pmatrix} h(x) \\ -g_1^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) \\ -\dfrac{1}{\sqrt{\varepsilon}}\, g_2^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) \end{pmatrix} \tag{34}$$

and then by using Theorem 2 and Theorem 10 it follows that Σ and (33) have L_2-gain less than or equal to γ. □

Remark 13

A similar result was derived in Reference 12 but there we considered instead of (33):

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\begin{pmatrix} g_1^{\mathrm{T}}(x) \\ \dfrac{1}{2\varepsilon}\, g_2^{\mathrm{T}}(x) \end{pmatrix} V_x^{\mathrm{T}}(x) \tag{35}$$

When applied to Σ this feedback also leads to a closed-loop system with L_2-gain less than γ but it is no min-max solution of the differential game problem corresponding to $\bar{\Sigma}$.

From this theorem we easily obtain the following condition under which the closed-loop system has L_2-gain less than γ.

Corollary 14

Consider the system Σ and let $\gamma > 0$. Suppose there exists constants $\varepsilon, u > 0$ such that there exists a nonnegative C^r-solution V ($k \geqslant r \geqslant 1$) to the Hamilton–Jacobi inequality:

$$V_x(x)f(x) + \tfrac{1}{2} V_x(x)\left[\left(\frac{1}{\gamma^2} + \mu\right)k(x)k^{\mathrm{T}}(x) - g_1(x)g_1^{\mathrm{T}}(x) - \frac{1}{\varepsilon}\, g_2(x)g_2^{\mathrm{T}}(x)\right]V_x^{\mathrm{T}}(x) + \tfrac{1}{2} h^{\mathrm{T}}(x)h(x) \leqslant 0 \tag{36}$$

$$V(0) = 0$$

then the closed-loop system for the feedback (33) has L_2-gain less than γ.

Proof. From (36) it follows that there exists a constant $0 \leqslant \bar{\gamma} < \gamma$ such that (32) is satisfied with γ replaced by $\bar{\gamma}$. Hence by Theorem 12 the closed-loop system has L_2-gain less than γ (less than or equal to $\bar{\gamma}$). □

For the converse result of Corollary 14 we use Assumption 1 (see Reference 12)

Theorem 15

Suppose there exists a state feedback (21) which satisfies Assumption 1 and which solves the state feedback H_∞ suboptimal control problem in the sense that one of the solutions $V \geqslant 0$ to the

Hamilton–Jacobi inequality (7) for the closed-loop system $\bar{\Sigma}$, (21) is C^1, then V is also a solution of (36) and hence by Corollary 14 the state feedback (33) also solves the state feedback H_∞ suboptimal control problem for the system Σ.

Until now we have not considered the stability of the closed-loop system. But the following theorem can be easily obtained from Theorem 4.

Theorem 16

Suppose there exists a solution $V \geqslant 0$ to (32). Assume the system (34) is zero-state observable. Then $V(x) > 0$ for $x \neq 0$ and the closed-loop system $\bar{\Sigma}$, (33) (with $d_1(t) \equiv 0$) is locally asymptotically stable. Also the closed loop-system Σ, (33) is asymptotically stable. Assume additionally that V is proper, then the closed-loop system Σ, (33) or $\bar{\Sigma}$, (33) is globally asymptotically stable.

Now we shall linearize the system (20) around the equilibrium $x = 0$. This leads to

$$\begin{aligned} \dot{\bar{x}} &= F\bar{x} + G_1\bar{u}_1 + G_2\bar{u}_2 + K\bar{d}_1 \\ \bar{y} &= \bar{x} \\ \bar{z} &= \begin{pmatrix} H\bar{x} \\ \bar{u}_1 \end{pmatrix} \end{aligned} \tag{37}$$

Straightforwardly from Van der Schaft[18] the following results are obtained.

Proposition 17

Suppose the L_2-gain of (20), (21) is less than γ, and assume $F + GL$ with $L = (\partial l/\partial x)(0)$ is asymptotically stable, then there exists a neighbourhood W of 0 and a smooth function $V \geqslant 0$ on W satisfying (32).

