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Parameter Identification for Hyperbolic Stochastic Systems

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Parameter identification is studied for hyperbolic stochastic systems. Explicit sufficient conditions ensuring consistency of the maximum likelihood estimate are given.

1. INTRODUCTION

With the rapid progress in the study of flexible structures, identification and control of hyperbolic systems have become increasingly important. Although parameter identification for infinite dimensional systems has been studied extensively in the literature (see the survey of Polis [1]), relatively little attention has been paid to the consistency question of parameter estimates for stochastic hyperbolic systems. Sufficient conditions for establishing strong consistency of maximum likelihood estimates for general linear infinite dimensional systems have been given by Bagchi and Borkar [2]. These conditions, however, are hard to verify, except for finite dimensional systems. Our purpose in this paper is to give sufficient conditions, in terms of the partial differential operator, ensuring strong consistency of parameter estimates in stochastic hyperbolic systems.

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2. System Model

Let V and H be two Hilbert spaces, $V \subset H$, V dense in H. Let $\|\cdot\|$ and $|\cdot|$ denote the norms in V and H, respectively, and let $(\cdot, \cdot)$ be the scalar product in H. We identify H with its dual and, V' denoting the dual of V, we have

$$V \subset H \subset V',$$

where we assume that the injection of V into H is compact.

Let $\theta \in \Theta$ be the finite dimensional vector of all the unknown parameters and $A(\theta) \in \mathscr{L}(V; V')$ satisfy

$$\langle A(\theta)\phi, \phi \rangle \geqslant \alpha \|\phi\|^2, \qquad |\langle A(\theta)\phi, \phi \rangle| \leqslant \tilde{\alpha} \|\phi\|^2 \tag{A-1}$$

for $\alpha, \tilde{\alpha} > 0$, independent of $\theta \in \Theta$, $\forall \phi \in V$, and

$$A(\theta) = A(\theta)^*, \tag{A-2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between V and V'. Assume that the domain $D(A(\theta))$ is independent of θ. We omit θ in the subsequent discussions when there is no confusion. An operator A satisfying (A-1) and (A-2) is called *regular.*

We consider the stochastic hyperbolic system

$$\Sigma_1: \begin{cases} \dfrac{d^2u(t)}{dt^2} + k\dfrac{du(t)}{dt} + Au(t) = \dfrac{dW(t)}{dt}, & 0 \leqslant t < \infty \\ u(0) = u_0, \qquad \dfrac{du}{dt}(0) = \dot{u}_0, \end{cases}$$

where k is a positive constant and $W(t)$ is a Brownian motion process in H with nuclear covariance

$$E\{|W(t)|^2\} = \int_0^t \mathrm{Tr}[Q]\, ds < \infty. \tag{2.1}$$

The precise form of Σ_1 then becomes

$$\Sigma_2: \begin{cases} (u(t), \phi_1) = (u_0, \phi_1) + \displaystyle\int_0^t (\dot{u}(s), \phi_1)\, ds, & \forall \phi_1 \in H \\ (\dot{u}(t), \phi_2) + \displaystyle\int_0^t k(\dot{u}(s), \phi_2)\, ds + \int_0^t \langle Au(s), \phi_2 \rangle\, ds & \\ \qquad = (\dot{u}_0, \phi_2) + (W(t), \phi_2), & \forall \phi_2 \in V. \end{cases}$$

The following theorem is a special case of a more general result established by Pardoux [3]:

THEOREM 2.1. *$(\Omega, \mathscr{J}, P)$ is the basic probability space. Under assumptions* (A–1) *and* (A–2) *and with*

$$u_0 \in L^2(\Omega; V) \qquad \text{and} \qquad \dot{u}_0 \in L^2(\Omega; H)$$

there exists a unique continuous solution to Σ_2 such that

$$u \in L^2(\Omega; C([0, T]; V)) \tag{2.2}$$

and

$$\dot{u} \in L^2(\Omega; C([0, T]; H)) \tag{2.3}$$

for all $T < \infty$.

For convenience of description, we rewrite the hyperbolic system Σ_2 in the vector form. We set

$$\mathscr{H} = V \times H. \tag{2.4}$$

If $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in \mathscr{H}$, with $\boldsymbol{\phi} = (\phi_1, \phi_2)'$ and $\boldsymbol{\psi} = (\psi_1, \psi_2)'$, define the scalar product on $\mathscr{H}$ by

$$[\boldsymbol{\phi}, \boldsymbol{\psi}] = \langle A\phi_1, \psi_1 \rangle + (\phi_2, \psi_2). \tag{2.5}$$

Identifyting $\mathscr{H}$ with its dual and introducing

$$\mathscr{V} = V \times V \tag{2.6}$$

we have

$$\mathscr{V} \subset \mathscr{H} \subset \mathscr{V}',$$

where

$$\mathscr{V}' = V \times V'. \tag{2.7}$$

For $\boldsymbol{\phi} \in \mathscr{V}$, define

$$\mathscr{A}\boldsymbol{\phi} = (-\phi_2, A\phi_1 + k\phi_2)', \tag{2.8}$$

that is,

$$\mathscr{A} = \begin{pmatrix} 0 & -1 \\ A & k \end{pmatrix} \in \mathscr{L}(\mathscr{V}; \mathscr{V}'). \tag{2.9}$$

