ELSEVIER

Discrete Mathematics 185 (1998) 231–238

DISCRETE MATHEMATICS

Note

Closure concepts for claw-free graphs

H.J. Broersma*, H. Trommel

Faculty of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands

Received 17 October 1996; received in revised form 18 July 1997; accepted 4 August 1997

Abstract

Recently, Ryjáček introduced an interesting new closure concept for claw-free graphs, and used it to prove that every nonhamiltonian claw-free graph is a spanning subgraph of a nonhamiltonian line graph (of a triangle-free graph). We discuss the relationship between Ryjáček's closure and the K_4-closure introduced by the first author. Our main result deals with a variation on the K_4-closure. It implies a simpler proof of Ryjáček's closure theorem, and yields a more general closure concept which is not restricted to claw-free graphs only.

Keywords: Closure; Claw-free graph; Circumference; (Long, Hamilton) cycle

AMS classifications: 05C45; 05C75

1. Introduction

The results in this paper are motivated by a recent closure result due to Ryjáček [5] (Theorem 1 below), and its connection with a closure concept introduced by one of the authors in [2].

We use [1] for terminology and notation not defined here and consider finite simple graphs only.

Let $G=(V,E)$ be a graph on n vertices with vertex set V and edge set E. Let $N(v)$ denote the set of neighbors of a vertex $v \in V$, and let $d(v)=|N(v)|$ denote the *degree* of v. The *neighborhood* of v is the subgraph of G induced by $N(v)$. The *local completion of G at a vertex v* is the operation of joining all pairs of nonadjacent vertices in $N(v)$, i.e. replacing the neighborhood of v by the complete graph on $N(v)$. The graph G is *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$.

* Corresponding author. E-mail: broersma@math.utwente.nl.

PII S0012-365X(97)00210-0

If $S \subseteq V$, we denote by $G[S]$ the subgraph of G induced by the vertices of S. By $d_G(u,v)$ we denote the distance between to vertices u and v in G.

In [5] Ryjáček proved the following result.

Theorem 1. *Let G be a claw-free graph, v a vertex of G whose neighborhood is connected, and G' the graph obtained from G by local completion at v. Then*
(i) *G' is claw-free, and*
(ii) *for every cycle C' of G' there exists a cycle C of G such that $V(C') \subseteq V(C)$.*

For a claw-free graph G, we define the *Ryjáček closure* $C_R(G)$ of G as the graph obtained from G by iteratively performing local completions at vertices with connected neighborhoods until no more edges can be added. As shown in [5], $C_R(G)$ is uniquely determined by G, and $C_R(G)$ is the line graph of a triangle-free graph. Moreover, in [5] it is shown that Theorem 1 has the following consequences. Let $c(G)$ denote the *circumference* of G, i.e. the length of a longest cycle of G.

Theorem 2. *Let G be a claw-free graph on n vertices. Then*
(i) $c(C_R(G)) = c(G)$.
(ii) *If $C_R(G)$ is complete and $n \geqslant 3$, then G is hamiltonian.*
(iii) *Every nonhamiltonian claw-free graph is a spanning subgraph of a nonhamiltonian line graph.*

Theorem 2(ii) implies a result due to Oberly and Sumner [4], who proved that a connected claw-free graph on $n \geqslant 3$ vertices is hamiltonian if every vertex has a connected neighborhood. Theorem 2(iii) together with a result of Zhan [7] implies that every 7-connected claw-free graph is hamiltonian. Moreover it yields the equivalence of two conjectures due to Thomassen [6] and Matthews and Sumner [3], respectively: every 4-connected line graph is hamiltonian if and only if every 4-connected claw-free graph is hamiltonian.

The main result of [5] was obtained during the Workshop on the Hamiltonicity of 2-Tough Graphs held at Enschede, The Netherlands in November 1995, sponsored by EIDMA (the Euler Institute for Discrete Mathematics and its Applications). At the same meeting we discussed whether the Ryjáček closure could be obtained using a closure concept introduced in [2] based on the following result in [2].

