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Symmetries and Conservation Laws of the System: $u_x = vw_x, \quad v_y = uw_y, \quad uv + w_{xx} + w_{yy} = 0$

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Abstract. After a short exposition of the theory of local and nonlocal symmetries and conservation laws for systems of PDE's, results on these and the recursion operator are listed for the system of PDE's $u_x = vw_x$, $v_y = uw_y$, $uv + w_{xx} + w_{yy} = 0$. In between the methods of computation are explained.

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1. Introduction

The following system of PDE's plays an important role in geometry [11]

$$\frac{\partial u}{\partial x} + (u-v)\frac{\partial w}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} - (u-v)\frac{\partial w}{\partial y} = 0,$$
$$uv\,\mathrm{e}^{2w} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0. \tag{1.1}$$

The underlying geometry is defined as the manifold of local surfaces which admit nontrivial isometries conserving principal curvatures, the so-called isothermic surfaces. Sym and Cieśliński intend to write a paper on the soliton solution found, the Bäcklund transformations of the system as well of the surfaces, and to give a proof of integrability by the Painlevé test. In this report, it is proven that this system (1.1) admits an infinite hierarchy of commuting symmetries and conservation laws. Results will be computed for the system (1.1) simplified by the transformation $u \mapsto u\,\mathrm{e}^w$, $v \mapsto v\,\mathrm{e}^w$, i.e. the system

$$\frac{\partial u}{\partial x} - v\frac{\partial w}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} - u\frac{\partial w}{\partial y} = 0,$$
$$uv + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0. \tag{1.2}$$

A symmetry equation package recently developed [1] has been applied for computation of symmetries, conservation laws and nonlocal symmetries of system (1.2). Insight gained by the theory of deformations [6] led to the recursion operator for symmetries.

All these results have been established in the differential geometric framework. Several authors (e.g. Olver [8] and Vinogradov [13]) have discussed the various aspects involved and in this paper we will follow Vinogradov's work [13]. Section 2 is dedicated to the algebraic notion of a system of PDE's, for which the jet bundle formulation is essential. Aspects of symmetries and conservation laws are discussed in Section 3. In Section 4, the theory of coverings set up by Krasil'shchik and Vinogradov [7] is explained and used to define nonlocal symmetries and conservation laws properly. In Section 5, the methods applied for computation of symmetries and conservation laws are described. In Section 6 results attained by the methods described in Section 5 are listed, i.e. local and nonlocal symmetries and conservation laws and the recursion operator for symmetries of system (1.2).

2. The Geometrical Framework of Systems of PDE's

In differential geometric theory a system of PDE's is defined as an smooth submanifold of some jet bundle. We will explain this for a nth order system Δ involving p independent variables $x = (x_1, \ldots, x_p)$ and q dependent variables $u = (u^1, \ldots, u^q)$. Multi-indices $I = (i_1, \ldots, i_p)$ enable a compact notation for derivatives of u

$$u_I^\alpha \equiv \frac{\partial^{|I|}}{\partial x^I} u^\alpha = \frac{\partial^{i_1+\cdots+i_p}}{\partial x_1^{i_1} \cdots \partial x_p^{i_p}} u^\alpha, \quad \text{e.g.} \quad u^2_{(2,0,1)} \equiv \frac{\partial^3 u^2}{\partial x_1^2 \partial x_3}$$

and $u^2_{(2,0,1)}$ is denoted by u^2_{201} if no confusion is possible. The kth order jet bundle J^k is a manifold with local coordinate representation $(x, u_I)_{|I| \leqslant k}$. The nth order system Δ is the submanifold $\mathcal{Y} = \{F = 0\} \subset J^n$, e.g. the Korteweg–DeVries equation is given by

$$\begin{aligned} \mathcal{Y} &= \{u_{01} = u u_{10} + u_{30}\} \subset J^3 \\ &= \{(x, t, u, u_{10}, u_{01}, u_{20}, u_{11}, u_{02}, u_{30},\ u_{21}, u_{12}, u_{03})\}. \end{aligned}$$

The infinite jet bundle J^∞ will be of most interest to us.

The algebraic analogue of the partial derivative $\partial/\partial x_k$ is given by the total derivative $\mathcal{D}_k$

$$\mathcal{D}_k = \frac{\partial}{\partial x_k} + \sum_{\alpha=1}^{q} \sum_{|I| \geqslant 0} u^\alpha_{I,k} \frac{\partial}{\partial u^\alpha_I}$$

$$\text{for } k = 1, \ldots, p \quad \text{with } u^\alpha_{I,k} \equiv \frac{\partial^{|I|+1}}{\partial x^I \partial x_k} u^\alpha. \tag{2.1}$$

Likewise, the higher order partial derivative $\partial^{|I|}/\partial x^I$ is represented in the algebraic setting by $\mathcal{D}^I = \mathcal{D}_1^{i_1} \circ \cdots \circ \mathcal{D}_p^{i_p}$, where $\mathcal{D}_k^i = \mathcal{D}_k^{i-1} \circ \mathcal{D}_k$. Differential consequences of the nth order system $\mathcal{Y} = \{F = 0\} \subset J^n$ are given by prolongation. The kth order prolongation of the system $\mathcal{Y}$ is given by $\mathcal{Y}_k = \{\mathcal{D}^I F = 0\colon\ 0 \leqslant |I| \leqslant k\} \subset J^{k+n}$. The infinite prolongation of $\mathcal{Y}$, denoted by $\mathcal{Y}_\infty \subset J^\infty$, is of most interest in what follows.

Local coordinates for $\mathcal{Y}_\infty$ are selected among the coordinates of J^∞; the selected ones are called internal coordinates. The remaining coordinates of J^∞ are called external coordinates; on $\mathcal{Y}_\infty$ the external coordinates are expressed in the internal coordinates. Clearly, internal coordinates may be chosen in many different ways, e.g. internal coordinates for the KDV can be chosen to be x, t, u_{i0} for $i \geqslant 0$ or x, t, u_{ij} for $i = 0, 1, 2$ and $j \geqslant 0$. In the first case t-derivates, i.e. u_{ij} with $j \geqslant 1$, are the external coordinates and, in the second case, third-order x-derivatives, i.e. u_{ij} with $i \geqslant 3$, are the external coordinates.

Geometrically, the total derivative operators (2.1) may be treated as vector fields on J^∞ and these vector fields are tangent to the (arbitrary) system of PDE's $\mathcal{Y}_\infty$. In each point $\theta \in \mathcal{Y}_\infty$ the total derivative operators generate an n-dimensional plane $\mathcal{C}_\theta = \{\sum \lambda_k \mathcal{D}_k\colon\ \lambda_1, \ldots, \lambda_p \in \mathbb{R}\}$ and all together these planes form the Cartan distribution or contact structure on $\mathcal{Y}_\infty$. This contact structure on $\mathcal{Y}_\infty$ is denoted by $\mathcal{C}$ and is a completely integrable finite-dimensional distribution.

An n-dimensional manifold $L \subset \mathcal{Y}_\infty$ is said to be a solution of the system if its tangent space $T_\theta L = \mathcal{C}_\theta$ for every $\theta \in L$. Solution manifolds correspond to solutions of the system of PDE's, for in local coordinates L can be parametrized by x and if done so $u_I^\alpha = (\partial^{|I|}/\partial x^I) f^\alpha(x)$ where f is a solution.

Functions on J^∞ depend, by definition, on a finite number of variables. Operators of the form

$$\sum_{|I| \leqslant k} F_I \mathcal{D}^I \tag{2.2}$$

constitute a natural class of operators on J^∞, called C-differential operators. Here, k is a finite number and each F_I is a $m \times n$ matrix whose elements are functions on J^∞. Both functions and C-differentials are restrictable to submanifolds of the form $\mathcal{Y}_\infty \subset J^\infty$. In the algebraic approach of a system of PDE's as an object by itself these restrictions would be required, but we will do without them.