Then the system Σ_2 can be equivalently written as

$$x(t)+\int_0^t \mathscr{A}x(s)\,ds = x_0 + \mathscr{B}W(t), \tag{2.10}$$

where $x(t)=(u(t), \dot{u}(t)) \in L^2(\Omega; C([0, T]; \mathscr{H}))$ and

$$\mathscr{B}W(t)=(0, W(t))'.$$

Corresponding to the system (2.10), we set the observation mechanism

$$y(t)=\int_0^t Cu(s)\,ds + V(t), \tag{2.11}$$

where $C \in \mathscr{L}(H; \mathbb{R}^q)$. We can write Eq. (2.11) equivalently as

$$y(t)=\int_0^t \mathscr{C}x(s)\,ds + V(t), \tag{2.12}$$

where $\mathscr{C} \in \mathscr{L}(\mathscr{H}; \mathbb{R}^q)$ defined by $\mathscr{C}\boldsymbol{\phi} = C\phi_1$, and $V(t)$ is a vector-valued Brownian motion on $\mathbb{R}^q$ with known incremental covariance. Without loss of generality, we can assume this to be the identity matrix.

Let us now explain the identification problem. θ denotes the finite dimension vector of all the unknown parameters in A. Let θ_0 denote their true values and we assume that the parameter set $\Theta \subset \mathbb{R}^k$ is compact. We want to determine the "best" estimate of θ_0, denoted by $\hat{\theta}_T$, in Θ based on the observation $y(t)$, $0 \leqslant t \leqslant T$.

THEOREM 2.2. *Assume that $A(\theta)$ are regular such that $A(\theta)\phi \in H$ for $\phi \in H$ and $\forall \theta \in \Theta$. Then $-\mathscr{A}(\theta)$ generates a strongly continuous semigroup $S(t; \theta)$ such that*

$$\|S(t;\theta)\| \leqslant e^{-\omega_1 t}, \tag{2.13}$$

where $\omega_1 > 0$ is independent of θ.

Proof. From [4, Theorem 2.2.3] we know that $D(A^{1/2}(\theta)) = V$, under the assumptions of the theorem. It follows from [5] (see also [6, p. 353]) that $\mathscr{A}(\theta)$, with domain $D(A) \times V$ generates a strongly continuous contraction semigroup $S(t; \theta)$. To prove (uniform) exponential stability, we use [6, Lemma 2.1]. To apply this result, we need to verify that $\omega(-A) \triangleq \sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(-A)\} < 0$, where $\sigma(-A)$ denotes the spectrum of $-A$. Because of assumption (A-1), $\omega(-A(\theta)) < -\alpha$ for all $\theta \in \Theta$. Theorem 2.2 then follows directly from [6, Lemma 2.1]. ∎

Because of Theorem 2.2, we know that the system will reach a steady state. From now on, we assume that the system has already reached a steady state. This assumption is strictly not necessary for studying consistency of the parameter estimates, but will simplify some expressions encountered later in the paper.

We use the method of maximum likelihood for estimating the unknown parameters. Let P_θ, $P_{\theta T}$, $\theta \in \Theta$, $T \in [0, \infty)$, denote the probability measures induced on $C([0, \infty); \mathbb{R}^q)$, $C([0, T]; \mathbb{R}^q)$, respectively, by $y(t; \theta)$. Then $P_{\theta T}$ is absolutely continuous with respect to $P_{\theta_0 T}$ for all $T \in [0, \infty)$ and, as shown in [2],

$$\frac{dP_{\theta T}}{dP_{\theta_0 T}}(y) = \exp -\frac{1}{2}\left(\int_0^T \|\mathscr{C}\hat{x}(t;\theta) - \mathscr{C}\hat{x}(t;\theta_0)\|^2\, dt\right.$$
$$\left. -2\int_0^T [(\mathscr{C}\hat{x}(t;\theta) - \mathscr{C}\hat{x}(t;\theta_0)), d\tilde{W}(t;\omega)]\right), \qquad P_{\theta_0} \text{ a.s.}, \tag{2.14}$$

where $\hat{x}(t; \theta_0) \triangleq E[x(t; \theta_0) \mid y(s),\ 0 \leqslant s \leqslant t]$ which is a functional of $y(s)$, $0 \leqslant s \leqslant t$, $\hat{x}(t; \theta)$ is given by the same functional expression with θ replacing θ_0 and $\tilde{W}(t; \omega)$, $t \geqslant 0$, is a P_{θ_0}-Brownian motion.