Theorem 3. *Let $G = (V,E)$ be a graph and let $\{x,y,u,v\}$ be a subset of four vertices of V such that $uv \notin E$, $\{x,y\} \subseteq N(u) \cap N(v)$ and $xy \in E$. If $N(x) \cup N(y) \subseteq N(u) \cup N(v) \cup \{u,v\}$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Based on Theorem 3, we say that a graph H is a *K_4-supergraph* of a graph G if H can be obtained from G by iteratively joining pairs $\{u,v\}$ satisfying the condition in Theorem 3 for some $\{x,y\} \subseteq N(u) \cap N(v)$ with $xy \in E$, and a *K_4-closure* of G if, moreover, H contains no such pairs.

As noted in [2], a graph can have different K_4-closures, but obtaining a K_4-closure of G can be helpful to answer the question whether G is hamiltonian, for instance, if K_n is a K_4-closure of G. If G has a unique K_4-closure, then we denote it by $K_4(G)$.

Theorem 3 has the following obvious consequence for claw-free graphs.

Corollary 4. *Let $G=(V,E)$ be a claw-free graph and let $\{x,y,u,v\}$ be a subset of four vertices of V such that $G[\{x,y,u,v\}]=K_4-uv$. Then G is hamiltonian if and only if $G+uv$ is hamiltonian.*

Motivated by the above results, it is natural to investigate the possible connections between the Ryjáček closure and the K_4-closure, and to look for more general closure concepts. In fact, our research led to an easy proof of a result which is slightly stronger than Theorem 1, using a variation on the K_4-closure concept. We present this proof in Section 2. In Section 3 we show that, alternatively, the Ryjáček closure can be obtained using a combination of the K_4-closure and a similar closure concept defined on K_5 minus an edge.

2. A variation on the K_4-closure

We start this section by introducing some additional notation. If P is a path with a fixed orientation from one of the end vertices to the other and $u,v\in V(P)$, then $u\overrightarrow{P}v$ denotes the consecutive vertices of P between u and v (if any) in the direction specified by the fixed orientation, while $v\overleftarrow{P}u$ denotes the same vertices in reverse order. We will consider $u\overrightarrow{P}v$ and $v\overleftarrow{P}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on P (if any) in the direction specified by the fixed orientation, and u^- to denote its predecessor (if any).

We define a new closure based on the following result.

Theorem 5. *Let $G=(V,E)$ be a graph and let $\{x,y,u,v\}$ be a subset of four vertices of V such that $uv\notin E$ and $\{x,y\}\subseteq N(u)\cap N(v)$. If $N(x)\subseteq N(u)\cup N(v)\cup\{u,v\}$ and $N(y)\backslash(N(x)\cup\{x\})$ induces a complete graph (or is empty), then for every cycle C' of $G+uv$ there exists a cycle C of G such that $V(C')\subseteq V(C)$.*

Proof. Let G, x, y, u, and v be chosen as in the hypothesis of the theorem, and assume that C' is a cycle in $G+uv$ such that G has no cycle containing all vertices of C'. Consider the path $P=C'-uv$ in G, and orient it from u to v. Clearly $N(u)\cap N(v)\subseteq V(P)$. Hence x and y are on P. Without loss of generality, we may assume $x\in u\overrightarrow{P}y$, otherwise we reverse the orientation and interchange the roles of u and v in the arguments. If $xy\in E(P)$, then the cycle $uy\overrightarrow{P}vx\overleftarrow{P}u$ contains all vertices of $V(P)=V(C')$, a contradiction. We obtain the same contradiction if $x^+u\in E$. Hence, by the hypothesis, $x^+v\in E$. If $y^-\in N(x)$, then the cycle $u\overrightarrow{P}xy^-\overleftarrow{P}x^+v\overleftarrow{P}yu$ gives a contradiction. If $y^+\in N(x)$, then the cycle $u\overrightarrow{P}xy^+\overrightarrow{P}vx^+\overrightarrow{P}yu$ contradicts the assumptions.