Alternatively, assume $f + gl$ is globally asymptotically stable. Define the Hamiltonian

$$H_\gamma(x, p) = p^{\mathrm{T}}[f(x) + g(x)l(x)] + \frac{1}{2}\frac{1}{\gamma^2}p^{\mathrm{T}}k(x)k^{\mathrm{T}}(x)p + \frac{1}{2}h^{\mathrm{T}}(x)h(x) + \frac{1}{2}l_1^{\mathrm{T}}(x)l_1(x) \tag{38}$$

and suppose X_{H_γ} is hyperbolic, and its stable invariant manifold is diffeomorphic to M under the canonical projection π: $T^*M \to M$. Then there exists a global solution $V \geqslant 0$ to (32).

Proposition 18

Let $\gamma > 0$. Suppose there exists a smooth feedback $u = l(x)$, $l(0) = 0$, for (2) such that the L_2-gain of the nonlinear closed-loop system (20), (21) is less than (or equal to) γ. Then the linear feedback $\bar{u} = L\bar{x}$, with $L = \partial l/\partial x(0)$, for (11) results in the linear closed-loop system

$$\begin{aligned} \dot{\bar{x}} &= (F + GL)\bar{x} + K\bar{d}_1 \\ \begin{bmatrix} \bar{y} \\ \bar{u}_1 \end{bmatrix} &= \begin{pmatrix} H \\ L_1 \end{pmatrix}\bar{x} \end{aligned} \tag{39}$$

which also has L_2-gain less than (or equal to) γ.

Proof. The linearization of (20), (21) is equal to (39). Then the result follows from Reference 18. □

Now we will derive the converse result. Suppose the feedback $\bar{u} = L\bar{x}$ solves the H_∞ problem for the linearized system (37). What can we say about the H_∞ problem for the nonlinear system? (See Reference 12)

Theorem 19

Consider the linearized system (11). Let $\gamma > 0$. Suppose there exists a feedback $\bar{u} = L\bar{x}$ such that the L_2-gain of the closed-loop system (from $\bar{d}$ to $\bar{z}$) is less than γ and the closed-loop system is asymptotically stable.

Then there exists a neighbourhood W of 0 and a smooth function $V \geq 0$ defined on W such that V is a solution of the Hamilton–Jacobi inequality (32). Furthermore, the feedback

$$u = -\begin{pmatrix} I_{m_1} & 0 \\ 0 & \frac{1}{\varepsilon} I_{m-m_1} \end{pmatrix} g^{\mathrm{T}}(x) V_x^{\mathrm{T}}(x)$$

for (2) has the property that the closed-loop system has locally L_2-gain less than γ, in the sense that for all $x \in W$ there exists a constant $K(x)$, $0 \leq K(x) < \infty$, with $K(0) = 0$, such that

$$\int_0^T (\|y(t)\|^2 + \|u_1(t)\|^2)\,\mathrm{d}t \leq \gamma^2 \int_0^T \|d_1(t)\|^2\,\mathrm{d}t + K(x) \tag{40}$$

for all $T \geq 0$ and all $d \in L_2(0, T)$ such that the state-space trajectories $x(t)$ starting from $x(0) = x$ do not leave W (i.e., the state feedback H_∞-control problem for γ is solved on W).

We will summarize some of the above results in the following theorem

Theorem 20

Consider the nonlinear system (20) and its linearization (37). Then the following statements are equivalent.

(a) There exists a linear feedback

$$\bar{u} = L\bar{x} \tag{41}$$

such that the L_2-gain of the closed-loop system (37), (41) is less than γ and $F + GL$ is asymptotically stable.

(b) There exists a positive definite solution P to the algebraic Riccati equation

$$F^{\mathrm{T}}P + PF + \frac{1}{\gamma^2} PKK^{\mathrm{T}}P - PG_1G_1^{\mathrm{T}}P - \frac{1}{\varepsilon} PG_2G_2^{\mathrm{T}}P + H^{\mathrm{T}}H + \varepsilon Q = 0 \tag{42}$$

for all $Q>0$ and for some $\varepsilon>0$ such that also

$$F+\frac{1}{\gamma^2}KK^{\mathrm{T}}P-G_1G_1^{\mathrm{T}}P-\frac{1}{\varepsilon}G_2G_2^{\mathrm{T}}P \tag{43}$$

is asymptotically stable.