Let $\hat{\theta}_T(y) \in \Theta$ be such that

$$\left.\frac{dP_{\theta T}}{dP_{\theta_0 T}}(y)\right|_{\theta = \hat{\theta}_T} \geqslant \frac{dP_{\theta' T}}{dP_{\theta_0 T}}(y) \tag{2.15}$$

for all $\theta' \in \Theta$, $T \in [1, \infty)$ and for a.e. $y \cdot \hat{\theta}_T$ is a maximum likelihood estimate of θ_0 based on $y(s)$, $0 \leqslant s \leqslant T$.

3. The Filtering Problem

We omit the explicit mention of θ_0 when working with the true parameter. In steady state, the filter $\hat{x}(t)$ is given by

$$\hat{x}(t) + \int_0^t \mathscr{A}\hat{x}(s)\, ds = \int_0^t \mathscr{K}\, dz(s) \tag{3.1}$$

under the assumption that x_0 is zero-mean Gaussian, independent of W and V. $z(t)$, the so-called *innovation process* is defined by

$$z(t) = y(t) - \int_0^t \mathscr{C}\hat{x}(s)\, ds \tag{3.2}$$

and the gain $\mathscr{K}$ is given by

$$\mathscr{K} = \mathscr{P}\mathscr{C}^*. \tag{3.3}$$

$\mathscr{P}$ is the unique solution of the algebraic Riccati equation

$$-\mathscr{A}\mathscr{P}-\mathscr{P}\mathscr{A}^*+\mathscr{Q}-\mathscr{P}\mathscr{C}^*\mathscr{C}\mathscr{P}=0, \tag{3.4}$$

where

$$\mathscr{Q}=\begin{pmatrix}0 & 0\\ 0 & \mathscr{Q}\end{pmatrix} \tag{3.5a}$$

and the adjoint operators of $\mathscr{A}$ and $\mathscr{C}$ are defined by

$$\mathscr{A}^*=\begin{pmatrix}0 & 1\\ -A & k\end{pmatrix} \tag{3.5b}$$

and, for $z\in\mathbb{R}^q$,

$$\mathscr{C}^*z=(A^{-1}C^*z, 0)'. \tag{3.5c}$$

Consider now the parameter dependent algebraic Riccati equation

$$-\mathscr{A}(\theta)\,\mathscr{P}(\theta)-\mathscr{P}(\theta v)\,\mathscr{A}(\theta)^*+\mathscr{Q}-\mathscr{P}(\theta)\,\mathscr{C}^*\mathscr{C}\mathscr{P}(\theta)=0. \tag{3.4'}$$

THEOREM 3.1. *Under the conditions of Theorem* 2.1 *and* $C\in\mathscr{L}(H;\mathbb{R}^q)$, *assuming that all coefficients of* $A(\theta)$ *are regular such that* $A(\theta)\phi\in H$ *for* $\phi\in V$, *the solution of* (3.4′) *satisfies*

$$\mathscr{P}(\theta)=\begin{pmatrix}\mathscr{P}_0(\theta) & \mathscr{P}_1(\theta)\\ \mathscr{P}_2(\theta) & \mathscr{P}_3(\theta)\end{pmatrix} \tag{3.6}$$

where $\mathscr{P}_0\in\mathscr{L}(V;V)$, $\mathscr{P}_1\in\mathscr{L}(H;V)$, $\mathscr{P}_2\in\mathscr{L}(V;H)$ *and* $\mathscr{P}_3\in\mathscr{L}(H;H)$.

Proof. In order to study the regularity property of solution to the operator Riccati equation, we use the technique proposed by Lions [7, p. 307]. Thus, we consider the optimal control problem for the deterministic system,

$$\frac{d\boldsymbol{\phi}(t)}{dt}+\mathscr{A}^*\boldsymbol{\phi}(t)+\mathscr{C}^*f(t)=0 \qquad \text{in } (s, T) \tag{3.7a}$$

$$\boldsymbol{\phi}(s)=\mathbf{g}\in\mathscr{H}, \tag{3.7b}$$

where we do not mention the dependence of $\mathscr{A}^*$ on θ, and the cost function is given by

$$J(f)=\int_s^T \{[\mathscr{Q}\boldsymbol{\phi}(t), \boldsymbol{\phi}(t)]+|f(t)|^2_{\mathbb{R}^q}\}\,dt, \tag{3.8}$$

where $f \in L^2([s, T]; \mathbb{R}^q)$. Setting $\boldsymbol{\phi}(t) = (\phi_1(t), \phi_2(t))$ and $\mathbf{g} = (g_1, g_2)$, Eq. (3.7) can be rewritten as

$$\phi_1(t) = A^{-1}\left(\frac{d\phi_2(t)}{dt} + k\phi_2(t)\right) \tag{3.9a}$$

$$\frac{d^2\phi_2(t)}{dt^2} + k\frac{d\phi_2(t)}{dt} + A\phi_2(t) + C^*f(t) = 0 \tag{3.9b}$$

$$\phi_1(s) = g_1, \qquad \phi_2(s) = g_2$$

the initial conditions of Eq. (3.9b) are given by

$$\phi_2(s) = g_2 \in H$$

and $d\phi_2(s)/dt = Ag_1 - kg_2 \in V'$.