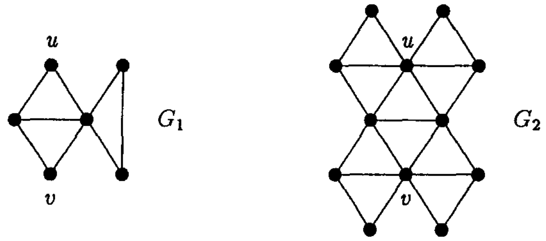

Fig. 1. The graphs G_1 and G_2.

Hence $y^- \notin N(x) \cup \{x\}$ and $y^+ \notin N(x) \cup \{x\}$. The hypothesis of the theorem implies $y^-y^+ \in E$. But then the cycle $u\overrightarrow{P}y^-y^+\overrightarrow{P}vyu$ contains all vertices of $V(P) = V(C')$, our final contradiction. □

Remark that x and y can be nonadjacent in Theorem 5, and the graph G need not be claw-free. Note, however, that if G is a claw-free graph, then the conditions of Theorem 5 are always satisfied if x and y are adjacent.

Based on Theorem 5, a graph H is called a *K_4^*-closure* of a graph G if H can be obtained from G by iteratively joining pairs $\{u,v\}$ satisfying the conditions in Theorem 5 for some $\{x,y\} \subseteq N(u) \cap N(v)$, and if H contains no such pairs. As with the K_4-closure, a graph can have different K_4^*-closures. In fact, the example given in [2] to show that a graph can have different K_4-closures, also has different K_4^*-closures. We omit the details. If G has a unique K_4^*-closure, then we denote it by $K_4^*(G)$.

The examples in Fig. 1 show that there exists a graph G_1 such that $K_4(G_1) = G_1$ and $K_4^*(G_1) = G_1 + uv$ for two nonadjacent vertices u and v of G_1, as well as a graph G_2 such that $K_4^*(G_2) = G_2$ and $K_4(G_2) = G_2 + uv$ for two nonadjacent vertices u and v of G_2. We leave the details to the reader.

As an obvious consequence of Theorem 5, we obtain the following result.

Corollary 6. *For any graph G and any K_4^*-closure H of G, $c(H) = c(G)$.*

We next show that for every claw-free graph G, the Ryjáček closure $C_R(G)$ is contained in some K_4^*-closure of G. Before we give a proof of this, we first present a useful observation.

Proposition 7. *Let $G = (V,E)$ be a claw-free graph and let $\{x,y,u,v\}$ be a subset of four vertices of V such that $uv \notin E(G)$ and $\{x,y\} \subseteq N(u) \cap N(v)$. If $xy \in E$, then $N(x) \subseteq N(u) \cup N(v) \cup \{u,v\}$ and $N(y) \backslash (N(x) \cup \{x\})$ induces a complete graph (or is empty).*

Proof. Let G and $\{x,y,u,v\}$ satisfy the hypothesis of the proposition. Suppose there exists a vertex $z \in N(x) \backslash (N(u) \cup N(v) \cup \{u,v\})$. Then clearly $G[\{x,z,u,v\}] = K_{1,3}$, a contradiction. Suppose there exist two nonadjacent vertices $z_1, z_2 \in N(y) \backslash (N(x) \cup \{x\})$. Then clearly $G[\{x,y,z_1,z_2\}] = K_{1,3}$. □

Theorem 8. *Let G be a claw-free graph. Then $C_R(G)$ is a spanning subgraph of some K_4^*-closure of G.*

It is clear from Theorem 1(i) and the definition of $C_R(G)$ that Theorem 8 follows by iteratively applying the next lemma as long as the graph under consideration contains vertices with a connected noncomplete neighborhood.

