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Philon's Line Generalized: An Optimization Problem from Geometry[1]

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Abstract. Consider in n-dimensional Euclidean space the intersection of a convex cone and a hyperplane through a given point. The problem is to minimize the $(n-1)$-volume of this intersection. A geometric interpretation of the first-order optimality condition is given. The special case $n=2$ is known as a characteristic property of Philon's line.

Key Words. Philon's line, optimality conditions, convex cones, centroids.

1. Introduction

Philon's line appears in a geometric construction for the Delian problem of duplicating a cube. This line has a characteristic extremum property. An analysis of the problem together with historical notes and an extension to non-Euclidean geometry in the plane can be found in a recent article by Coxeter and Van De Craats (Ref. 1). The aim of this paper is to show how that extremum property of a line segment in the plane can be generalized to an extremum property of a hyperplane segment in n-dimensional Euclidean space.

2. Philon's Line

In the Euclidean plane, consider two straight lines l_1 and l_2 intersecting at a point 0. Let $P \neq 0$ be a point in the plane and m a line through P but

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not through 0, intersecting l_1 at R_1 and l_2 at R_2. The perpendicular from 0 on m meets m in Q. The line m is called a Philon line if the distances R_1P and QR_2 are equal (P and Q between R_1 and R_2 or vice versa). If l_1 and l_2 are perpendicular, one such line exists for each $P \notin l_1 \cup l_2$, and the equation $x^3 = a$ can be solved by constructing that line for P with (l_1, l_2) coordinates $(1, a)$. For the Delian problem, choose $a = 2$. In Ref. 1, it is shown that for oblique l_1, l_2 one, two, or three lines m with Philon's property exist, depending on the position of P.

For an arbitrary line m as above, let $\Phi(m)$ be the intercept R_1R_2 (distance between R_1 and R_2). The extremum property mentioned in the introduction states that m is a Philon line if and only if Φ is stationary at m.

3. Generalization to Higher Dimension

For a Philon line as in Section 2, the segments R_1R_2 and PQ have the same midpoint. It is this property which allows a generalization. The following notation will be used: (x, y) is the standard inproduct in R^n, $\|x\| = (x, x)^{1/2}$ is the Euclidean norm, and $S = \{u \mid \|u\| = 1\}$ is the unit sphere.

As generalization of the sector between the lines l_1 and l_2 in the preceding section, we consider a closed convex cone $K \subset \mathbb{R}^n$ with vertex 0 and nonvoid interior K^0. We assume $K \backslash \{0\}$ to be contained in a half-space $\{x \mid (a, x) > 0\}$, or equivalently, $K \cap (-K) = \{0\}$. In Ref. 2, a convex cone with this property is called pointed. With our assumptions, the polar cone

$$K^p = \{y \mid (y, x) \leq 0, \text{ for all } x \in K\}$$

is also pointed and has nonvoid interior $(K^p)^0$. Since the set $-(K^p)^0$ will be needed repeatedly, we introduce the notation

$$K^+ = -(K^p)^0 = \{y \mid (y, x) > 0, \text{ for all } x \in K \backslash \{0\}\}.$$

The line m from Section 2 is generalized to a hyperplane

$$H(b) = \{x \mid (b, x) = 1\}.$$

Note the one-to-one correspondence between the vectors $b \neq 0$ and the hyperplanes in $\mathbb{R}^n$ which do not pass through 0. The intersection $K \cap H(b)$ is compact and nonvoid if and only if $b \in K^+$. The length $\Phi(m)$ of the segment R_1R_2 is generalized to the $(n-1)$-volume of this intersection,

$$\Phi(b) = \int_{K \cap H(b)} da,$$

with da the $(n-1)$-dimensional volume element on $H(b)$.

For $b \in K^+$, the projection from the origin $T(u) = u/(b, u)$ defines a bijection $T: K \cap S \to K \cap H(b)$. For integrable f, the substitution rule

$$\int_{K \cap H(b)} f(x)\, da = \|b\| \int_{K \cap S} f(u/(b, u))(b, u)^{-n}\, d\omega$$

holds. The easy proof using exterior multiplication of suitable orthonormal vectors and their images under the derivative of T is omitted here. Intuitively, the change in $(n-1)$-volume can be explained as the product of two factors: $1/(b, u)^{n-1}$, due to the distance factor $1/(b, u)$ in $T(u)$; and $1/\cos \delta = \|b\| \cdot \|u\|/(b, u) = \|b\|/(b, u)$, due to the angle δ between the two normal vectors u of S and b of $H(b)$. The $(n-1)$-volume of $K \cap H(b)$ is now

$$\Phi(b) = (b, b)^{1/2} \int_{K \cap S} (b, u)^{-n}\, d\omega.$$

Let a point $p \neq 0$ be given, and consider the optimization problem to minimize $\Phi(b)$ under the constraint $(b, p) = 1$. A vector b which satisfies this constraint and the first-order optimality condition with Lagrange multiplier λ,