It follows that Eq. (3.9b) has a unique solution such that

$$\left(\phi_2, \frac{d\phi_2}{dt}\right) \in C((s, T); H \times V'). \tag{3.10}$$

From (3.9a) and (3.10), we conclude that

$$\boldsymbol{\phi} \in C((s, T); \mathscr{H}). \tag{3.11}$$

From the definition of $\mathscr{Q}$ and Eq. (3.11), we find that

$$[\mathscr{Q}\boldsymbol{\phi}(t), \boldsymbol{\phi}(t)] \leqslant \text{const}, \qquad \forall t \in [0, T].$$

This implies that the control problem (3.7)–(3.8) is well formulated. Using once more the optimal control theory of Lions [7], we derive the adjoint equation

$$-\frac{d\boldsymbol{\rho}(t)}{dt} + \mathscr{A}\boldsymbol{\rho}(t) = \mathscr{Q}\boldsymbol{\phi}(t) \tag{3.12a}$$

$$\boldsymbol{\rho}(T) = 0 \tag{3.12b}$$

and the optimal control $f^0(t)$ is given by $\mathscr{C}\boldsymbol{\rho}(t)$. In (3.12), setting $\boldsymbol{\rho}(t) = (\rho_1(t), \rho_2(t))'$, we have

$$\rho_2(t) = -\frac{d\rho_1(t)}{dt} \tag{3.13a}$$

$$\frac{d^2\rho_1(t)}{dt^2} - k\frac{d\rho_1(t)}{dt} + A\rho_1(t) = Q\phi_2(t) \tag{3.13b}$$

$$\rho_1(T) = \rho_2(T) = 0. \tag{3.13c}$$

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Since $Q\phi_2 \in C((s, T); H)$, Eq. (3.13b) has a unique solution such that

$$\left(\rho_1, \frac{d\rho_1}{dt}\right) \in C((s, T); \mathscr{H}).$$

It follows that $\boldsymbol{\rho} \in C((s, T); \mathscr{H})$.

By using the decoupling technique, the solution of the nonstationary Riccati equation satisfies

$$\boldsymbol{\rho}(s) = \mathscr{P}(s)\mathbf{g}$$

and, therefore, we have

$$\mathscr{P}(s) \in \mathscr{L}(\mathscr{H}; \mathscr{H}).$$

The remaining half of the proof consists of studying the case $s = -\infty$. Noting that $-\mathscr{A}^*$ generates an exponentially stable semigroup (see Theorem 2.2), we can apply the arguments in [7, Chap. 3, Sect. 3]. Furthermore, from [8], it follows that Eq. (3.4′) has a unique nonnegative solution. ∎

For later development, we need the following result:

THEOREM 3.2. *Under conditions of Theorem* 3.1, *there exists a strongly continuous semigroup* $Y(t; \theta)$ *such that*

$$\frac{dY(t;\theta)}{dt} = (-\mathscr{A}(\theta) - \mathscr{P}(\theta)\mathscr{C}^*\mathscr{C}))\ Y(t;\theta) \tag{3.14a}$$

$$Y(0;\theta) = I \tag{3.14b}$$

and for $x \in \mathscr{H}$,

$$|Y(t;\theta)x|_{\mathscr{H}}^2 \leqslant d_2 e^{-\omega_2 t} |x|_{\mathscr{H}}^2, \tag{3.15}$$

where $d_2 \geqslant 1$ *and* $\omega_2 > 0$ *are independent of* θ.

Proof. Using the approach of Lions [7, p. 309, Lemma 5.4] we first obtain the uniform bound

$$[\mathscr{P}(\theta)x, x]_{\mathscr{H}} \leqslant c|x|_{\mathscr{H}}^2, \tag{3.16}$$

where the constant c is independent of θ. The fact that c is independent of θ follows from the uniform exponential bound (2.13) proved in Theorem 2.2.

Now to show (3.15), introduce the following evolution equation

$$\frac{d\bar{x}(t)}{dt} = (-\mathscr{A}^*(\theta) - \mathscr{C}^*\mathscr{C}\mathscr{P}(\theta))\,\bar{x}(t)$$
$$\bar{x}(0) = x \in \mathscr{H}. \tag{3.17}$$

Choosing $[\mathscr{P}(\theta)\,\bar{x}(t),\ \bar{x}(t)]_{\mathscr{H}}$ as a Lyapunov function and using the algebraic Riccati Eq. (3.4′), we obtain

$$[\mathscr{P}(\theta)\,\bar{x}(t), \bar{x}(t)]_{\mathscr{H}} + \int_0^t \{|\mathscr{C}\mathscr{P}(\theta)\,\bar{x}(s)|^2_{\mathbb{R}^q} + [\mathscr{Q}\bar{x}(s), \bar{x}(s)]_{\mathscr{H}}\}\,ds$$
$$= [\mathscr{P}(\theta)x, x]_{\mathscr{H}}. \tag{3.18}$$