(c) There exists a neighbourhood $W\subset M$ of 0, and a nonlinear feedback $u=l(x)$ as in (21) defined on W, such that $F+GL$, with $L=\partial l/\partial x(0)$, is asymptotically stable and the closed-loop system (20), (21) has locally L_2-gain less than γ on W.

Remark 21

In the regular case,[18] $m_1=m$, there was the extra assumption that (H,F) must be detectable. In the present case $H^{\mathrm{T}}H+\varepsilon Q>0$, and hence there exists a nonsingular matrix $\tilde{H}$ such that

$$\tilde{H}^{\mathrm{T}}\tilde{H}=H^{\mathrm{T}}H+\varepsilon Q \tag{44}$$

and $(\tilde{H},F)$ is always detectable.

5. SINGULAR NONLINEAR MEASUREMENT FEEDBACK H_∞ CONTROL

Consider the singular dynamic measurement feedback H_∞ problem for affine nonlinear systems of the form:

$$\Sigma_m:\begin{cases}\dot{x}=f(x)+g_1(x)u_1+g_2(x)u_2+k(x)d_1\\ y=c(x)+d_2\\ z=\begin{pmatrix}h(x)\\ u_1\end{pmatrix}\end{cases} \tag{45}$$

As mentioned in the introduction we search for a dynamic affine nonlinear compensator

$$\begin{aligned}\dot{\xi}&=w(\xi)+p(\xi)y\\ u&=\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}q_1(\xi)\\ q_2(\xi)\end{pmatrix}=q(\xi)\end{aligned} \tag{46}$$

with $w(0)=0$ and $q(0)=0$, such that the closed-loop system has L_2-gain less than γ. We make a similar assumption as in Section 3 about the finiteness of the L_2-gain between d_1, d_2 and $q_2(\xi)$.

Assumption 2

The L_2-gain from d_1, d_2 to q_2 is finite, i.e., there exists a constant $N>0$ such that for all $x\in M$, $\xi\in M_c$ there exists a constant $K_c(x,\xi)$, $0\leqslant K_c(x,\xi)<\infty$, with $K_c(0,0)=0$ such that

$$\int_0^T\|q_2(\xi(t))\|^2\,\mathrm{d}t\leqslant N\int_0^T(\|d_1(t)\|^2+\|d_2(t)\|^2)\,\mathrm{d}t+K_c(x,\xi) \tag{47}$$

for all $T>0$ and all d_1, $d_2\in L_2[0,T]$, where $x(t)$ is the solution of the state equation of the closed-loop system Σ_m, (46).

Also in this case it is easy to prove that the compensator (46) applied to the system:

$$\bar{\Sigma}_m: \begin{cases} \dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 + k(x)d_1 \\ y = c(x) + d_2 \\ \bar{z} = \begin{pmatrix} h(x) \\ u_1 \\ \sqrt{\varepsilon}u_2 \end{pmatrix} \end{cases} \tag{48}$$

if ε is sufficiently small leads to a closed-loop system with L_2-gain less than γ, and vice versa.

Theorem 22

Consider a compensator (46). Then (ii) implies (i) and under Assumption 2 also (i) implies (ii):

(i) The closed-loop system Σ_m, (46) has L_2-gain less than γ.
(ii) The closed-loop system $\bar{\Sigma}_m$, for ε sufficiently small, with the compensator (46) has L_2-gain less than γ.

The proof is similar as in Section 4 and will therefore be omitted.

Hence also in this case we can, under Assumption 2, restrict ourselves to finding a compensator for the system $\bar{\Sigma}$. We will do this using the worst-case certainty equivalence principle. We give a brief exposition of this principle (for more details see References 2 and 19).

We start with the finite horizon H_∞ problem, i.e., we consider the L_2-gain on some fixed interval $[T_1, T_2]$. This comes down to looking for max-min solution of the performance criterion

$$\frac{1}{2}\int_{T_1}^{T_2} (u_1^{\mathrm{T}}u_1 + \varepsilon u_2^{\mathrm{T}}u_2 + h^{\mathrm{T}}h - \gamma^2 d_1^{\mathrm{T}}d_1 - \gamma^2 d_2^{\mathrm{T}}d_2)\,\mathrm{d}t \tag{49}$$

where $u(t)$ may depend on $y(\tau)$ for $\tau \le t$. This problem can be considered in two parts. First we look at the state feedback H_∞ control problem, considered in Section 4 for the infinite horizon case. This leads in the finite horizon case to the nonstationary Hamilton–Jacobi equation

