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A note on K_4-closures in hamiltonian graph theory

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Abstract

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Let $G=(V, E)$ be a 2-connected graph. We call two vertices u and v of G a K_4-pair if u and v are the vertices of degree two of an induced subgraph of G which is isomorphic to K_4 minus an edge. Let x and y be the common neighbors of a K_4-pair u,v in an induced K_4-e. We prove the following result: If $N(x)\cup N(y)\subseteq N(u)\cup N(v)\cup\{u,v\}$, then G is hamiltonian if and only if $G+uv$ is hamiltonian. As a consequence, a claw-free graph G is hamiltonian if and only if $G+uv$ is hamiltonian, where u,v is a K_4-pair. Based on these results we define socalled K_4-closures of G. We give infinite classes of graphs with small maximum degree and large diameter, and with many vertices of degree two having complete K_4-closures.

1. Introduction

We use Bondy and Murty [4] for terminology and notation not defined here and consider simple graphs only.

About 15 years ago the now well-known paper 'A method in graph theory' by Bondy and Chvátal [3] was published. The closure concept they introduced in this paper opened a new horizon for the research on hamiltonian and related properties. We focus on the hamiltonian problem and mention their main results.

In the sequel let G be a graph on n vertices, and let $d(x)$ denote the degree of the vertex x of G.

Lemma 1 [3]. *Let u and v be distinct nonadjacent vertices in G such that $d(u)+d(v)\geqslant n$. Then G is hamiltonian if and only if $G+uv$ is hamiltonian.*

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Bondy and Chvátal defined the *n-closure* $C_n(G)$ of G as the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains. They showed that $C_n(G)$ is well defined and that G is hamiltonian if and only if $C_n(G)$ is hamiltonian. As a consequence, if G is a graph on at least 3 vertices and $C_n(G)$ is complete, then G is hamiltonian.

Inspired by these results several other closure concepts were developed (e.g. in [1, 2, 5, 6, 8–12]). Most of these concepts are based on conditions for nonadjacent pairs of vertices which include some 'global' parameters of the graph, i.e., the number of vertices or the cardinality of a maximum independent set of vertices containing the pair. Recently, Hasratian and Khachatrian [7] introduced a closure concept based on 'local' conditions concerning the extended neighborhood structure. Here we discuss a condition which takes into account the local structure of the graph.

2. Main results

Denote by K_4-e the complete graph on four vertices minus an arbitrary edge. A pair of vertices $\{u,v\}\subseteq V(G)$ is a *K_4-pair* of G if u and v are the two nonadjacent vertices in an induced subgraph H of G which is isomorphic to K_4-e; the pair of vertices of degree 3 in H is called a *copair of u and v*. By $N(v)$ we denote the set of neighbors of a vertex v of G.

Theorem 2. *Let $\{u,v\}$ be a K_4-pair of G with copair $\{x,y\}$ such that $N(x)\cup N(y)\subseteq N(u)\cup N(v)\cup\{u,v\}$. Then G is hamiltonian if and only if $G+uv$ is hamiltonian.*

Proof. If G is hamiltonian, then obviously $G+uv$ is also hamiltonian. Conversely, assume $G+uv$ is hamiltonian and G is not hamiltonian for some K_4-pair $\{u,v\}$ with copair $\{x,y\}$ satisfying $N(x)\cup N(y)\subseteq N(u)\cup N(v)\cup\{u,v\}$. Then G contains a Hamilton path $v_1v_2\cdots v_{n-1}v_n$ with $v_1=u$ and $v_n=v$. Without loss of generality assume $x=v_i$ and $y=v_j$ for some i and j with $2\leqslant i<j\leqslant n-1$. Clearly $i<j-1$, otherwise

$$v_1v_2\cdots v_iv_nv_{n-1}\cdots v_{i+1}v_1$$

is a Hamilton cycle in G. By similar arguments $v_{i+1}\notin N(u)$ and $v_{j-1}\notin N(v)$.

Hence $i<j-2$, $v_{i+1}\in N(v)$ and $v_{j-1}\in N(u)$. But now

$$v_1v_{j-1}v_{j-2}\cdots v_{i+1}v_nv_{n-1}\cdots v_jv_iv_{i-1}\cdots v_1$$

is a Hamilton cycle in G, a contradiction. □

Theorem 2 has the following consequence for *claw-free* graphs, i.e., graphs that do not contain an induced subgraph isomorphic to $K_{1,3}$.

Corollary 3. *Let $\{u,v\}$ be a K_4-pair of a claw-free graph G. Then G is hamiltonian if and only if $G+uv$ is hamiltonian.*

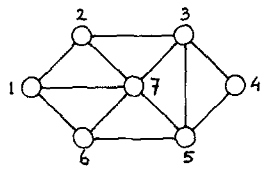


Fig. 1. G_1.

Proof. Let $\{u,v\}$ be a K_4-pair of a claw-free graph G with copair $\{x,y\}$. If $w\in N(x)$ (or $N(y)$), then $w\in N(u)\cup N(v)\cup\{u,v\}$, otherwise the subgraph of G induced by $\{w,x,u,v\}$ (or $\{w,y,u,v\}$) is isomorphic to $K_{1,3}$. Hence, $N(x)\cup N(y)\subseteq N(u)\cup N(v)\cup\{u,v\}$, and the result follows from Theorem 2. $\square$

Note that, in fact, we only need that the subgraphs of G induced by $(N(x)\cup\{x\})-\{y\}$ and $(N(y)\cup\{y\})-\{x\}$ are claw-free.