We conclude this section with a short summary: 'A system of PDE's is a manifold $\mathcal{Y}_\infty$ ($\subset J^\infty$) provided with a contact structure $\mathcal{C}$ (induced by the total derivatives)'.

3. Symmetries and Conservation Laws

DEFINITION 1. A symmetry of a system of PDE's $\mathcal{Y}_\infty$ is a smooth contact transformation $\phi\colon\ \mathcal{Y}_\infty \to \mathcal{Y}_\infty$. A contact transformation on $\mathcal{Y}_\infty$ is a transformation that conserves the contact structure $\mathcal{C}$ on $\mathcal{Y}_\infty$, i.e. for all $\theta \in \mathcal{Y}_\infty$: $\phi_*(\mathcal{C}_\theta) = \mathcal{C}_{\phi(\theta)}$.

Symmetries map solution manifolds onto solution manifolds. In the following we need the infinitesimal variant of symmetries ([13, p. 8]).

An infinitesimal transformation on $\mathcal{Y}_\infty$, i.e. $x_k \mapsto x_k + \varepsilon a_k$ and $u_I^\alpha \mapsto u_I^\alpha + \varepsilon a_I^\alpha$, where a_k and a_I^α are functions on $\mathcal{Y}_\infty$, may be thought of as a vector field on $\mathcal{Y}_\infty$

$$X = \sum_{k=1}^{p} a_k \frac{\partial}{\partial x_k} + \sum_{\alpha=1}^{q} \sum_{|I|=1}^{\infty} a_I^\alpha \frac{\partial}{\partial u_I^\alpha}. \tag{3.1}$$

The infinitesimal transformation on $\mathcal{Y}_\infty$ has been expressed in the local coordinates of J^∞ and is well-defined if the vector field (3.1) is tangent to $\mathcal{Y}_\infty$. For $\mathcal{Y} = \{F = 0\}$ this implies that for all I the expressions $X(\mathcal{D}^I F)$ vanish on $\mathcal{Y}_\infty$.

Conservation of the contact structure on $\mathcal{Y}_\infty$ implies X to be of the form

$$X = \ni_\varphi + \sum_{k=1}^{p} \mu_k \mathcal{D}_k, \tag{3.2}$$

where $\varphi = (\varphi_1, \ldots, \varphi_q)$, φ_α and μ_k are functions on $\mathcal{Y}_\infty$ and

$$\ni_\varphi = \sum_{\alpha=1}^{q} \sum_{|I|=0}^{\infty} \mathcal{D}^I(\varphi_\alpha) \frac{\partial}{\partial u_I^\alpha}. \tag{3.3}$$

The infinitesimal contact transformation $X = \sum \mu_k \mathcal{D}_k$ is trivial. In the sequel, this part is omitted from (3.2) and the remaining part $X = \ni_\varphi$ is said to be the evolution differentiation corresponding to the generating function φ.

A useful property of the evolution differentiation (3.3) is that its action commutes with total differentiation: $\ni_\varphi(\mathcal{D}_k f) = \mathcal{D}_k(\ni_\varphi f)$, where f is an arbitrary function on $\mathcal{Y}_\infty$. This is used in the following theorem ([13, p. 10]).

THEOREM 1. *The vector field* $\ni_\varphi$ (3.3) *is a symmetry of* $\mathcal{Y} = \{F = 0\}$ *if and only if*

$$\ni_\varphi(F) = 0 \quad \textit{on } \mathcal{Y}_\infty. \tag{3.4}$$

Another way to express this Theorem 1 is in terms of the universal linearization operator l_F, which is introduced by putting

$$l_F(f) = \ni_f(F), \quad f = (f_1, \ldots, f_q) f_i \, \mathrm{e} C^\infty(\widetilde{\mathcal{Y}}_\infty). \tag{3.5}$$

Probably the reader is more familiar with the notion of the Fréchet derivative l_F, usually defined by $l_F(f)(\theta) = (\mathrm{d}/\mathrm{d}\epsilon)|_{\epsilon=0} F(x_k, u_I^\alpha + \epsilon[\mathcal{D}^I f^\alpha](\theta))$, considered $\theta = (x_k, u_I^\alpha) \in \mathcal{Y}_\infty$.

Using the local coordinates of J^∞ one obtains the following expression for l_F, where s is the number of equations

$$l_F = \sum_{|I|=1}^{\infty} \begin{bmatrix} \frac{\partial F^1}{\partial u_I^1} & \cdots & \frac{\partial F^1}{\partial u_I^q} \\ \vdots & & \vdots \\ \frac{\partial F^s}{\partial u_I^1} & \cdots & \frac{\partial F^s}{\partial u_I^q} \end{bmatrix} \mathcal{D}^I. \tag{3.6}$$

Clearly, l_F is a C-differential (2.2). The symmetry equation (3.4) may be rewritten in the form

$$l_F(\varphi) = 0 \quad \text{on } \mathcal{Y}_\infty. \tag{3.7}$$

THEOREM 2. *The commutator or Lie bracket of two symmetries $\ni_\varphi$ and $\ni_{\varphi'}$, which is denoted by $[\ni_\varphi, \ni_{\varphi'}] = \ni_\varphi \circ \ni_{\varphi'} - \ni_{\varphi'} \circ \ni_\varphi$, also is a symmetry, whose generating function is given by $\ni_\varphi(\varphi') - \ni_{\varphi'}(\varphi)$.*

DEFINITION 2. A conservation law for the system $\mathcal{Y}_\infty$ is given by the vanishing of the divergence of the tuple of functions $\Omega = (\omega_1, \ldots, \omega_p)$ defined on $\mathcal{Y}_\infty$

$$\operatorname{div}\Omega \stackrel{\text{def}}{=} \sum_{k=1}^{p} \mathcal{D}_k(\omega_k) = 0 \quad \text{on } \mathcal{Y}_\infty. \tag{3.8}$$

In fact, conservation laws are considered up to equivalence induced by trivial conservation laws like $\omega_k = \sum_i \mathcal{D}_i(a_{ik})$, where $a_{ik} = -a_{ki}$ are arbitrary functions. Because conservation laws vanish on $\mathcal{Y}_\infty$, (3.8) may be rewritten in the form

$$\operatorname{div}\Omega = A(F), \tag{3.9}$$

where A is a C-differential (2.2) of the form

$$A = \sum_{|I|=0}^{\infty} [a_{1I} \cdots a_{sI}] \mathcal{D}^I. \tag{3.10}$$

Partial integration of the right side of (3.9) leads to the characteristic form of the conservation law

$$\operatorname{div}\Omega = \sum_{i=1}^{s} \sum_{|I|=0}^{\infty} (-1)^{|I|} \mathcal{D}^I(a_{iI}) F^i + \operatorname{div}\Omega', \tag{3.11}$$

where Ω' vanishes on $\mathcal{Y}_\infty$.