Noting that $\mathscr{Q} \geqslant 0$, we take the limit in (3.18) as $t \to \infty$ to obtain

$$\int_0^\infty |\mathscr{C}\mathscr{P}(\theta)\,\bar{x}(s)|^2_{\mathbb{R}^q} \leqslant [\mathscr{P}(\theta)x, x]_{\mathscr{H}} \leqslant c\,|x|^2_{\mathscr{H}}. \tag{3.19}$$

Now let $Y^*(t;\theta)$ be the semigroup generated by the operator $(-\mathscr{A}^*(\theta) - \mathscr{C}^*\mathscr{C}\mathscr{P}(\theta))$; that is,

$$\frac{dY^*(t;\theta)}{dt} = (-\mathscr{A}^*(\theta) - \mathscr{C}^*\mathscr{C}\mathscr{P}(\theta))\,Y^*(t;\theta)$$
$$Y^*(0;\theta) = I. \tag{3.20}$$

The semigroup $Y^*(t;\theta)$ can be represented by

$$Y^*(t;\theta)x = S^*(t;\theta)x - \int_0^t S^*(t-s;\theta)\,\mathscr{C}^*\mathscr{C}\mathscr{P}(\theta)\,Y^*(s;\theta)x\,ds, \tag{3.21}$$

where $S^*(t;\theta)$ is the semigroup generated by $-\mathscr{A}^*(\theta)$, and we have from Theorem 2.2,

$$|S^*(t;\theta)x|_{\mathscr{H}} \leqslant d_1 e^{-\omega_1 t}|x|_{\mathscr{H}}. \tag{3.22}$$

It follows from (3.21) and (3.22) that

$$|Y^*(t;\theta)x|_{\mathscr{H}} \leqslant |S^*(t;\theta)x|_{\mathscr{H}} + \int_0^t |S^*(t-s;\theta)\,\mathscr{C}^*\mathscr{C}\mathscr{P}(\theta)\,Y^*(s;\theta)x|_{\mathscr{H}}\,ds$$
$$\leqslant d_1 e^{-\omega_1 t}|x|_{\mathscr{H}} + \textstyle\int_0^t \tilde{d}e^{-\omega_1(t-s)}|\mathscr{C}\mathscr{P}(\theta)\,Y^*(s;\theta)x|_{\mathbb{R}^q}\,ds$$
$$\text{(for some } \tilde{d} > 0)$$
$$= d_1 e^{-\omega_1 t}|x|_{\mathscr{H}} + \int_0^t \tilde{d}e^{-\omega_1(t-s)}|\mathscr{C}\mathscr{P}(\theta)\,\bar{x}(s)|_{\mathbb{R}^q}\,ds.$$

Young's inequality (see [8, p. 256]) implies that

$$\left(\int_0^\infty |Y^*(t;\theta)x|_{\mathscr{H}}^2\,dt\right)^{1/2}$$

$$\leqslant d_1\left(\int_0^\infty e^{-\omega_1 t}\,dt\right)^{1/2}|x|_{\mathscr{H}}+\tilde{d}\left(\int_0^\infty e^{-\omega_1\tau}\,d\tau\right)\left(\int_0^\infty |\mathscr{C}\mathscr{P}(\theta)\,\bar{x}(s)|_{\mathscr{H}}^2\,ds\right)^{1/2}$$

$$\leqslant d_2|x|_{\mathscr{H}},\qquad \text{using (3.19)}.$$

This implies, following the proof of the important result of Datko [9, pp. 614–615, Theorem and Corollary] that $Y^*(t;\theta)$ is bounded by $d_2 e^{-\omega_2 t}$, where $d_2 \geqslant 1$ and $\omega_2 > 0$ are independent of θ. This proves the desired result. ▮

4. Consistency of Maximum Likelihood Estimates

We assume that the coefficients of the operator $A(\theta)$ are differentiable with respect to θ. Furthermore, assume that there exists a constant N such that

$$\sup_{\theta\in\Theta}\left|\frac{\partial A(\theta)}{\partial\theta_i}\right|_{\mathscr{L}(V;V')}\leqslant N,\qquad \forall i=1,\dots,k. \tag{B–1}$$

Example 4.1. Let $V=H_0^1(0,1)$ and $V'=H^{-1}(0,1)$. Consider the following operator

$$\langle A(\theta)\phi,\psi\rangle=\int_0^1 a(x,\theta)\frac{\partial\phi(x)}{\partial x}\frac{\partial\psi(x)}{\partial x}\,dx,\qquad \text{for}\quad \phi,\psi\in V. \tag{4.1}$$

In this example, for condition (B–1) to hold, it is sufficient to assume that

$$\sup_{x\in[0,1],\theta\in\Theta}\left|\frac{\partial a(x;\theta)}{\partial\theta}\right|\leqslant N. \tag{4.2}$$