$$\begin{aligned} &V_t(x,t) + V_x(x,t)f(x) + \tfrac{1}{2}h^{\mathrm{T}}(x)h(x) \\ &+ \tfrac{1}{2}V_x(x,t)\left[\left(\frac{1}{\gamma^2} + \delta\right)k(x)k^{\mathrm{T}}(x) - g_1(x)g_1^{\mathrm{T}}(x) - \frac{1}{\varepsilon}g_2(x)g_2^{\mathrm{T}}(x)\right]V_x^{\mathrm{T}}(x,t) = 0 \\ &V(x, T_2) = 0 \end{aligned} \tag{50}$$

with resulting suboptimal state feedback

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = -\begin{pmatrix} g^{\mathrm{T}}(x) \\ \frac{1}{\varepsilon}g_2^{\mathrm{T}}(x) \end{pmatrix} V_x^{\mathrm{T}}(x,t) \tag{51}$$

Secondly, let $\tau \in [T_1, T_2]$, and let $\bar{u}_1(t)$, $\bar{u}_2(t)$ and $\bar{y}(t)$, $t \in [T_1, \tau]$ be a given pair of inputs and corresponding measured output trajectories of the system $\bar{\Sigma}_m$. Then we look for the maximal solution $x(T_1)$ and $d_1(t)$, $d_2(t)$ of the performance criterion

$$\frac{1}{2}\int_{T_1}^{\tau} (\bar{u}_1^{\mathrm{T}}\bar{u}_1 + \varepsilon\bar{u}_2^{\mathrm{T}}\bar{u}_2 + h^{\mathrm{T}}h - \gamma^2 d_1^{\mathrm{T}}d_1 - \gamma^2 d_2^{\mathrm{T}}d_2)\,\mathrm{d}t + V(x(\tau), \tau) \tag{52}$$

which satisfies the constraint that the measured output equals $\bar{y}(t)$. We assume that this maximization problem has a unique solution for every $\tau \in [T_1, T_2]$. Then we define for every τ

$$\begin{pmatrix} \bar{u}_1(\tau) \\ \bar{u}_2(\tau) \end{pmatrix} = -\begin{pmatrix} g_1^{\mathrm{T}}(\hat{x}(\tau)) \\ \dfrac{1}{\varepsilon}\, g_2^{\mathrm{T}}(\hat{x}(\tau)) \end{pmatrix} V_x^{\mathrm{T}}(\hat{x}(\tau), \tau) \tag{53}$$

where $\hat{x}(\cdot)$ is the state trajectory corresponding to the maximizing solution of (52). Now $\hat{x}(\cdot)$ depends on $\bar{u}(\cdot)$ and by (53) we have defined a causal mapping from $\bar{u}(t)$ to $\bar{u}(t)$, $t \in [T_1, T_2]$. Denote the fixed point of this mapping by $\hat{u}(t)$. This fixed point only depends on $\bar{y}$, in a causal way. Now

$$\begin{pmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{pmatrix} = -\begin{pmatrix} g_1^{\mathrm{T}}(\hat{x}(t)) \\ \dfrac{1}{\varepsilon}\, g_2^{\mathrm{T}}(\hat{x}(t)) \end{pmatrix} V_x^{\mathrm{T}}(\hat{x}(t), t) \tag{54}$$

is the solution of the considered optimization problem (49) (see Reference 2). This is called the *worst-case certainty equivalence solution.*

We still have to solve the maximization problem with the performance criterion (52). This can be done in a classical way by first maximizing the criterion (52) under the constraint that the final value of the state $x(\tau)$ equals x and after that we maximize the solution of this problem with respect to x. Furthermore, since the disturbance d_2 fully influences the observations y we can substitute $d_2 = c(x) - \bar{y}$ into the performance criterion such that the constraint is automatically satisfied. The second maximization with respect to x is equal to the maximum of $S(x, \tau) = V(x, \tau) - W(x, \tau)$ where $W \geqslant 0$ satisfies