The operator G^* conjugated to the C-differential $G = \sum_I G_I \mathcal{D}^I$ is defined as ([13, p. 15])

$$[G^*]_{ij} = \sum_{|I|=0}^{\infty} (-1)^{|I|} \mathcal{D}^I \circ [G_I]_{ji}. \tag{3.12}$$

DEFINITION 3. The generating function of the conservation law (3.9) is given by the s-tuple ψ

$$\psi = A^*(1) = \sum_{|I|=0}^{\infty} (-1)^{|I|} (\mathcal{D}^I(a_{1I}), \ldots, \mathcal{D}^I(a_{sI})). \tag{3.13}$$

Generating functions of conservation laws for $\mathcal{Y} = \{F = 0\}$ satisfy

$$l_F^*(\psi) = 0 \quad \text{on } \mathcal{Y}_\infty. \tag{3.14}$$

Solutions ψ of (3.14) are called adjoint symmetries [12]. An adjoint symmetry is the generating function of a conservation law for $\mathcal{Y}_\infty$, if the following additional condition, involving C-differentials B and C, is satisfied

$$l_\psi + B^* = C \circ l_F \quad \text{on } \mathcal{Y}_\infty, \quad \text{where } C = C^* \tag{3.15}$$

and where B is given by the equivalent form of (3.14)

$$l_F^*(\psi) = B(F). \tag{3.16}$$

Symmetries may be used to obtain (new) conservation laws from those already known.

THEOREM 3. *Let φ be the generating function of a symmetry of $\mathcal{Y} = \{F = 0\}$ and* $\operatorname{div}\Omega = 0$ *be a conservation law of $\mathcal{Y}_\infty$ with generating function ψ. A new conservation law is given by*

$$\operatorname{div}(\ni_\varphi(\Omega)) = \sum_{k=1}^{p} \mathcal{D}_k(\ni_\varphi(\omega_k)) = 0 \quad \textit{on } \mathcal{Y}_\infty. \tag{3.17}$$

The characteristic of the conservation law (3.17) is given by the restriction on $\mathcal{Y}_\infty$ of

$$\ni_\varphi\{\psi\} = \ni_\varphi(\psi) + D^*(\psi), \tag{3.18}$$

where the C-differential D is given by the equivalent form of (3.4)

$$\ni_\varphi(F) = D(F). \tag{3.19}$$

4. Coverings, Nonlocal Symmetries and Conservation Laws

The symmetries and conservation laws discussed in the previous section are fully obtained in the (internal) local coordinates of $\mathcal{Y}_\infty$. Other interesting symmetries and conservation laws involve nonlocal variables, i.e. integrals, which are introduced properly by the theory of coverings set up by Krasil'shchik and Vinogradov [7]. Interesting cases concern nonlocal variables that correspond to integrability conditions of the system.

The theory of coverings is based on the concept of a system of PDE's as a (sub)manifold provided with a contact structure. This concept has been explained in Section 2. A simplified, but strong enough definition of a covering is the following

DEFINITION 4. A covering of the system $\mathcal{Y}_\infty$ is given by a projection $\tau: M \to \mathcal{Y}_\infty$, whereby M is a manifold provided with a contact structure, such that the contact structure of M is projected onto the contact structure of $\mathcal{Y}_\infty$. We say M covers $\mathcal{Y}_\infty$.

Coverings of a system $\mathcal{Y}_\infty$ are found by searching for extensions of the form $M = \mathcal{Y}_\infty \times \mathbb{R}^s$, whereby s may be taken infinite. Local coordinates for M are given by the internal local coordinates for $\mathcal{Y}_\infty$ extended by the nonlocal variables $\omega = (\omega^1, \dots, \omega^s)$. The contact structure on M is generated by the total derivatives of $\mathcal{Y}_\infty$ extended by the components for ω

$$\mathcal{D}_k^M = \mathcal{D}_k + \sum_{\alpha=1}^{s} X_\alpha^k \frac{\partial}{\partial \omega_\alpha} \quad \text{for } k = 1, \dots, p. \tag{4.1}$$

The coefficients X_α^k, each of which depends on a finite number of variables in M only, have to satisfy the integrability conditions $[\mathcal{D}_i^M, \mathcal{D}_j^M] = 0$, also written as

$$\mathcal{D}_i^M(X_\alpha^j) = \mathcal{D}_j^M(X_\alpha^i) \quad \text{for } i, j = 1, \dots, p \quad \text{and } \alpha = 1, \dots, s. \tag{4.2}$$

Some integrability conditions provide no information of the system at all and the corresponding covering is therefore called trivial ([7, p. 169]). This gives rise to the following equivalence classes: isomorphic coverings are called equivalent. When describing coverings over $\mathcal{Y}_\infty$, it is natural to consider the equivalence classes only.

Once a covering $\tau: M \to \mathcal{Y}_\infty$ has been established, one might wonder if M corresponds to a system of PDE's, i.e. can be written as $\widetilde{\mathcal{Y}}_\infty$. The answer is positive, the covering system $\widetilde{\mathcal{Y}}$ is given by ([7, p. 168])

$$\mathcal{Y} \wedge \frac{\partial w^\alpha}{\partial x_k} = X_\alpha^k \quad \text{for } k = 1, \dots, p \quad \text{and } \alpha = 1, \dots, s. \tag{4.3}$$

The system $\widetilde{\mathcal{Y}}$ is well-defined, for the integrability conditions

$$\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} w^\alpha = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} w^\alpha$$

are in algebraic sense equivalent to (4.2). Clearly, solutions of $\widetilde{\mathcal{Y}}$ give rise to solutions of $\mathcal{Y}$, i.e. $\widetilde{\mathcal{Y}}$ covers $\mathcal{Y}$. The generators of the contact structure on $\widetilde{\mathcal{Y}}_\infty$ will be denoted by $\widetilde{\mathcal{D}}_k$ for $k = 1, \dots, p$.

DEFINITION 5 ([7, p. 185]). Symmetries of the covering system are called nonlocal symmetries of the covered system.

A weaker version of this definition yields the interesting nonlocal τ solutions.

DEFINITION 6 ([7, p. 195]). A nonlocal τ solution of (3.7) with values in C^∞ $(\widetilde{\mathcal{Y}}_\infty)$, is a tuple of functions φ defined on $\widetilde{\mathcal{Y}}_\infty$, which satisfies

$$\widetilde{\ni}_\varphi(F) = 0, \quad \text{where } \widetilde{\ni}_\varphi = \sum_{\alpha=1}^{q} \sum_{|I|=0}^{\infty} \widetilde{\mathcal{D}}^I(\varphi_\alpha) \frac{\partial}{\partial u_I^\alpha}. \tag{4.4}$$

The projection of the nonlocal symmetry, e.g.

$$\ni_{(\varphi,\psi)} = \sum_{\alpha=1}^{q} \varphi_\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\alpha=1}^{s} \psi_\alpha \frac{\partial}{\partial w^\alpha} + \textit{prolong.}$$

is called the shadow, i.e.

$$\ni_\varphi = \sum_{\alpha=1}^{q} \varphi_\alpha \frac{\partial}{\partial u^\alpha} + \textit{prolong.},$$

φ being a nonlocal τ-solution.

A shadow is a nonlocal solution, but is a nonlocal solution a shadow? In other words, does there exist a covering in which the nonlocal solution is the shadow of a local symmetry? The answer to this question has been given by Vinogradov, Krasil'shchik and Khor'kova in the following theorem:

THEOREM 4 ([7, p. 198]). *To any nonlocal solution φ corresponds a nonlocal symmetry in an appropriate extension of the covering system.*

5. Methods of Computation

In this section the methods applied for computation of symmetries and conservation laws for system (1.2)

$$u_x - vw_x = 0, \qquad v_y - uw_y = 0, \qquad w_{yy} + uv + w_{xx} = 0 \tag{5.1}$$

will be explained. In the first subsection an important feature of system (1.2), which is being used in the methods of computation, is introduced. In the second subsection the working-method for computation of symmetries is explained. The third subsection is concerned with the computation of conservation laws. Adjoint symmetries can be computed in a way similar to the computation of symmetries, but in order to compute the divergence form of the conservation law (Definition 2) straightforward computations are implemented.

The methods of computation have been implemented in the symbolic manipulation program 'reduce'. Results on these computations are listed in the next section.