$$D_b\Phi = \lambda p, \tag{1}$$

is called stationary. Note that the assumption $p \neq 0$ is the linear independency constraint qualification from optimization theory. The derivative $D_b\Phi$ is easily calculated to be

$$D_b\Phi = b(b, b)^{-1/2} \int_{K \cap S} (b, u)^{-n}\, d\omega - (b, b)^{1/2} n \int_{K \cap S} u(b, u)^{-(n+1)}\, d\omega$$
$$= q \int_{K \cap H(b)} da - n \int_{K \cap H(b)} x\, da,$$

with

$$q = b/(b, b) \quad \text{and} \quad x = T(u) = u/(b, u).$$

Now, q is a multiple of b and satisfies $(b, q) = 1$. Hence, q lies on $H(b)$ and is the foot of the perpendicular from the origin on $H(b)$. The integral in the second term is equal to $c \cdot \Phi(b)$, with c the centroid of $K \cap H(b)$. The optimality condition (1) can be written as

$$(q - nc)\Phi(b) = \lambda p.$$

With

$$(b, p) = (b, q) = (b, c) = 1,$$

we find that

$$\lambda = -(n-1)\Phi(b).$$

From our assumption $K^0 \neq \phi$, it follows that $\Phi(b) > 0$. Hence, the optimality condition (1) is equivalent to

$$c = (1/n)q + [(n-1)/n]p.$$

Result 3.1. $\Phi(b)$ is stationary under the constraint $p \in H(b)$ if and only if the centroid c of $K \cap H(b)$ lies on the line through p and the foot q of the perpendicular from the origin on $H(b)$ and divides the line segment between q and p in the ratio $(n-1)/1$, or in the special case $p = q$, the three points coincide.

For $n = 2$, this is the property of Philon's line from Section 2, the points c, p, q being trivially collinear, with c the midpoint of the line segment pq.

4. Open Questions and Remarks

Several open questions remain: What is the number of stationary points b for a given p? Which types of stationary points occur (local or global extrema, saddle-points)? What can be said about second-order optimality conditions? A short informal discussion of these questions follows.

(i) A difference between the problems considered in the Sections 2 and 3 must be mentioned. In Section 2 and in Ref. 1, the lines l_1, l_2 divide $\mathbb{R}^2$ into four sectors and the segment R_1R_2 may lie in any of these sectors. The cone K in Section 3 corresponds to only one of the four sectors. The obvious reason lies in the required convexity of K and compactness of $K \cap H(b)$. The results in Ref. 1 imply for the case $n = 2$ in our problem that exactly one stationary (minimum) point exists for $p \in K^0$ (interior of K) and that, for $p \notin K$, the number of stationary points is 0, 1, or 2.

(ii) For a given p, write

$$A(p) = \{b \mid (b, p) = 1\} \cap K^+.$$

Suppose that $p \in K^0$. Then, the closure $\bar{A}(p)$ is compact, and the $(n-1)$-volume $\Phi(b)$ is finite and continuous for $b \in A(p)$ and tends to infinity when b approaches the relative boundary of $A(p)$. Hence, $\Phi(b)$ has a global minimum on $A(p)$. We have just observed that, for $n = 2$, this minimum occurs at the unique stationary $b \in A(p)$. The situation is different for $n \geq 3$. An analysis of the second-order optimality condition for the optimization problem of Section 3 suggests that several stationary points should exist if K is a narrow simplicial cone (spanned by n linearly independent vectors lying almost in one hyperplane) and $p \in K^0$ close to one of the spanning vectors. It is easily verified that, for the cone $K \subset \mathbb{R}^3$ spanned by $(1, -1, -0.2)$, $(1, 1, -0.2)$, $(1, 0, 0.01)$, and the point $p = (1, 0, 0)$, two local minima and one saddle-point of $\Phi(b)$ exist.

(iii) The ratio $(n-1)/1$ in Section 3 is no surprise. Considering a special case (p on the boundary of a simplicial cone K), we obtain as a corollary of our result the well-known statement: the centroid of an $(n-1)$-simplex divides the line segment between a vertex and the centroid of the opposite facet in that ratio.

(iv) Consider the related problem to minimize the n-dimensional volume of the cone segment cut off by a hyperplane through a given point. Using $\Phi(b)$ from Section 3, it is an easy exercise to show that this volume is stationary if and only if $p = c$. This statement holds even more generally without the assumption that K should be a cone; see Ascoli, Ref. 3.

References

1. COXETER, H. S. M., and VAN DE CRAATS, J., *Philon Lines in Non-Euclidean Planes*, Journal of Geometry, Vol. 48, pp. 26–55, 1993.
2. STOER, J., and WITZGALL, C., *Convexity and Optimization in Finite Dimensions, Vol. 1*, Springer, Berlin, Germany, 1970.
3. ASCOLI, G., *Sui Baricentri delle Sezioni Piane di un Dominio Spaziale Connesso*, Bolletino della Unione Matematica Italiana., Vol. 10, pp. 123–128, 1931.