Lemma 9. *Let G be a claw-free graph and let x be a vertex of G with a connected noncomplete neighborhood. Then the local completion G^* of G at x can be obtained by iteratively joining pairs $\{u,v\} \subseteq N(x)$ satisfying the conditions in Theorem 5 for some $y \in N(u) \cap N(v)$.*

Proof. Consider the subgraph H_x of G induced by $N(x) \cup \{a \in V(G) \mid ab \in E(G)$ for some $b \in N(x)\}$. Note that $x \in V(H_x)$. Clearly H_x is a claw-free graph. Hence, by Proposition 7, $N(x) \subseteq N(u) \cup N(v) \cup \{u,v\}$ and $N(y) \backslash (N(x) \cup \{x\})$ induces a complete graph (or is empty) in H_x for all $y \in N(x)$. Since we only join nonadjacent pairs in $N(x)$, $N(x)$ and $N(y)$ will keep these properties for all $y \in N(x)$. □

Comparing the Ryjáček closure and the K_4^*-closure, it is clear that the latter is more generally applicable, since it can be applied to graphs containing induced claws (see graph G in Fig. 1 for an example). Unfortunately, we lose the nice property of unicity.

The proof of Theorem 5 (and the corresponding result for hamiltonian graphs) is easier and shorter than the proof of Theorem 1 in [5]. One of the reasons for this is that in the proof of Theorem 1, all missing edges in a connected noncomplete neighborhood are added at the same time, while in the proof of Theorem 5, only one edge is added. In this last sense the proof of Theorem 5 is similar to the proofs of many known closure results.

3. A closure concept for $K_5 - e$

Let $K_4 - uv$ be an induced subgraph of a graph G for two different vertices u and v. In Theorem 3 a condition is given to add the edge uv, such that G is hamiltonian if and only if $G + uv$ is hamiltonian. It is not difficult to adapt the proof of Theorem 3 in [2] to show that, under the same conditions, in fact $c(G + uv) = c(G)$.

A natural question that arose during the workshop mentioned before, is whether the Ryjáček closure $C_R(G)$ of a graph G is always contained in some K_4-closure of G. This is not the case. As an example consider the graph G_3 of Fig. 2. One readily checks that $C_R(G_3) = K_6$, while $K_4(G_3) = K_6 - E(K_3)$. We omit the details.

However, if we combine the condition on subgraphs isomorphic to $K_4 - e$ with the following condition on subgraphs isomorphic to $K_5 - e$, we obtain a closure concept, different from the K_4^*-closure, which is more general than the Ryjáček closure.

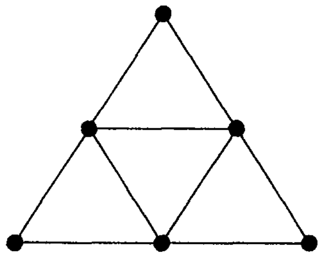

Fig. 2. The graph G_3.

Theorem 10. *Let $G=(V,E)$ be a graph and let $\{x,y_1,y_2,u,v\}$ be a subset of five vertices of V such that $G[\{x,y_1,y_2,u,v\}]=K_5-uv$. If*
(1) $N(x)\subseteq N(u)\cup N(v)\cup\{u,v\}$,
(2) $N(y_i)\subseteq N(u)\cup N(v)\cup N(y_{3-i})\cup\{u,v\}$ $(i=1,2)$, *and*
(3) $N(y_i)\backslash(N(x)\cup\{x\})$ *induces a complete graph (or is an empty set)* $(i=1,2)$,
then for every cycle C' of $G+uv$ there exists a cycle C of G such that $V(C')\subseteq V(C)$.