Remark on Notation. The multi-index subscript is combined with the x and y subscript for denoting partial derivatives, e.g. $\mathcal{D}_2(w_{ij}) = \mathcal{D}_y(w_{ij}) = w_{ij,y} \equiv (\partial/\partial x)^i(\partial/\partial y)^{j+1}w$.

5.1. THE DEGREE OF THE SYSTEM

A very important and useful feature of system (5.1) is the presence of the scaling invariant T_ε

$$x \mapsto \mathrm{e}^{-\varepsilon} x, \qquad y \mapsto \mathrm{e}^{-\varepsilon} y, \qquad u \mapsto \mathrm{e}^{\varepsilon} u, \qquad v \mapsto \mathrm{e}^{\varepsilon} v. \tag{5.2}$$

This scaling invariant is in fact a symmetry of (1.2), for

$$u = \mathrm{e}^{\varepsilon} f(\mathrm{e}^{\varepsilon} x,\, \mathrm{e}^{\varepsilon} y), \qquad v = \mathrm{e}^{\varepsilon} g(\mathrm{e}^{\varepsilon} x,\, \mathrm{e}^{\varepsilon} y), \qquad w = h(\mathrm{e}^{\varepsilon} x,\, \mathrm{e}^{\varepsilon} y) \tag{5.3}$$

is a solution whenever $u = f(x, y)$, $v = g(x, y)$, $w = h(x, y)$ is.

In the algebraic context the scaling invariant is represented by the degrees of the independent and dependent variables

$$\deg(x) = \deg(y) = -1, \qquad \deg(u) = \deg(v) = 1, \qquad \deg(w) = 0. \tag{5.4}$$

All objects of interest for system (5.1), like symmetries and conservation laws, turn out to be homogeneous with respect to the induced degrees, e.g. $\deg(u_{ij}) = \deg(u) - i\deg(x) - j\deg(y) = 1 + i + j$ and $\deg(uv\partial_{w_{11}}) = \deg(u) + \deg(v) - \deg(w_{11}) = 0$.

5.2. COMPUTATION OF SYMMETRIES

For computation of symmetries the symmetry equation (3.7) is used. The universal linearization operator for system (5.1) is

$$l_F = \begin{bmatrix} \mathcal{D}_x & -w_x & -v\mathcal{D}_x \\ -w_y & \mathcal{D}_y & -u\mathcal{D}_y \\ v & u & \mathcal{D}_x^2 + \mathcal{D}_y^2 \end{bmatrix}. \tag{5.5}$$

Though the C-differential l_F has a degree, the coefficients of l_F do not have the same degree, e.g. $\deg(\mathcal{D}_x) = 1$ whereas $\deg(-v\mathcal{D}_x) = 2$. Still the system $l_F(\varphi)$ is homogeneous with respect to the degree, for the symmetry with generating function $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ is homogeneous with respect to the degree, i.e. $\deg(\varphi^1\partial_u) = \deg(\varphi^2\partial_v) = \deg(\varphi^3\partial_w)$, leading to the required degree of φ

$$\deg(\varphi^1) = \deg(\varphi^2) = \deg(\varphi^3) + 1. \tag{5.6}$$

Internal coordinates of $\mathcal{Y}_\infty$, where $\mathcal{Y}$ is system (5.1), are choosen to be x, y, u_{0j} for $j \in \mathbb{N}$, v_{i0} for $i \in \mathbb{N}$, w_{i0} and w_{i1} for $i \in \mathbb{N}$. Thus $\mathcal{Y}_\infty$ is solved for u_x, v_y, w_{yy} and their differential consequences. With this choice of internal coordinates the symmetry equation (3.7) reads

$$\begin{aligned} &\mathcal{D}_x(\varphi^1) - w_{10}\varphi^2 - v\mathcal{D}_x(\varphi^3) = 0, \\ &-w_{01}\varphi^1 + \mathcal{D}_y(\varphi^2) - u\mathcal{D}_y(\varphi^3) = 0, \\ &v\varphi^1 + u\varphi^2 + \mathcal{D}_x^2(\varphi^3) + \mathcal{D}_y^2(\varphi^3) = 0, \end{aligned} \tag{5.7a}$$

if

$$\left.\begin{aligned} u_{i+1,j} &= \mathcal{D}_y^j(\mathcal{D}_x^i(vw_{10})) \\ v_{i,j+1} &= \mathcal{D}_y^j(\mathcal{D}_x^i(uw_{01})) \\ w_{i,j+2} &= \mathcal{D}_y^j(\mathcal{D}_x^i(-uv-w_{20})) \end{aligned}\right\} \quad \text{for all } \ i,j\in\mathbb{N}. \tag{5.7b}$$

Now we will explain the working-method implemented for computation of symmetries, i.e. solutions of (3.7). In a nutshell, the working-method repeats certain actions and these actions solve the symmetry equation (or better, what is left of it) for its variables of highest degree. Though the implementations are problem independent, this will be explained in the light of the symmetry equation (5.7).

The generating function $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ depends on a finite number of internal coordinates, for φ is defined on $\mathcal{Y}_\infty$. Dependencies for the generating function are selected with respect to degree, i.e. φ depends on the internal coordinates of degree n or less. According to (5.6) this means that φ^3 depends on internal coordinates of degree $n-1$ or less.

Besides the formal total derivative operators (2.1), also its truncated for degree versions

$$\mathcal{D}_k^{(d)} = \frac{\partial}{\partial x_k} + \sum_{\deg(u_I^\alpha)\leqslant d} u_{I,k}^\alpha \frac{\partial}{\partial u_I^\alpha} \quad \text{for } k=1,\dots,p \tag{5.8}$$

have been implemented. The number d in $\mathcal{D}_k^{(d)}$ is called the degree of truncation.

Let φ depend on internal coordinates of degree 2 or less, the symmetry equation (5.7) is written as

$$\begin{aligned} &\mathcal{D}_x^{(2)}(\varphi^1) - w_{10}\varphi^2 - v\mathcal{D}_x^{(1)}(\varphi^3) = 0, \\ &-w_{01}\varphi^1 + \mathcal{D}_y^{(2)}(\varphi^2) - u\mathcal{D}_y^{(1)}(\varphi^3) = 0, \\ &v\varphi^1 + u\varphi^2 + \mathcal{D}_x^{(2)}(\mathcal{D}_x^{(1)}(\varphi^3)) + \mathcal{D}_y^{(2)}(\mathcal{D}_y^{(1)}(\varphi^3)) = 0, \end{aligned} \tag{5.9a}$$

if

$$\left.\begin{aligned} u_{i+1,j} &= \mathcal{D}_y^j(\mathcal{D}_x^i(vw_{10})) \\ v_{i,j+1} &= \mathcal{D}_y^j(\mathcal{D}_x^i(uw_{01})) \\ w_{i,j+2} &= \mathcal{D}_y^j(\mathcal{D}_x^i(-uv-w_{20})) \end{aligned}\right\} \quad \text{for all } \ i+j\leqslant 1. \tag{5.9b}$$

A simple statement suffices system (5.9a) to be written explicitly in its variables of highest degree, i.e. of degree 3. In fact, just the total derivative operators are written explicitly in these variables, but because φ is independent of variables of degree 3 this is sufficient. For system (5.9a) this means that $\mathcal{D}_x^{(2)}$ and $\mathcal{D}_y^{(2)}$ are substituted by respectively

$$\mathcal{D}_x^{(1)} + \sum_{\deg(u_I^\alpha)=2} u_{I,x}^\alpha \frac{\partial}{\partial u_I^\alpha} \quad \text{and} \quad \mathcal{D}_y^{(1)} + \sum_{\deg(u_I^\alpha)=2} u_{I,y}^\alpha \frac{\partial}{\partial u_I^\alpha}. \tag{5.10}$$

To be precisely, the summing in (5.10) is taken over u_{10}, u_{01}, v_{10}, v_{01}, w_{20}, w_{11} and w_{12}.