$$\begin{aligned} &W_t(x, t) + W_x(x, t)[f(x) + g_1(x)\bar{u}_1(t) + g_2(x)\bar{u}_2(t)] \\ &\quad + \frac{1}{2}\frac{1}{\gamma^2} W_x(x, t)k(x)k^{\mathrm{T}}(x)W_x^{\mathrm{T}}(x, t) + \frac{1}{2} h^{\mathrm{T}}(x)h(x) \\ &\quad - \frac{1}{2}\gamma^2 c^{\mathrm{T}}(x)c(x) + \gamma^2 c^{\mathrm{T}}(x)\bar{y}(t) - \frac{1}{2}\gamma^2\|\bar{y}(t)\|^2 + \frac{1}{2}\|\bar{u}_1(t)\|^2 + \frac{1}{2}\varepsilon\|\bar{u}_2(t)\|^2 = 0 \\ &\qquad W(x, T_1) = 0 \end{aligned} \tag{55}$$

Assume that this maximum is determined by $S_x(\hat{x}(t), t) = 0$ and that the Hessian is nondegenerate. Then the state equation for $\hat{x}(\cdot)$ can be found by differentiation of this equality.[19]

The resulting compensator which solves the *infinite* horizon H_∞ problem can be found by letting $T_2 \to \infty$ while imposing that $x(t) \to 0$ for $t \to \infty$ and $T_1 \to -\infty$ while $x(t) \to 0$ for $t \to -\infty$.

A finite-dimensional approximation to the constructed nonlinear controller which locally solves the H_∞ problem is given by

$$\begin{aligned}\dot{\xi} &= f(\xi) - g_1(\xi)g_1^{\mathrm{T}}(\xi)V_\xi^{\mathrm{T}}(\xi) - \frac{1}{\varepsilon} g_2(\xi)g_2^{\mathrm{T}}(\xi)V_\xi^{\mathrm{T}}(\xi) + \frac{1}{\gamma^2} k(\xi)k^{\mathrm{T}}(\xi)V_\xi^{\mathrm{T}}(\xi) \\ &\quad + \gamma^2[W_{\xi\xi}(\xi) - V_{\xi\xi}(\xi)]^{-1} \frac{\partial c^{\mathrm{T}}}{\partial \xi}(\xi)(y(t) - c(\xi)) \\ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= -\begin{pmatrix} g^{\mathrm{T}}(\xi) \\ \frac{1}{\varepsilon} g_2^{\mathrm{T}}(\xi) \end{pmatrix} V_\xi^{\mathrm{T}}(\xi)\end{aligned} \tag{56}$$

where $V(\xi)$ is a solution of (32) with equality and $W(\xi)$ is a solution of the stationary version of (55) ($\bar{u}(t) = 0$; $\bar{y}(t) = 0$)

$$W_x(x)f(x) + \frac{1}{2}\frac{1}{\gamma^2} W_x(x)k(x)k^{\mathrm{T}}(x)W_x^{\mathrm{T}}(x) + \frac{1}{2} h^{\mathrm{T}}(x)h(x) - \frac{1}{2}\gamma^2 c^{\mathrm{T}}(x)c(x) = 0 \tag{57}$$
$$W(0) = 0$$

such that

$$\begin{aligned}& f - g_1g_1^{\mathrm{T}}V_x^{\mathrm{T}} - \frac{1}{\varepsilon} g_2g_2^{\mathrm{T}}V_x^{\mathrm{T}} + \frac{1}{\gamma^2} kk^{\mathrm{T}}V_x^{\mathrm{T}} \quad \text{is exponentially stable} \\ & -\left(f + \frac{1}{\gamma^2} kk^{\mathrm{T}}W_x^{\mathrm{T}}\right) \quad \text{is exponentially stable} \\ & W_{xx}(x) > V_{xx}(x), \quad \forall x\end{aligned} \tag{58}$$

Hence we have the following result.

Theorem 23

Consider the system Σ_m, and suppose there exist a constant $\varepsilon > 0$ and solutions $V \geq 0$, $W \geq 0$ to (32) with equality respectively (57), satisfying (58). Then the closed-loop system Σ_m, (56) has locally L_2-gain less than γ.

Also a converse result can be obtained.[19] Therefore we need the Assumption 2.