Corollary 4. *Let $\{u,v\}$ be a K_4-pair of G such that $|N(u)\cup N(v)|=n-2$. Then G is hamiltonian if and only if $G+uv$ is hamiltonian.*

Proof. Let $\{u,v\}$ be a K_4-pair of G with $|N(u)\cup N(v)|=n-2$. Then $N(u)\cup N(v)=V(G)-\{u,v\}$, and obviously any copair $\{x,y\}$ of $\{u,v\}$ satisfies $N(x)\cup N(y)\subseteq N(u)\cup N(v)\cup\{u,v\}$. The result follows from Theorem 2. $\square$

Based on the above results, we say that a graph H is a *K_4-closure* of G if H can be obtained from G by recursively joining K_4-pairs that satisfy the condition of Theorem 2 and H contains no such pairs. A graph can have different K_4-closures. Consider, e.g., the graph G_1 of Fig. 1. It is easy to check that we can recursively join the K_4-pairs $\{2,6\},\{1,3\},\{1,5\},\{1,4\},\{2,5\},\{3,6\},\{4,7\},\{2,4\}$ and $\{4,6\}$ to obtain K_7 as a K_4-closure of G_1, whereas, on the other hand, we can recursively join the K_4-pairs $\{1,3\},\{1,5\},\{1,4\},\{2,5\},\{3,6\}$ and $\{4,7\}$ to obtain K_7 minus the edges of a triangle as another K_4-closure of G_1.

Although a unique K_4-closure may not exist, obtaining a K_4-closure of G can be helpful to answer the question whether G is hamiltonian.

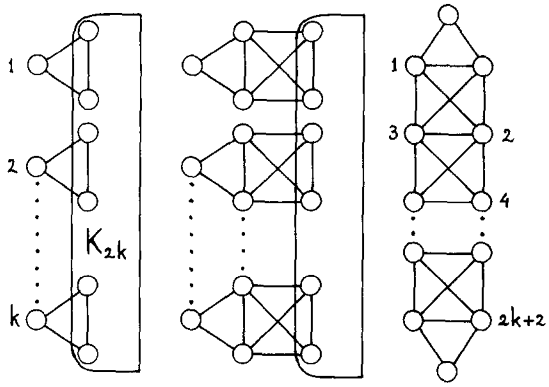
The following results are obvious consequences of Theorem 2.

Theorem 5. *Let H be a K_4-closure of G. Then G is hamiltonian if and only if H is hamiltonian.*

Corollary 6. *Let G be a graph on $n\geqslant 3$ vertices. If K_n is a K_4-closure of G, then G is hamiltonian.*

3. Applications

In this section we consider some classes of graphs that have K_n as a K_4-closure.



Z_k, $k \geqslant 2$ T_k, $k \geqslant 1$ L_{2k+6}, $k \geqslant 0$

Fig. 2.

Proposition 7. *Let G be a connected graph with complete subgraphs $H_1, H_2, \ldots, H_t$ such that*:

(i) $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_t)$,

(ii) $V(H_i) \cap V(H_j) = \emptyset$ *or* $|V(H_i) \cap V(H_j)| \geqslant 2$ $(1 \leqslant i < j \leqslant t)$,

(iii) $V(H_i) \cap V(H_j) \cap V(H_k) = \emptyset$ $(1 \leqslant i < j < k \leqslant t)$.

Then K_n is a K_4-closure of G.

Proof. By induction on t. For $t=1$ the statement is trivial. Assume the statement is true for $t \leqslant s$ and let G be a graph satisfying the conditions of the proposition with $t = s+1 \geqslant 2$. Since G is connected, there exist i and j such that H_i and H_j have a nonempty intersection. By (ii) and (iii), $V(H_i) \cap V(H_j) \cap V(G)$ contains a pair $\{x, y\}$ such that $N(x) \cup N(y) \subseteq V(H_i) \cup V(H_j) \subseteq N(u) \cup N(v) \cup \{u, v\}$ for all $\{u, v\}$ with $u \in V(H_i)$ and $v \in V(H_j)$. Using Theorem 2 we can recursively join all nonadjacent pairs $\{u, v\}$ with copair $\{x, y\}$ to obtain a complete subgraph H_{ij} of the new graph G' with $V(H_{ij}) = V(H_i) \cup V(H_j)$. It is not difficult to see that G' is a connected graph with complete subgraphs $H'_1, H'_2, \ldots, H'_s$ satisfying (i), (ii) and (iii). By the induction hypothesis, K_n is a K_4-closure of G' and hence of G. $\square$

Examples. Fig. 2 shows some examples of classes of graphs that can be shown to be hamiltonian using Corollary 6 and Proposition 7 (or an algorithm based on these results). Note that one third of the vertices of the graph Z_k, $k \geqslant 2$ has degree 2, that the diameter of the graph L_{2r}, $r \geqslant 3$ is r and that the maximum degree of this graph is 5. If we delete the edges of the edge set $\{\{i, i+1\} | i = 1, 3, \ldots, 2k+1\}$ from L_{2k+6}, $k \geqslant 0$, then the remaining graph has maximum degree 4 and also has K_{2k+6} as a K_4-closure.

It is clear from the above examples that Corollary 6 can be useful in cases where counterparts of Corollary 6 based on degree conditions or neighborhood conditions are not useful. Therefore, it seems to be worthwhile to try to obtain more results on local structure conditions. However, it is also clear that the above methods are not useful in general. Of course it is possible to combine several closure conditions to obtain more powerful concepts, but which combination gives 'the best' concept? Is there a unifying condition that covers all known closure conditions?

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