Another simple statement sets the external coordinates of degree 3 equal to their corresponding expressions according to (5.9b), e.g. $u_{20} = \mathcal{D}_x(vw_{10}) = v_{10}w_{10} + vw_{20}$ and $u_{11} = \mathcal{D}_y(vw_{10}) = v_{01}w_{10} + vw_{11}$. At this stage the symmetry equation (5.9b) turns out to be a polynomial in the internal coordinates of degree 3. Even though the total derivative operators are not written in full, witness the presence of $\mathcal{D}_x^{(1)}$ and $\mathcal{D}_y^{(1)}$, the coefficients of the monomials in the internal coordinates of degree 3 are fixed. For this it is essential that the external coordinates are expressed in variables of the same or lesser degree, as is satisfied by (5.9b).

The symmetry equation splits for the monomials in the internal coordinates of degree 3; the coefficients of these monomials have to vanish, giving rise to new equations. In the ideal case these new equations can be solved and their solutions fix the dependencies of the generating function in the internal coordinates of degree 2. More general, experience has shown out that these equations are about the first to be solved. In our case these new equations are

$$\frac{\partial \varphi^1}{\partial v_{10}} = 0, \qquad \frac{\partial \varphi^2}{\partial u_{01}} = 0, \qquad \frac{\partial \varphi^3}{\partial u} = 0 \quad \text{and} \quad \frac{\partial \varphi^3}{\partial v} = 0. \tag{5.11}$$

These new PDE's are readily solved, but their solutions do not fix all the dependencies of degree 2, i.e. φ^1 has not been fixed for u_{01} and φ^2 has not been fixed for v_{10}. Next the symmetry equation, or better what is left of it, needs to be solved for the variables of degree 2.

Though unspecified dependencies of degree 2 are still present, the two previously mentioned statements are executable. In fact they are always executable, but the symmetry equation may split for internal coordinates the functions are independent of only. Some insight in the symmetry equation is useful, for if at this stage the first and second pde of (5.9a) are differentiated twice with respect to v_{10} and u_{01} respectively, we attain

$$w_{10}\frac{\partial^2 \varphi^1}{\partial v_{10}^2} = 0 \quad \text{and} \quad w_{01}\frac{\partial^2 \varphi^2}{\partial u_{01}^2} = 0. \tag{5.12}$$

Clearly, the solutions of these equations fix the remaining unspecified dependencies of degree 2 and so allow us to solve the symmetry equation for variables of degree 2. In the same way the symmetry equation is solved for variables of degree 1 and so on.

Two remarks concerning the working-method implemented are due here.

(1) The presence of the total derivative operators and its truncated versions is indispensable, for they compactify the many irrelevant terms and so prevent an overflow of computers memory.

(2) The presence of a scale invariant is useful, but not essential to the working-method for computation of symmetries. All that is required is that the vari-

ables can be ordered in rank of elimination from the symmetry equation and that the solutions of the new PDE's allow the symmetry equation to be solved for the variables of the next rank.

An explanation of the implementations involved is far beyond the scope of this report, for it would require a thorough understanding of the symbolic manipulation program 'reduce'. A complete description of the aspects of implementations can be found in the symmetry equation package for reduce [1]. Another package applied is the integrator package for reduce, developed by Gragert [3] and Kersten [5] and improved by Roelofs [9] for solving the frequently occurring special types of linear PDE's among the new PDE's, like (5.11) and (5.12). Certain tools used by these two packages are developed in the tools package for reduce [10].

5.3. COMPUTATION OF CONSERVATION LAWS

Adjoint symmetries are computed in the same way as symmetries, using Equation (3.14). This time the components of the adjoint symmetry $\psi = (\psi^1, \psi^2, \psi^3)$ have equal degree

$$\deg(\psi^1) = \deg(\psi^2) = \deg(\psi^3). \tag{5.13}$$

If ψ depends on the internal coordinates of degree n or less, Equation (3.14) reads

$$\begin{aligned} &-\mathcal{D}_x^{(n)}(\psi^1) - w_{01}\psi^2 + v\psi^3 = 0,\\ &-w_{10}\psi^1 - \mathcal{D}_y^{(n)}(\psi^2) + u\psi^3 = 0,\\ &\mathcal{D}_x^{(n)}(v\psi^1) + \mathcal{D}_y^{(n)}(u\psi^2) + \mathcal{D}_x^{(n+1)}(\mathcal{D}_x^{(n)}(\psi^3)) +\\ &\quad +\mathcal{D}_y^{(n+1)}(\mathcal{D}_y^{(n)}(\psi^3)) = 0, \end{aligned} \tag{5.14a}$$

if

$$\left.\begin{aligned} u_{i+1,j} &= \mathcal{D}_y^j(\mathcal{D}_x^i(vw_{10}))\\ v_{i,j+1} &= \mathcal{D}_y^j(\mathcal{D}_x^i(uw_{01}))\\ w_{i,j+2} &= \mathcal{D}_y^j(\mathcal{D}_x^i(-uv - w_{20})) \end{aligned}\right\} \quad \text{for all } \ i + j \leqslant n. \tag{5.14b}$$

The working-method for computation of symmetries is appropriate for solving the adjoint symmetry equation (5.14) too. Computations involving the additional condition (3.15) for adjoint symmetries in order to be a generating function of a conservation law, are (at least look) complicated and do not yield the more desired form (3.8)

$$\begin{aligned} &\mathcal{D}_x F^x + \mathcal{D}_y F^y = 0\\ &\text{if } \left\{\begin{aligned} u_{i+1,j} &= \mathcal{D}_y^j(\mathcal{D}_x^i(vw_{10}))\\ v_{i,j+1} &= \mathcal{D}_y^j(\mathcal{D}_x^i(uw_{01}))\\ w_{i,j+2} &= \mathcal{D}_y^j(\mathcal{D}_x^i(-uv - w_{20})) \end{aligned}\right. \qquad \text{for all } \ i, j \in \mathbb{N}. \end{aligned} \tag{5.15}$$

However, the presence of the degree allow for straightforward computations of solutions of (5.15). The implementations involved will be explained now.

First of all, we restrict ourselves to solutions F^x and F^y of (5.15) that are independent of x and y.

Furthermore, we consider F^x and F^y homogeneous of a certain degree, e.g. homogeneous of degree 2, and write them in full

$$F^x = \sum_{\deg(mon)=2} c^x_{mon} mon \quad \text{and} \quad F^y = \sum_{\deg(mon)=2} c^y_{mon} mon. \tag{5.16}$$

The summing is taken over all monomials of degree 2 composed of internal coordinates, i.e. u_{01}, v_{10}, w_{20}, w_{11}, u^2, uv, uw_{10}, uw_{01}, v^2, vw_{10}, vw_{01}, w_{10}^2, $w_{10}w_{01}$ and w_{01}^2. The coefficients c^x_{mon} and c^y_{mon} are functions of w.

The expressions for F^x and F^y (5.16) are inserted in (5.15) and the external coordinates present are expressed in the internal coordinates according to $\mathcal{Y}_\infty$. The resulting equation splits for all internal coordinates present, except w, giving rise to many new equations. These new equations are readily solved for the coefficients c^x_{mon} and c^y_{mon}.

A drawback of this implementation explained so far, is the presence of trivial conservation laws, i.e. $F^x = \mathcal{D}_y(F^0)$ and $F^y = -\mathcal{D}_x(F^0)$, like $\mathcal{D}_x(u_{01}) + \mathcal{D}_y(-u_{10}) = 0$. These trivial forms are eliminated in the following way.