Theorem 24

Suppose the H_∞ suboptimal control problem for $\gamma \geq 0$ is solvable by a compensator (46) in the sense that there exists a C^1-solution $V(x, \xi) \geq 0$ to the corresponding Hamilton–Jacobi inequality

$$\begin{aligned}& V_x(x,\xi)[f(x) + g_1(x)q_1(\xi) + g_2(x)q_2(\xi)] + V_\xi(x,\xi)[w(\xi) + p(\xi)c(x)] \\ &\quad + \frac{1}{2}\frac{1}{\gamma^2} V_x(x,\xi)k(x)k^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x,\xi) + \frac{1}{2}\frac{1}{\gamma^2} V_\xi(x,\xi)p(\xi)p^{\mathrm{T}}(\xi)V_\xi(x,\xi) \\ &\quad + \frac{1}{2} h^{\mathrm{T}}(x)h(x) + \frac{1}{2} q_1^{\mathrm{T}}(\xi)q_1(\xi) + \frac{1}{2} \varepsilon q_2^{\mathrm{T}}(\xi)q_2(\xi) \leq 0\end{aligned} \tag{59}$$
$$V(0,0) = 0$$

and suppose Assumption 2 is satisfied. Furthermore assume that the equation $V_\xi(x, \xi) = 0$ has a C^1-solution $\xi = F(x)$, $F(0) = 0$, with $F: M \to M_c$. Then there exists nonnegative solutions $V(x)$ and $W(x)$ to the Hamilton–Jacobi inequality (32) and (57) with = replaced by $\leq$, as well as the coupling condition

$$V(x) \leq W(x) \tag{60}$$

near 0.

6. PARAMETER UNCERTAINTY

Consider the system

$$\dot{x} = f(x, \theta) + g(x)u \tag{61}$$

where we first assume that we can measure the full state $(y = x)$. The matrix θ contains the uncertain parameters.

We assume the following linear dependency of f on θ

$$f(x, \theta) = f(x, 0) + k(x)\theta h(x) \tag{62}$$

for some known smooth functions k and h. This assumption is for linear systems equivalent to the structured perturbations considered by Hinrichsen and Pritchard.[7]

Under assumption (62) the perturbed system (61) can be rewritten as

$$\Sigma_P: \begin{cases} \dot{x} = f(x, 0) + g(x)u + k(x)d_1 \\ z = h(x) \\ y = x \end{cases} \tag{63}$$

where d_1 is given by $d_1 = \theta z$ with θ the constant matrix specifying the parameter uncertainties. Then the robust stabilization problem is to find a feedback

$$F: u = \sigma(x) \tag{64}$$

such that the L_2-gain of the closed-loop system (63) and (64) from d_1 to z is minimized to γ^*.

Using the small-gain theorem (see, for example, Reference 4) this means that the above overall system is L_2-stable for all perturbations θ with L_2-gain (largest singular value of θ) strictly less than $1/\gamma^*$. In this we also include complex perturbations θ. Because θ is in practice real we may obtain conservative bounds on θ.[8,9,16]

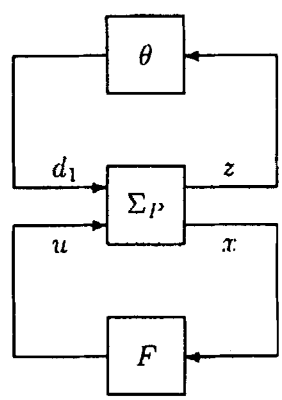

Figure 1. Parameter perturbed system with feedback

The problem of minimizing the L_2-gain from d_1 to $z = h(x)$ is a singular state feedback H_∞ nonlinear optimal control problem as studied in Section 4. For the solution to the suboptimal problem we can formulate the following result.

Theorem 25

Suppose there exists an constant $\varepsilon > 0$ and a solution $V \geqslant 0$ to

$$V_x(x)f(x,0) + \tfrac{1}{2}V_x(x)\left[\frac{1}{\gamma^2}k(x)k^{\mathrm{T}}(x) - \frac{1}{\varepsilon}g(x)g^{\mathrm{T}}(x)\right]V_x^{\mathrm{T}}(x) + \tfrac{1}{2}h^{\mathrm{T}}(x)h(x) \leqslant 0 \qquad V(0) = 0 \tag{65}$$

then the feedback

$$u = -\frac{1}{\varepsilon}g^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) \tag{66}$$

local asymptotically stabilizes the origin of the closed-loop system (63), and (66) for every perturbation θ having L_2-gain less than $1/\gamma$ when the closed-loop system is zero-state observable.