Theorem 4.1. *Under* (B–1) *and conditions stated in Theorem* 3.1, *the solution* $\mathscr{P}(\theta)$ *of Eq.* (3.4′) *is differentiable with respect to* θ *and satisfies, for* $i=1,\dots,k$, *the equation*

$$(-\mathscr{A}(\theta)-\mathscr{P}(\theta)\mathscr{C}^*\mathscr{C})\frac{\partial\mathscr{P}(\theta)}{\partial\theta_i}+\frac{\partial\mathscr{P}(\theta)}{\partial\theta_i}(-\mathscr{A}(\theta)^*-\mathscr{C}^*\mathscr{C}\mathscr{P}(\theta))$$

$$=\begin{pmatrix}0 & 0\\ \dfrac{\partial A(\theta)}{\partial\theta_i} & 0\end{pmatrix}\mathscr{P}(\theta)+\mathscr{P}(\theta)\begin{pmatrix}0 & 0\\ -\dfrac{\partial A(\theta)}{\partial\theta_i} & 0\end{pmatrix} \tag{4.3}$$

with

$$\frac{\partial \mathscr{P}(\theta)}{\partial \theta_i} \in \mathscr{L}(\mathscr{H}; \mathscr{H}). \tag{4.4}$$

Proof. With the aid of the properties of the operators $\mathscr{P}_i$, $i = 0, 1, 2, 3$, the term on the r.h.s. of (4.3) belongs to $\mathscr{L}(\mathscr{H}; \mathscr{H})$. From Theorem 3.2, we derive (4.4). ∎

It is a simple exercise to give the explicit form for the estimate $\hat{x}$ (see Balakrishnan [10]),

$$\hat{x}(t; \theta) = \int_0^t \ell(t-s; \theta)(I + \mathscr{k}(t, s, \theta_0))\, dz(s), \tag{4.5}$$

where

$$\ell(t-s; \theta) \triangleq Y(t-s; \theta)\, \mathscr{P}(\theta) \mathscr{C}^* \tag{4.6}$$

and

$$\mathscr{k}(t, s; \theta_0) \triangleq \int_s^t S(\tau - s; \theta_0)\, \mathscr{P}(\theta_0) \mathscr{C}^*\, d\tau. \tag{4.7}$$

THEOREM 4.2. *There exists a constant* N_1 (*independent of* θ *and* θ_0) *such that*

$$|\mathscr{k}(t, s; \theta_0)|^2_{\mathscr{L}(\mathbb{R}^q; \mathbb{R}^q)} \leqslant N_1 \tag{4.8a}$$

and

$$|\ell(t-s; \theta)|^2_{\mathscr{V}} \leqslant N_1 e^{-\omega_2(t-s)}. \tag{4.8b}$$

Proof. From Theorems (3.1) and (2.2), estimate (4.8a) can be readily obtained. Furthermore, Theorem 3.2 implies that

$$\begin{aligned}
|\ell(t-s; \theta)|^2_{\mathscr{V}} &\leqslant d_2 e^{-\omega_2(t-s)} |\mathscr{P}(\theta) \mathscr{C}^*|^2_{\mathscr{V}} \\
&= d_2 e^{-\omega_2(t-s)} \{ |\mathscr{P}_0(\theta)\, A^{-1}(\theta) \mathscr{C}^*|^2_{\mathscr{L}(\mathbb{R}^q; V)} + |\mathscr{P}_2(\theta)\, A^{-1}(\theta) \mathscr{C}^*|^2_{\mathscr{L}(\mathbb{R}^q; V)} \} \\
&\leqslant N_1 e^{-\omega_2(t-s)}
\end{aligned}$$

on account of Theorem 3.1 and the fact that $\mathscr{C}^* \in \mathscr{L}(\mathbb{R}^q; H)$.

THEOREM 4.3. *Under* (B–1), *the estimate holds,*

$$\left| \frac{\partial \ell(t-s; \theta)}{\partial \theta_i} \right|^2_{\mathscr{L}(\mathbb{R}^q; \mathscr{V}')} \leqslant N_2 e^{-\omega_3(t-s)}, \tag{4.9}$$

where $\omega_3 > 0$ *and* N_2 *are independent of* θ.