Before inserting the expressions for F^x and F^y (5.16) in (5.15), F^x and F^y are considered up to $\mathcal{D}_y(F^0)$ and $-\mathcal{D}_x(F^0)$, respectively. Hereby, F^0 is written in full for its degree, e.g.

$$F^0 = \sum_{\deg(mon)=1} c^0_{mon} mon = c^0_u u + c^0_v v + c^0_{w_{10}} w_{10} + c^0_{w_{01}} w_{01}. \tag{5.17}$$

The coefficients c^0_{mon} are expressed in the coefficients c^x_{mon} and c^y_{mon} as follows. The equations $F^x - \mathcal{D}_y(F^0) = 0$ and $F^y + \mathcal{D}_x(F^0) = 0$ are split for the internal coordinates and solved not completely, but just for the coefficients c^0_{mon}. If done so, F^x and F^y are set equal to $F^x - \mathcal{D}_y(F^0)$ and $F^y + \mathcal{D}_x(F^0)$; trivial conservation laws are eliminated.

6. Results Computed

In succession, the following results have been computed for the system (1.2):

(1) Symmetries.
(2) Adjoint symmetries.
(3) Conservation laws.
(4) Nonlocal variables.
(5) Nonlocal symmetries.
(6) Nonlocal conservation laws.
(7) Recursion operator for symmetries and adjoint symmetries.
(8) Remarks.

The methods of computations applied have been explained in the previous section. Large expressions are inserted in the Appendix.

6.1. SYMMETRIES

Solutions of the symmetry equation (5.7) have been computed for the generating function φ depending on the internal coordinates of degree 6 or less. Here the generating functions of the symmetries are listed. There are two symmetries of degree 0

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u + xvw_{10} + yu_{01} \\ v + xv_{10} + yuw_{01} \\ xw_{10} + yw_{01} \end{pmatrix}. \tag{6.1}$$

The second symmetry in (6.1) corresponds to the scale invariant T_ε (5.2). Other symmetries appear in pairs of degree 1, 3 and 5. The symmetries of degree 1

$$\begin{pmatrix} u_{01} \\ uw_{01} \\ w_{01} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} vw_{10} \\ v_{10} \\ w_{10} \end{pmatrix} \tag{6.2}$$

are the evolutionary or vertical forms of $\partial/\partial y$ and $\partial/\partial x$, respectively. The symmetries of degree 3 are

$$\begin{pmatrix} 6u^2vw_{01} + 3u^2u_{01} + 6uw_{01}w_{20} + 3u_{01}w_{10}^2 - 3u_{01}w_{01}^2 + 2u_{03} \\ u^3w_{01} + 3uw_{10}^2w_{01} - 3uw_{01}^3 - 2uw_{21} + 2u_{01}w_{20} + 2w_{01}u_{02} \\ u^2w_{01} - 2vu_{01} + 3w_{10}^2w_{01} - w_{01}^3 - 2w_{21} \end{pmatrix} \tag{6.3a}$$

and

$$\begin{pmatrix} 3v^3w_{10} - 3vw_{10}^3 + 3vw_{10}w_{01}^2 + 2vw_{30} + 2w_{10}v_{20} - 2w_{20}v_{10} \\ 3v^2v_{10} - 6vw_{10}w_{20} - 3w_{10}^2v_{10} + 3w_{01}^2v_{10} + 2v_{30} \\ 3v^2w_{10} - w_{10}^3 + 3w_{10}w_{01}^2 + 2w_{30} \end{pmatrix}. \tag{6.3b}$$

The two symmetries of degree 5 have been inserted in the appendix. Apart from the second symmetry in (6.1) these symmetries commute, i.e. $[\ni_\varphi, \ni_{\varphi'}] = 0$. The Lie bracket with the symmetry (6.1b) acts like multiplication by the degree of the symmetry.

6.2. ADJOINT SYMMETRIES

The adjoint symmetries have been computed using formula (5.14). The function ψ is choosen to be dependent of internal coordinates of degree 6 or less. There is one solution ψ of degree 0, which is

$$\begin{pmatrix} yu \\ -xv \\ -xw_{01} + yw_{10} \end{pmatrix}. \tag{6.4}$$

Other solutions appear to be in pairs of odd degrees. The two adjoint symmetries of degree 1 are

$$\begin{pmatrix} 0 \\ v \\ w_{01} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u \\ 0 \\ w_{10} \end{pmatrix}. \tag{6.5}$$

Note that the differential consequences of (6.4) with respect to x and y yield (6.5). The solutions ψ of degree 3 are

$$\begin{pmatrix} -u^3 - uw_{10}^2 + 3uw_{01}^2 - 2u_{02} \\ -2uw_{11} + 2w_{10}u_{01} \\ -u^2w_{10} + 2uv_{10} + 2v^2w_{10} - w_{10}^3 + 3w_{10}w_{01}^2 + 2w_{30} \end{pmatrix} \tag{6.6a}$$

and

$$\begin{pmatrix} 2vw_{11} - 2w_{01}v_{10} \\ v^3 - 3vw_{10}^2 + vw_{01}^2 + 2v_{20} \\ v^2w_{01} - 3w_{10}^2w_{01} + w_{01}^3 + 2w_{21} \end{pmatrix}. \tag{6.6b}$$

The two solutions of degree 5 are given in the Appendix.

6.3. CONSERVATION LAWS

Conservation laws of the form (5.15) are computed for the degree of F^x and F^y of 0 up to and including 6. For degree 2 the computer gave two solutions, namely

$$\begin{aligned} &F^x = w_{10}w_{01} \wedge F^y = (v^2 - w_{10}^2 + w_{01}^2)/2, \\ &F^x = (u^2 + w_{10}^2 - w_{01}^2)/2 \wedge F^y = w_{10}w_{01}. \end{aligned} \tag{6.7}$$

Also degree 4 yields two conservation laws, which are

$$\begin{aligned}
F^x &= (u^4 - 4u^3v + 4u^2v^2 + 2u^2w_{10}^2 - 6u^2w_{01}^2 - 4u^2w_{20} + \\
&\qquad + 8uvw_{20} + 8uw_{01}u_{01} + w_{10}^4 - 6w_{10}^2w_{01}^2 + w_{01}^4 + \\
&\qquad + 4w_{20}^2 - 4u_{01}^2 - 4w_{11}^2)/4, \\
F^y &= u^2w_{10}w_{01} - u^2w_{11} - 2uvw_{10}w_{01} + 2uvw_{11} + \\
&\quad + w_{10}^3w_{01} - w_{10}w_{01}^3 + 2w_{20}w_{11}, \\
F^x &= -2uvw_{10}w_{01} + v^2w_{10}w_{01} - v^2w_{11} - w_{10}^3w_{01} + \\
&\quad + w_{10}w_{01}^3 - 2w_{20}w_{11}, \\
F^y &= (v^4 - 6v^2w_{10}^2 + 2v^2w_{01}^2 + 4v^2w_{20} + 8vw_{10}v_{10} + w_{10}^4 - \\
&\qquad - 6w_{10}^2w_{01}^2 + w_{01}^4 + 4w_{20}^2 - 4w_{11}^2 - 4v_{10}^2)/4.
\end{aligned} \tag{6.8}$$

Two conservation laws, for which F^x and F^y have degree 6, have been inserted in [2].