Remark 26

If the solution $V \geqslant 0$ of (65) is also proper then the feedback (66) even globally stabilizes the closed-loop system.

Under Assumption 1 we can also formulate a converse result:

Theorem 27

Suppose there exists a feedback (64) which locally stabilizes the closed-loop system (63) and (64). Assume that Assumption 1 is satisfied. Then there exists a constant $\varepsilon > 0$ and a solution $V \geqslant 0$ to the Hamilton–Jacobi inequality (65).

When the full state cannot be measured we consider the following system

$$\begin{aligned} \dot{x} &= f(x,\theta) + g(x)u \\ y &= c(x,\eta) \end{aligned} \tag{67}$$

where the uncertainty is given by the matrix θ and by the vector η. The same assumption (62) on $f(x,\theta)$ is made and further we assume that the structure of c is defined as:

$$c(x,\eta) = c(x,0) + \eta h(x) \tag{68}$$

Then the perturbed system is

$$\Sigma_P\colon \begin{cases} \dot{x} = f(x,0) + g(x)u + k(x)d_1 \\ z = h(x) \\ y = c(x,0) + d_2 \end{cases} \tag{69}$$

where d_1, d_2 are the output of an arbitrary nonlinear system Δ with input z

$$\Delta:\begin{cases} \dot{\varphi} = \alpha(\varphi, z) \\ \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} \beta_1(\varphi, z) \\ \beta_2(\varphi, z) \end{pmatrix} = \beta(\varphi, z) \end{cases} \tag{70}$$

Then the robust stabilization problem is to find a compensator C

$$C:\begin{cases} \dot{\xi} = p(\xi, y); & p(0,0) = 0 \\ u = q(\xi, y); & q(0,0) = 0 \end{cases} \tag{71}$$

such that the L_2-gain of the closed-loop system (69) and (71) from d_1, d_2 to z is minimized.

The suboptimal problem is as we have seen in Section 5 locally solvable if there exists a constant $\varepsilon > 0$ and a solution $V \geqslant 0$ to (65) with equality and a solution $W \geqslant 0$ to (57) with satisfy that

$$\begin{aligned} & f - \frac{1}{\varepsilon} gg^{\mathrm{T}} V_x^{\mathrm{T}} + \frac{1}{\gamma^2} kk^{\mathrm{T}} V_x^{\mathrm{T}} \quad \text{is exponentially stable} \\ & -\left(f + \frac{1}{\gamma^2} kk^{\mathrm{T}} W_x^{\mathrm{T}}\right) \quad \text{is exponentially stable} \\ & W_{xx}(x) > V_{xx}(x), \quad \forall x \end{aligned} \tag{72}$$

Hence:

Theorem 28

Suppose there exists a constant $\varepsilon > 0$ and a solution $V \geqslant 0$ (65) with equality and a solution $W \geqslant 0$ to (57) with satisfies (72). Assume that the certainty equivalence principle holds for Σ_{P}. Then the controller

$$\begin{aligned} \dot{\xi} &= f(\xi, 0) - \frac{1}{\varepsilon} g(\xi) g^{\mathrm{T}}(\xi) V_\xi^{\mathrm{T}}(\xi) + \frac{1}{\gamma^2} k(\xi) k^{\mathrm{T}}(\xi) V_\xi^{\mathrm{T}}(\xi) \\ &\quad + \gamma^{\mathrm{T}} [W_{\xi\xi}(\xi) - V_{\xi\xi}(\xi)]^{-1} \frac{\partial c^{\mathrm{T}}}{\partial \xi}(\xi)(y(t) - c(\xi, 0)) \\ u &= -\frac{1}{\varepsilon} g^{\mathrm{T}}(\xi) V_\xi^{\mathrm{T}}(\xi) \end{aligned} \tag{73}$$

locally stabilizes the closed-loop system (69), (70) and (73) for every perturbation system Δ as in (70), having L_2-gain less than γ.