Proof. From (4.6), we have

$$\frac{d\ell(t-s;\theta)}{dt}=(-\mathscr{A}(\theta)-\mathscr{P}(\theta)\,\mathscr{C}^*\mathscr{C})\,\ell(t-s;\theta)$$

$$\ell(0;\theta)=\mathscr{P}(\theta)\mathscr{C}^*.$$

Differentiating with respect to θ_i, obtain we

$$\frac{d}{dt}\left(\frac{\partial\ell(t-s;\theta)}{\partial\theta_i}\right)=(-\mathscr{A}(\theta)-\mathscr{P}(\theta)\,\mathscr{C}^*\mathscr{C})\frac{\partial\ell(t-s;\theta)}{\partial\theta_i}+\left(-\frac{\partial\mathscr{A}(\theta)}{\partial\theta_i}-\frac{\partial\mathscr{P}(\theta)}{\partial\theta_i}\mathscr{C}^*\mathscr{C}\right)\ell(t-s;\theta)$$

$$\frac{\partial\ell(0;\theta)}{\partial\theta_i}=\frac{\partial\mathscr{P}(\theta)}{\partial\theta_i}\mathscr{C}^*.$$

Using the semigroup $Y(t-s;\theta)$, the above equation can be rewritten as

$$\frac{\partial\ell(t-s;\theta)}{\partial\theta_i}=Y(t-s;\theta)\frac{\partial\mathscr{P}(\theta)}{\partial\theta_i}\mathscr{C}^*+\int_s^t Y(t-\tau;\theta)\times\left(-\frac{\partial\mathscr{A}(\theta)}{\partial\theta_i}-\frac{\partial\mathscr{P}(\theta)}{\partial\theta_i}\mathscr{C}^*\mathscr{C}\right)\ell(\tau-s;\theta)\,d\tau. \tag{4.10}$$

Then, applying Theorems 2.2 and 3.2, we have

$$\begin{aligned}\left|\frac{\partial\ell(t-s;\theta)}{\partial\theta_i}\right|_{\mathscr{L}(\mathbb{R}^q;\mathscr{V}')}&\leqslant d_1e^{-\omega_2(t-s)}\left|\frac{\partial\mathscr{P}(\theta)}{\partial\theta_i}\mathscr{C}^*\right|_{\mathscr{L}(\mathbb{R}^q;\mathscr{V}')}\\&+\int_0^t d_2e^{-\omega_2(t-\tau)}\left\{\left|\begin{pmatrix}0&0\\-\dfrac{\partial A(\theta)}{\partial\theta_i}&0\end{pmatrix}\ell(\tau-s;\theta)\right|_{\mathscr{L}(\mathbb{R}^q;\mathscr{V}')}\right.\\&\left.+\left|\frac{\partial\mathscr{P}(\theta)}{\partial\theta_i}\mathscr{C}^*\mathscr{C}\ell(\tau-s;\theta)\right|_{\mathscr{L}(\mathbb{R}^q;\mathscr{V}')}\right\}d\tau.\end{aligned}$$

Writing $\ell(t-s;\theta)=(\ell_1(t-s;\theta),\ \ell_2(t-s;\theta))'$ and using (B–1),

$$\begin{aligned}\left|\begin{pmatrix}0&0\\-\dfrac{\partial A(\theta)}{\partial\theta_i}&0\end{pmatrix}\ell(\tau-s;\theta)\right|_{\mathscr{L}(\mathbb{R}^q;\mathscr{V}')}&=\left|\frac{\partial A(\theta)}{\partial\theta_i}\ell_1(\tau-s;\theta)\right|_{\mathscr{L}(\mathbb{R}^q;\mathscr{V}')}\\&\leqslant N|\ell_1(\tau-s;\theta)|_{\mathscr{L}(\mathbb{R}^q;\mathscr{V})}\\&\leqslant N|\ell_1(\tau-s;\theta)|_{\mathscr{L}(\mathbb{R}^q;\mathscr{H})}\end{aligned}$$

and from Theorem 4.1, it follows that

$$\left|\frac{\partial \mathscr{P}(\theta)}{\partial \theta_i} \mathscr{C}^* \mathscr{C} \ell(\tau - s; \theta)\right|_{\mathscr{L}(\mathbb{R}^q; \mathscr{V}')}$$
$$\leqslant N_3 |\ell(\tau - s; \theta)|_{\mathscr{L}(\mathbb{R}^q; \mathscr{H})}, \qquad \text{for some } N_3 > 0.$$

We, therefore, have

$$\left|\frac{\partial \ell(t - s; \theta)}{\partial \theta_i}\right|_{\mathscr{L}(\mathbb{R}^q; \mathscr{V}')}$$
$$\leqslant N_4 \left\{ e^{-\omega_2(t-s)} + \int_s^t e^{-\omega_2(t-\tau)} |\ell(\tau - s; \theta)|_{\mathscr{L}(\mathbb{R}^q; \mathscr{H})} \, d\tau \right\}. \qquad (4.11)$$

Applying the bound obtained in Theorem 4.2 to Eq. (4.11), the estimate (4.9) can be derived. ∎

Consistency of the maximum likelihood estimate, which is the main result in this paper, is established next.