Proof. Let G, x, y_1, y_2, u, and v be chosen as in the hypothesis of the theorem, and assume that C' is a cycle in $G+uv$ such that G has no cycle containing all vertices of C'. Consider the path $P=C'-uv$ in G, and orient it from u to v. Clearly $uv\notin E$ and $N(u)\cap N(v)\subseteq V(P)$. Hence x, y_1 and y_2 are on P. Without loss of generality, we may assume $y_2\in y_1\overrightarrow{P}v$ and $x\in u\overrightarrow{P}y_2$; otherwise we interchange the roles of u and v, or y_1 and y_2, in the arguments. We distinguish two cases.

Case 1. $x\in u\overrightarrow{P}y_1$. As in the proof of Theorem 5, $xy_1\notin E(P)$ and $x^+u\notin E$. Hence, by (1), $x^+v\in E$. As in the proof of Theorem 5, $y_i^-\notin N(x)$ $(i=1,2)$. If $y_1^-\in N(u)$, then the cycle $u\overrightarrow{P}xy_1\overrightarrow{P}vx^+\overrightarrow{P}y_1^-u$ gives a contradiction. If $y_1^-\in N(v)$, then the cycle $u\overrightarrow{P}y_1^-v\overleftarrow{P}y_1u$ contradicts the assumptions. Hence, by (2), $y_1^-\in N(y_2)$, and, by (3), $y_1^-y_2^-\in E$. Now $u\overrightarrow{P}y_1^-y_2^-\overleftarrow{P}y_1v\overleftarrow{P}y_2u$ contradicts the assumptions.

Case 2. $x\in y_1\overrightarrow{P}y_2$. As in Case 1, $x^-\neq y_1$, $x^+\neq y_2$, $x^+v\in E$, and, by symmetry, $x^-u\in E$. As in Case 1, $y_2^-\notin N(x)$, and, by symmetry, $y_1^+\notin N(x)$. Clearly $y_1^+\notin N(u)$, and, as in Case 1, by symmetry, $y_1^+\notin N(v)$. Hence, by (2), $y_1^+\in N(y_2)$, and, by (3), $y_1^+y_2^-\in E$. Now $u\overrightarrow{P}y_1xy_2\overrightarrow{P}vx^+\overrightarrow{P}y_2^-y_1^+\overrightarrow{P}x^-u$ contradicts the assumptions. □

Based on Theorem 10, a graph H is called a K_5-closure of a graph G if H can be obtained from G by iteratively joining pairs $\{u,v\}$ satisfying the condition in Theorem 10 for some $\{x,y_1,y_2\}$ such that $G[\{x,y_1,y_2,u,v\}]=K_5-uv$, and if H contains no such pairs. Clearly, we get the following analogue of Corollary 6.

Corollary 11. *For any graph G and any K_5-closure H of G, $c(H)=c(G)$.*

Note that in a claw-free graph G the conditions (1), (2) and (3) of Theorem 10 are satisfied for any $\{x,y_1,y_2,u,v\}\subseteq V(G)$ with $G[\{x,y_1,y_2,u,v\}]=K_5-uv$. We now show how the Ryjáček closure $C_{\mathrm{R}}(G)$ of a claw-free graph G can be obtained from G

using a combination of the K_4-closure and K_5-closure. This is based on the following lemmas.

Lemma 12. *Let G be a claw-free graph, x a vertex of G whose neighborhood is connected, and u and v two nonadjacent vertices in $N(x)$. Then*
(i) $d_{G[N(x)]}(u,v)=2$ *or* 3.
(ii) *If* $d_{G[N(x)]}(u,v)=2$, *and* $y \in N(x)\cap N(u)\cap N(v)$, *then* $G[\{x,y,u,v\}]$ *is a* K_4-uv *satisfying the hypothesis of Theorem* 3.
(iii) *If* $d_{G[N(x)]}(u,v)=3$, *and* $y_1 \in N(x)\cap N(u)$, $y_2 \in N(x)\cap N(y_1)\cap N(v)$, *then for some* K_4*-supergraph* H *of* G, $H[\{x,y_1,y_2,u,v\}]$ *is a* K_5-uv *satisfying the hypothesis of Theorem* 10 (*with* H *instead of* G).