6.4. NONLOCAL VARIABLES

The conservation laws given in the previous subsection allow for introduction of nonlocal variables. The conservation laws (6.7) give rise to two nonlocal variables of degree 1, called p and q

$$\begin{aligned}
p_x &= (-v^2 + w_{10}^2 - w_{01}^2)/2 \wedge p_y = w_{10}w_{01}, \\
q_x &= -w_{10}w_{01} \wedge q_y = (u^2 + w_{10}^2 - w_{01}^2)/2.
\end{aligned} \tag{6.9}$$

To the conservation laws (6.8) correspond two nonlocal variables of degree 3, called r and s

$$\begin{aligned}
r_x &= -u^2w_{10}w_{01} + u^2w_{11} + 2uvw_{10}w_{01} - 2uvw_{11} - \\
&\quad - w_{10}^3w_{01} + w_{10}w_{01}^3 - 2w_{20}w_{11}, \\
r_y &= (u^4 - 4u^3v + 4u^2v^2 + 2u^2w_{10}^2 - 6u^2w_{01}^2 - 4u^2w_{20} + \\
&\qquad + 8uvw_{20} + 8uw_{01}u_{01} + w_{10}^4 - 6w_{10}^2w_{01}^2 + w_{01}^4 + \\
&\qquad + 4w_{20}^2 - 4u_{01}^2 - 4w_{11}^2)/4, \\
s_x &= (-v^4 + 6v^2w_{10}^2 - 2v^2w_{01}^2 - 4v^2w_{20} - 8vw_{10}v_{10} - \\
&\qquad - w_{10}^4 + 6w_{10}^2w_{01}^2 - w_{01}^4 - 4w_{20}^2 + 4w_{11}^2 + 4v_{10}^2)/4, \\
s_y &= -2uvw_{10}w_{01} + v^2w_{10}w_{01} - v^2w_{11} - w_{10}^3w_{01} + \\
&\quad + w_{10}w_{01}^3 - 2w_{20}w_{11}.
\end{aligned} \tag{6.10}$$

6.5. NONLOCAL SYMMETRIES

The solutions of the nonlocal symmetry equation, for which the generating function of the symmetry depends on the internal coordinates of degree 6 or less including p, q, r and s, are given by, besides the local solutions, two nonlocal symmetries of degree 2 and 4 respectively. The nonlocal symmetry of degree 2 is shown here, the other one is inserted in the appendix. The nonlocal variables present, i.e. p and q, are printed in bold type. Note that the 'coefficients' of p, q, x and y are symmetries too, cf. (6.2) and (6.3).

$$\begin{aligned}
\varphi^1 = {} & -2\mathbf{p}vw_{10} - 2\mathbf{q}u_{01} + \\
& + x(3v^3w_{10} - 3vw_{10}^3 + 3vw_{10}w_{01}^2 + 2vw_{30} + 2w_{10}v_{20} - 2w_{20}v_{10}) + \\
& + y(-6u^2vw_{01} - 3u^2u_{01} - 6uw_{01}w_{20} - 3u_{01}w_{10}^2 + 3u_{01}w_{01}^2 - 2u_{03}) - \\
& - 2u^3 - 2uv^2 - 4uw_{10}^2 + 6uw_{01}^2 - 2vw_{20} + 4w_{10}v_{10}, -6u_{02}, \\
\varphi^2 = {} & -2\mathbf{p}v_{10} - 2\mathbf{q}uw_{01} + \\
& + x(3v^2v_{10} - 6vw_{10}w_{20} - 3w_{10}^2v_{10} + 3w_{01}^2v_{10} + 2v_{30}) + \\
& + y(-u^3w_{01} - 3uw_{10}^2w_{01} + 3uw_{01}^3 + 2uw_{21} - 2u_{01}w_{20} - 2w_{01}u_{02}) - \\
& - 2uw_{20} + 2v^3 - 6vw_{10}^2 + 4vw_{01}^2 - 4u_{01}w_{01} + 6v_{20}, \\
\varphi^3 = {} & -2\mathbf{p}w_{10} - 2\mathbf{q}w_{01} + \\
& + x(3v^2w_{10} - w_{10}^3 + 3w_{10}w_{01}^2 + 2w_{30}) + \\
& + y(-u^2w_{01} + 2vu_{01} - 3w_{10}^2w_{01} + w_{01}^3 + 2w_{21}) + \\
& + 2uv + 4w_{20}.
\end{aligned} \tag{6.11}$$

The nonlocal symmetry shown here is the recursive symmetry, i.e. by Theorem 2 the local symmetries of odd degree, cf. subsection 'symmetries', are generated. To be precisely, multiple commutators of the nonlocal symmetry (6.11) and the local symmetries of degree 1 yield all other local odd symmetries and each time the degree of the local symmetry of odd degree is increased by 2.

6.6. NONLOCAL CONSERVATION LAWS

Just one nonlocal conservation law, for which F^x and F^y depend on the internal coordinates of degree 4 including p, q, r and s, has been found

$$\mathcal{D}_x(q) + \mathcal{D}_y(p) = 0. \tag{6.12}$$

The local conservation laws computed previously, cf. subsection 'conservation laws', have not been found. This is in agreement with the theory, for the introduction of the nonlocal variables have turned those conservation laws into trivial ones.

6.7. RECURSION OPERATORS

In this subsection the nonlocal recursion operator $\mathcal{R}$ corresponding to the nonlocal symmetry $\ni_{\varphi_{nl}}$ (6.11) is established, i.e. for all symmetries $\ni_\varphi$ holds

$$[\ni_{\varphi_{nl}}, \ni_\varphi] = \ni_{\mathcal{R}(\varphi)}.$$

Nonlocal recursion operators are of a more general nature than C-differential operators, for inverses of the total derivative operators are admitted too. Motivated by the results of the theory of deformations [6] and [4], we constructed the following expression for $\mathcal{R}$

$$\begin{aligned}\mathcal{R} = &\begin{bmatrix} \mathcal{D}_y^2 + u^2 + w_{10}^2 - w_{01}^2 & -w_{10}\mathcal{D}_x + uv + w_{20} & uw_{10}\mathcal{D}_x - 2uw_{01}\mathcal{D}_y \\ w_{01}\mathcal{D}_y + w_{20} & -\mathcal{D}_x^2 - v^2 + w_{10}^2 - w_{01}^2 & 2vw_{10}\mathcal{D}_x - vw_{01}\mathcal{D}_y \\ 0 & u & \mathcal{D}_y^2 \end{bmatrix} + \\ &+ \begin{bmatrix} 0 & 0 & vw_{10}\mathcal{D}_y^{-1} \circ (w_{01}\mathcal{D}_x + w_{10}\mathcal{D}_y) - u_{01}\mathcal{D}_x^{-1} \circ (w_{01}\mathcal{D}_x + w_{10}\mathcal{D}_y) \\ 0 & 0 & v_{10}\mathcal{D}_y^{-1} \circ (w_{01}\mathcal{D}_x + w_{10}\mathcal{D}_y) - uw_{01}\mathcal{D}_x^{-1} \circ (w_{01}\mathcal{D}_x + w_{10}\mathcal{D}_y) \\ 0 & 0 & w_{10}\mathcal{D}_y^{-1} \circ (w_{01}\mathcal{D}_x + w_{10}\mathcal{D}_y) - w_{01}\mathcal{D}_x^{-1} \circ (w_{01}\mathcal{D}_x + w_{10}\mathcal{D}_y) \end{bmatrix}.\end{aligned} \tag{6.13}$$

That $\mathcal{R}$ indeed is a nonlocal recursion operator for symmetries, is checked by the following formula, in which $\mathcal{S}$ is a nonlocal C-differential

$$l_F \circ \mathcal{R} = \mathcal{S} \circ l_F \quad \text{on } \mathcal{Y}_\infty, \quad \text{where } \mathcal{Y} = \{F = 0\}. \tag{6.14}$$