LEMMA 4.1. *In addition to all the conditions stated in Section* 3, *we assume now that* (B–1) *also holds. Then we have*

$$\lim_{T \to \infty} \sup_{\theta \in \Theta} \left(\frac{1}{T} \int_0^T (\mathscr{C}\hat{x}(t; \theta) - \mathscr{C}\hat{x}(t, \theta_0)) \, d\tilde{W}(t) \right) = 0 \qquad a.s. \qquad (4.12)$$

Proof. Set

$$g(\theta, T) = \frac{1}{T} \int_0^T (\mathscr{C}\hat{x}(t, \theta) - \mathscr{C}\hat{x}(t; \theta_0)) \, d\tilde{W}(t). \qquad (4.13)$$

Using the argument of [11, p. 201, Lemma 4.1] it is easy to show that $\lim_{T \to \infty} g(\theta, T) = 0$ a.s. for fixed θ.

In order to derive (4.12), we must show that $g(\theta, T)$ is uniformly continuous in θ on Θ, uniformly in $T \in [1, \infty)$ a.s.

To show this, we note that

$$E\{|\mathscr{C}\hat{x}(t; \theta) - \mathscr{C}\hat{x}(t; \theta')|^2_{\mathbb{R}^q}\} \leqslant ME|\hat{x}(t, \theta) - \hat{x}(t; \theta')|^2_{\mathscr{V}'}.$$

Now, using Eq. (4.5), we have

$$E|\hat{x}(t; \theta) - \hat{x}(t; \theta)|^2_{\mathscr{V}'}$$
$$\leqslant M_1 \int_0^t |(\ell(t - s; \theta) - \ell(t - s; \theta'))([I + \ell(t, s; \theta_0))|^2_{\mathscr{L}(\mathbb{R}^q; \mathscr{V}')} \, ds$$

$$\leqslant M_2 \int_0^t |\ell(t-s;\theta)-\ell(t-s;\theta')|^2_{\mathscr{L}(\mathbb{R}^q;\mathscr{V}')}\,ds \qquad \text{(using (4.8a))}$$

$$= M_2 \int_0^t |\nabla_\theta \ell(t-s;\theta)|_{\theta=\tilde{\theta}}(\theta-\theta')|^2_{\mathscr{L}(\mathbb{R}^q;\mathscr{V}')}\,ds \qquad \text{for some } \tilde{\theta}\in\Theta$$

$$\leqslant M_3 \int_0^t e^{-\omega 3(t-s)}\,ds\,|\theta-\theta'|^2_{\mathbb{R}^k} \qquad \text{(using Theorem 4.3).}$$

We, therefore, obtain

$$E|\mathscr{C}\hat{x}(t;\theta)-\mathscr{C}\hat{x}(t;\theta')|^2_{\mathbb{R}^q} \leqslant M_4|\theta-\theta'|^2_{\mathbb{R}^k}. \tag{4.14}$$

Using now the argument given in Borkar and Bagchi [11] we can show, from (4.14), that $g(\theta, T)$ is uniformly continuous in θ. This establishes the lemma. ∎

THEOREM 4.4. *Let $\mathscr{N}$ be the set of trajectories y for which the statement of Lemma 4.1 fails; that is, $\mathscr{N}$ is the set of measure zero outside which the preceding lemma holds. For each $\omega\notin\mathscr{N}$ and for $T\in[1,\infty)$, let $\hat{\theta}_T$ be the maximum likelihood estimate of θ_0. Under all the conditions stated previously, we have*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |\mathscr{C}\hat{x}(t,\hat{\theta}_T)-\mathscr{C}\hat{x}(t;\theta_0)|^2_{\mathbb{R}^q}\,dt=0 \qquad a.s. \tag{4.15}$$

Proof. The maximum likelihood estimate $\hat{\theta}_T$ satisfies

$$g(\theta,T)|_{\theta=\hat{\theta}_T} \geqslant \left(\frac{1}{2T}\int_0^T |\hat{x}(t;\hat{\theta}_T)-\hat{x}(t;\theta_0)|^2_{\mathbb{R}^q}\,dt\right) \geqslant 0 \qquad \text{a.s.}$$

Applying Lemma 4.1, the desired result follows. ∎

The following result establishes consistency:

COROLLARY 4.1. $\hat{\theta}_T \to \{\theta \mid P_{\theta_0}\{\mathscr{C}\hat{x}(T;\theta)=\mathscr{C}\hat{x}(T;\theta_0)\ \forall T\}=1\}$ *a.s.*

Proof. Follows from the previous theorem and the fact that

$$\frac{1}{T}\int_0^T \|\mathscr{C}\hat{x}(t;\theta)-\mathscr{C}\hat{x}(t;\theta_0)\|^2_{\mathbb{R}^q}\,ds \to E\|\mathscr{C}\hat{x}(t;\theta)-\mathscr{C}\hat{x}(t;\theta_0)\|^2_{\mathbb{R}^q},$$

where the expectation is with respect to the stationary measure. ∎

5. Conclusion

An explicit sufficient condition for guaranteeing convergence of the maximum likelihood estimate to the true parameter value for stochastic hyperbolic systems is given. It is interesting to see how the technique developed in this paper may be used for studying consistency with other types of state noise models. For example, one can study Brownian motion with white noise in space [10] and/or Brownian motion in time with impulse in space as intensity, the so-called pointwise disturbance [12].

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