7. MULTIPLICATIVE PERTURBATIONS

Consider the system

$$\Sigma:\begin{cases} \dot{x} = f(x) + g(x) u \\ \bar{y} = h(x) \end{cases} \tag{74}$$

and suppose the output $\bar{y}$ is perturbed by a disturbance d_2. We assume that this disturbance is the

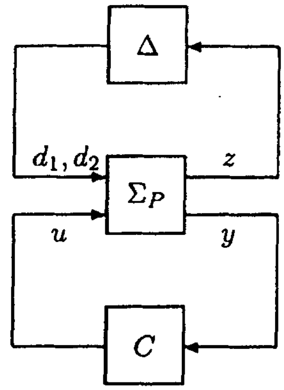

Figure 2. Parameter perturbed system with controller

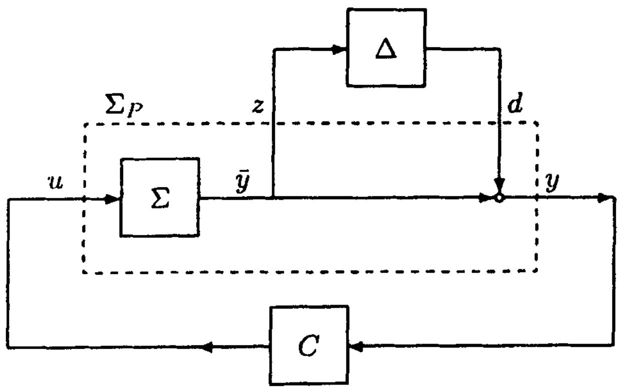

Figure 3. Multiplicative perturbed system with controller

output of an arbitrary nonlinear system with input $\bar{u}$:

$$\Delta : \begin{cases} \dot{\varphi} = \alpha(\varphi, \bar{y}) \\ d_2 = \beta(\varphi, \bar{y}) \end{cases} \tag{75}$$

Then the output perturbed system is given by

$$\Sigma_P : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) + d_2 \\ z = h(x) \end{cases} \tag{76}$$

Now the robust stabilization problem is, as in the previous section, to find a compensator

$$C : \begin{cases} \dot{\xi} = p(\xi, y) \\ u = q(\xi . y) \end{cases} \tag{77}$$

such that the L_2-gain of the closed-loop system (76) and (77) from d_2 to z is minimized.

In Section 5 we have seen that the strictly suboptimal problem is locally solvable if there exists a constant $\varepsilon > 0$ and a solution $V \geqslant 0$ to

$$V_x(x)f(x) - \frac{1}{2\varepsilon} V_x(x)g(x)g^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) + \tfrac{1}{2} h^{\mathrm{T}}(x)h(x) = 0 \qquad V(0) = 0 \tag{78}$$

and a solution $W \geqslant 0$ to

$$W_x(x)f(x) + \tfrac{1}{2}(1 - \gamma^2)h^{\mathrm{T}}(x)h(x) = 0 \qquad W(0) = 0 \tag{79}$$

such that

$$\begin{aligned} &f - \frac{1}{\varepsilon} gg^{\mathrm{T}}V_x^{\mathrm{T}} \quad \text{is exponentially stable} \\ &-f \quad \text{is exponentially stable} \\ &W_{xx}(x) > V_{xx}(x), \quad \forall x \end{aligned} \tag{80}$$

then a controller which solves the robust stabilization problem is

$$\begin{aligned} \dot{\xi} &= f(\xi) - \frac{1}{\varepsilon} g(\xi)g^{\mathrm{T}}(\xi)V_\xi^{\mathrm{T}}(\xi) + \gamma^{\mathrm{T}}[W_{\xi\xi}(\xi) - V_{\xi\xi}(\xi)]^{-1} \frac{\partial h^{\mathrm{T}}}{\partial \xi}(\xi)(y(t) - h(\xi)) \\ u &= -\frac{1}{\varepsilon} g^{\mathrm{T}}(\xi)V_\xi^{\mathrm{T}}(\xi) \end{aligned} \tag{81}$$

8. CONCLUSIONS

We have extended the theory about the regular nonlinear H_∞ control problem[18,19,10,1] to the setting of singular nonlinear H_∞ control by construction of a dynamic compensator on the basis of a min-max state feedback by using the certainty equivalence principle. This feedback and compensator are constructed by using a regularized system and under a certain assumption the solutions also solve the H_∞ problem for the singular system. It remains interesting to look for conditions for the singular nonlinear H_∞ problem to be solvable without using a regularization of the system.

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