Clearly, the right side vanishes for symmetries of $\mathcal{Y}_\infty$, i.e. solutions φ of $l_F(\varphi) = 0$ on $\mathcal{Y}_\infty$. Therefore the left side of (6.14) tells us that $\mathcal{R}(\varphi)$ is a symmetry if φ is. Computations performed by computer have shown that (6.14) holds by setting $\mathcal{S}$ equal to

$$\mathcal{S} = \begin{bmatrix} \mathcal{D}_y^2 + u^2 + w_{10}^2 - w_{01}^2 & -\mathcal{D}_x^2 - v^2 + w_{10}^2 - w_{01}^2 & -u\mathcal{D}_y - vw_{01} \\ -w_{01}\mathcal{D}_x - 2w_{11} & w_{10}\mathcal{D}_y + 2w_{11} & uw_{10} \\ 2(uv + w_{20})\mathcal{D}_x^{-1} \circ u - w_{01}\mathcal{D}_x^{-1}\mathcal{D}_y \circ u & 2w_{20}\mathcal{D}_y^{-1} \circ v + w_{10}\mathcal{D}_y^{-1}\mathcal{D}_x \circ v & \mathcal{S}_{33} \end{bmatrix}, \tag{6.15}$$

where $\mathcal{S}_{33}$, because of its size, is given separately

$$\begin{aligned}\mathcal{S}_{33} = {}& 2w_{20}\mathcal{D}_y^{-1} \circ w_{01} + w_{10}\mathcal{D}_y^{-1}\mathcal{D}_x \circ w_{01} + 2(uv + w_{20})\mathcal{D}_x^{-1} \circ w_{10} - \\ & - w_{01}\mathcal{D}_x^{-1}\mathcal{D}_y \circ w_{10} + \mathcal{D}_y^2 + w_{10}^2 - w_{01}^2.\end{aligned} \tag{6.16}$$

According to the adjoint variant of formula (6.14)

$$l_F^* \circ \mathcal{S}^* = \mathcal{R}^* \circ l_F^* \quad \text{on } \mathcal{Y}_\infty, \quad \text{where } \mathcal{Y} = \{F = 0\}, \tag{6.17}$$

the nonlocal C-differential $\mathcal{S}^*$ is the recursion operator for the adjoint symmetries. The expression for $\mathcal{S}^*$ is inserted in [2].

6.8. REMARKS

6.8.1. The results line up with the discrete symmetry T: $x \mapsto y$, $y \mapsto x$, $u \mapsto v$, $v \mapsto u$ of system (1.2).

6.8.2. The generating functions of the conservation laws (6.7) and (6.8) are given by the adjoint symmetries (6.5) and (6.6). Also the adjoint symmetry (6.4) is the generating function of a conservation law, namely

$$\begin{aligned}&\mathcal{D}_x(xw_{10}w_{01} - y(u^2 + w_{10}^2 - w_{01}^2)/2) + \\ &\quad + \mathcal{D}_y(x(v^2 - w_{10}^2 + w_{01}^2)/2 - yw_{10}w_{01}),\end{aligned} \tag{6.18}$$

which is equivalent to the nonlocal conservation law (6.12)

$$\mathcal{D}_x(q) + \mathcal{D}_y(p) = \mathcal{D}_x(xp_{01} - yq_{01}) + \mathcal{D}_y(-xp_{10} + yq_{10}) \equiv (6.18). \tag{6.19}$$

6.8.3. Though $\mathcal{S}^*$ is a nonlocal recursion operator for the adjoint symmetries, it is unknown if $\mathcal{S}^*$ is a nonlocal recursion operator for the generation functions of conservation laws.

6.8.4. In agreement with the results, application of Theorem 3 for φ given by (6.11) and (F^x, F^y) given by (6.18) yields a trivial conservation law.

6.8.5. The nonlocal solution (6.11) is the shadow of a symmetry for the system $\mathcal{Y}$ extended by the PDE's that define the nonlocal variables corresponding to the conservation laws $\ni_\varphi^k$ (6.7) for $k \in \mathbb{N}$ and φ given by (6.11), see Theorem 4.

7. Conclusion

An overview of local and nonlocal symmetries and conservation laws for the system (1.2)

$$u_x - vw_x = 0, \qquad v_y - uw_y = 0, \qquad w_{yy} + uv + w_{xx} = 0 \tag{7.1}$$

is shown in Figures 1 and 2. A filled dot represents a local object and an open square represents a nonlocal object. Symmetries have the form $\ni_\varphi$, whereas conservation laws $\mathcal{D}_x(F^x)+\mathcal{D}_y(F^y)$ have been inserted as 1-forms $F^y \mathrm{d}x - F^x \mathrm{d}y$, i.e. the degree of a conservation law is deg$(F^x) - 1$.

Note: Though the results indicate the existence of two infinite series of conservation laws, their existence has not been proven.

The nonlocal symmetries of degree -1, -3, $\ldots$, are given by the partial derivatives with respect to the nonlocal variables, i.e. $\partial/\partial p$, $\partial/\partial q$, $\partial/\partial r$, $\partial/\partial s$, ... Likewise, the nonlocal symmetry of degree 0 corresponds to the conservation law of degree 0.

Some aspects of the Lie algebra of symmetries have already been discussed. The Lie bracket of two symmetries vanishes, unless the scale symmetry (6.1b) or a nonlocal symmetry of positive degree is involved. In that case the Lie bracket acts like a vertical translation, e.g. the Lie bracket of a nonlocal symmetry of positive degree $2m$ and a symmetry of degree $2n + 1$ is a symmetry of degree $2(m + n) + 1$. For clearness' sake, Lie brackets involving the symmetry at the left, i.e. $\partial/\partial w$, or the nonlocal symmetry of degree 0 vanish.

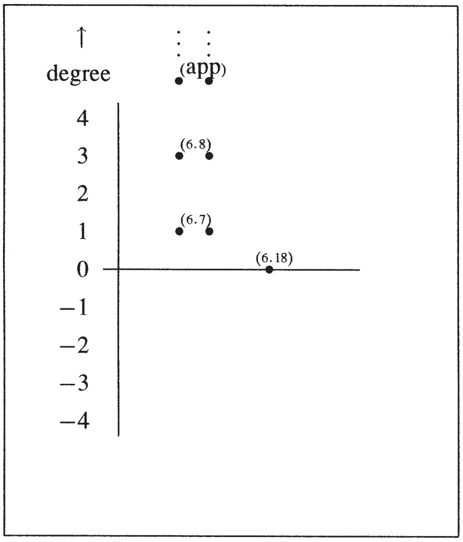

Figure 1. Symmetries.

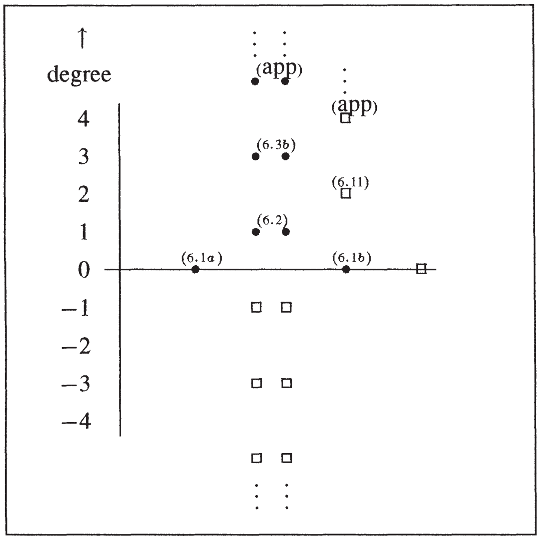

Figure 2. Conservation laws.

The recursion operator $\mathcal{R}$ (6.13) acts alike the Lie bracket of the nonlocal symmetry of degree 2, but now for the generating functions of the symmetries. Again, the generating function of the symmetry $\partial/\partial w$ is mapped onto zero and likewise the nonlocal symmetry of degree 0.

The two columns of conservation laws are generated by the nonlocal symmetry of degree 2, cf. Theorem 3.

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