Proof. Let G, x, u, and v be chosen as in the hypothesis of the lemma.
(i) If $d_{G[N(x)]}(u,v)\geqslant 4$, then for some vertex $w\in N(x)\backslash(N(u)\cup N(v))$, $G[\{x,u,v,w\}]=K_{1,3}$, a contradiction. Hence $d(u,v)=2$ or 3.
(ii) Suppose $d_{G[N(x)]}(u,v)=2$, and $y\in N(x)\cap N(u)\cap N(v)$. Then clearly $G[\{x,y,u,v\}]=K_4-uv$, and, since G is claw-free, $N(x)\cup N(y)\subseteq N(u)\cup N(v)\cup\{u,v\}$.
(iii) Suppose $d_{G[N(x)]}(u,v)=3$, $y_1\in N(x)\cap N(u)$, and $y_2\in N(x)\cap N(y_1)\cap N(v)$. Then by (ii), $G[\{x,y_1,y_2,u\}]$ is a K_4-uy_2 satisfying the hypothesis of Theorem 3, and $G[\{x,y_1,y_2,v\}]$ is a K_4-vy_1 satisfying the hypothesis of Theorem 3 (also after the addition of uy_2 to G). Hence, in the K_4-supergraph $H=G+\{uy_2,vy_1\}$ of G, $H[\{x,y_1,y_2,u,v\}]=K_5-uv$. Using the claw-freeness of G, it is easy to check that in H this K_5-uv satisfies the hypothesis of Theorem 10. □

The next lemma is an analogue of Lemma 9. We omit its proof.

Lemma 13. *Let G be a claw-free graph and x a vertex of G with a connected noncomplete neighborhood. Then the local completion G^* of G at x can be obtained by first joining pairs $\{u,v\}\subseteq N(x)$ with $d_{G[N(x)]}(u,v)=2$ (in G) satisfying the conditions in Theorem 3, and next joining pairs $\{u,v\}\subseteq N(x)$ with $d_{G[N(x)]}(u,v)=3$ (in G, if any) satisfying the conditions in Theorem 10 (in the new graph under consideration).*

Using Lemma 12 it is not difficult to check that we can obtain the Ryjáček closure $C_R(G)$ of a claw-free graph G by iteratively applying Lemma 13 as long as the graph under consideration contains vertices with a connected noncomplete neighborhood.

In order to show that the K_5-closure is also interesting in itself, consider a graph $G(p,q)$, obtained from p different copies of a K_5-e ($p\geqslant 2$) and a K_q ($q\geqslant 2p$) by identifying two vertices of degree 4 of each copy of the K_5-e with distinct vertices of the K_q. One easily checks that $G(p,q)$ has neither a complete K_4-closure nor a complete K_4^*-closure. $G(p,q)$ contains induced claws and vertices of low degree, hence has no complete Ryjáček closure and no complete closure based on degree conditions like, e.g., the classic Bondy-Chvátal closure. However, after applying the K_5-closure to

$G(p,q)$, each of the copies of the $K_5 - e$ is turned into a K_5, and the resulting graph has a complete K_4-closure.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York, (1976).

[2] H.J. Broersma, A note on K_4-closures in hamiltonian graph theory, Discrete Math. 121 (1993) 19–23.

[3] M.M. Matthews, D.P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, J. Graph Theory 8 (1984) 139–146.

[4] D.J. Oberly, D.P. Sumner, Every connected, locally connected nontrivial graph with no induced claw is hamiltonian, J. Graph Theory 3 (1979) 351–356.

[5] Z. Ryjáček, On a closure concept in claw-free graphs, Preprint, 1996.

[6] C. Thomassen, Reflections on graph theory, J. Graph Theory 10 (1986) 309–324.

[7] S. Zhan, On hamiltonian line graphs and connectivity, Discrete Math. 89 (1991) 89–95.