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Graded Differential Geometry in REDUCE: Supersymmetry Structures of the Modified KdV Equation

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Abstract. The description of a graded differential geometry package in REDUCE is given. The procedures are useful in the study of supersymmetric equations. The supersymmetric modified KdV equation is discussed as an application.

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0. Introduction

The interest of mathematical physicists in prolongation structures and symmetries has led to the construction of computer algebra programs to handle the massive computations arising from these problems on the computer.

Many problems have been handled using these programs [1, 2] and the reader is referred to them for further references. Schwarz [3] constructed a program which deals with the determination of point symmetries of differential equations, a program in the symbolic language REDUCE [4] which runs automatically.

Recently, Reid [5] constructed a program in Macsyma to construct the structure of the point symmetry algebra of equations without integration. A program constructed by Chanpagne *et al.* [6] is similar to [3] and has the disadvantage of being very time consuming due to expressionswell.

The introduction of the concept of supersymmetry [7], treating bosonic and fermionic fields on an equal footing, boiled down to the mathematical concept of graded differential geometry [8] and stimulated the authors to the subject of the present research.

In Section 1, we shall set down the general notions of graded differential geometry which have been taken from [8].

In Section 2, we give a general description of the procedures which have been constructed. For a more detailed description of the procedures, the reader is referred to [9].

In Section 3 we give an application of the developed software to construct bosonic and fermionic symmetries and conservation laws for the Manin–Radul supersymmetric modified Korteweg–de Vries equation. For other results obtained by the computer programs described below we refer to [13] where we derived higher-order symmetries, recursion operators, and higher-order supersymmetries and fermionic conservation laws for the KdV equation.

Roelofs and Van den Hijligenberg constructed a graded Lie algebra prolongation of the supersymmetric KdV equation [14]. Roelofs and Kersten [15] constructed supersymmetric extensions of the nonlinear Schrödinger equation and discussed the symmetry structure and prolongation algebra.

1. Basic Notions of Graded Differential Geometry

In this section, we give a short review of the notions of graded differential geometry which are needed for implementation on the computer by means of the procedure described in the next sections, i.e. graded commutative algebra, graded Lie algebra, graded manifold, graded derivation, graded vector field, graded differential form, exterior differentiation, interior differentiation (or contraction) by a vector field, and Lie differentiation by a vector field.

The notions and notations have been taken from Kostant [8] and the reader is referred to this paper for more details.

Throughout this section, the field is $\mathbb{R}$ or $\mathbb{C}$ and the grading will be with respect to $\mathbb{Z}_2 = \{0, 1\}$.

1. A vector space V over $\mathbb{R}$ is a *graded vector space* if one has V_0, V_1 subspaces of V, such that

   $$V = V_0 + V_1 \tag{1.1}$$

   is a direct sum.

   Elements of V_0 are called even, elements of V_1 are called odd; elements of V_0 or V_1 are called homogeneous elements. If $v \in V_i$ $(i = 1, 1)$, then i is called the degree of v, i.e.

   $$|v| = i \quad (i = 0, 1;\ i \in \mathbb{Z}_2) \tag{1.2}$$

   (*Convention*: $|\ \ |$ is used for homogeneous elements only.)

2. *A graded algebra* B is a graded vector space ($B = B_0 + B_1$) such that

   $$B_i B_j \subset B_{i+j} \quad (i, j = 0, 1 \in \mathbb{Z}_2). \tag{1.3}$$

3. A graded algebra B is called *graded commutative* if for any two homogeneous elements $x, y \in B$ we have

   $$xy = (-1)^{|x||y|} yx \tag{1.4}$$

4. V is *a left module for the graded algebra* B if V is a left module in the usual sense but V is also a graded vector space ($V = V_0 + V_1$) and

   $$B_i \cdot V_j \subseteq V_{i+j} \quad (i, j \in \mathbb{Z}_2); \tag{1.5}$$

the right modules are defined similarly.

5. If V is a left module for the *graded commutative algebra B*, then V inherits a *right module structure* where we define

$$v \cdot b = (-1)^{|b||v|} b \cdot v \quad (v \in V;\ b \in B). \tag{1.6}$$

Similarly, a left module structure is defined by a right module structure.

6. A graded vector space $\mathbf{g} = \mathbf{g}_0 + \mathbf{g}_1$, together with a bilinear operation $[*, *]$ on $\mathbf{g}$ such that $[x, y] \in \mathbf{g}_{|x|+|y|}$ is called a *graded Lie algebra* (GLA) if

$$(1)\ [x, y] = -(-1)^{|x||y|}[y, x], \tag{1.7a}$$
$$(2)\ (-1)^{|x||z|}[x, [y, z]] + (-1)^{|z||y|}[z, [x, y]] + (-1)^{|y||x|}[y, [z, x]] = 0. \tag{1.7b}$$

(1.7b) is called the (graded) Jacobi identity.

If V is a graded vector space, then End(V) has the structure of a GLA by

$$\alpha, \beta \in \mathrm{End}(V) \Rightarrow [\alpha, \beta] = \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha. \tag{1.8}$$

7. If B is a graded algebra, an operator $\alpha \in \mathrm{End}(B)_i$ is called a (graded) derivation of B if

$$\alpha(xy) = \alpha(x) \cdot y + (-1)^{|i||x|} x \cdot \alpha(y). \tag{1.9}$$

An operator $\alpha \in \mathrm{End}(B)$ is a derivation if its homogeneous components are so. The graded vector space of derivations of B, denoted Der(B) is a graded Lie subalgebra of End(B). (1.9) is called the (graded) Leibniz rule.

If B is a graded commutative algebra, then Der(B) is a left B-module where if $\zeta \in \mathrm{Der}(B)$, $f, g \in B$ then $f\zeta \in \mathrm{Der}(B)$, where

$$(f\zeta)g = f(\zeta g). \tag{1.10}$$

8. The local picture of a *graded manifold* is $U \subset \mathbb{R}^m$ open together with the *graded commutative algebra*

$$C^\infty(U) \otimes \Lambda(n) \tag{1.11}$$

where $\Lambda(n)$ is the antisymmetric (exterior) algebra on n elements

$$s_1, \ldots, s_n,\ |s_i| = 1 \quad (i = 1, \ldots, n); \qquad s_i s_j = -s_j s_i\ (i, j = 1, \ldots, n). \tag{1.12}$$

A particular element $f \in C^\infty \otimes \Lambda(n)$ is represented as

$$f = \sum_\mu f_\mu s_\mu, \tag{1.13}$$

where

$$\mu \in M_n = \{\mu = (\mu_1, \ldots, \mu_k) \mid \mu_i \in \mathbb{N},\ 1 \leqslant \mu_1 < \mu_2 \cdots < \mu_k \leqslant n\},$$
$$s_\mu = s_{\mu_1} \cdots s_{\mu_2}, \qquad f_\mu \in C^\infty(U), \tag{1.13a}$$

the elements s_μ will be represented by ALT($\mu_1, \mu_2, \ldots, \mu_k$) in the system (cf. Section 2).

9. *Graded vector fields* on a graded manifold $(U, C^\infty(U) \otimes \Lambda(n))$ are introduced as graded derivations of the algebra $C^\infty(U) \otimes \Lambda(n)$; they constitute a left $C^\infty(U) \otimes \Lambda(n)$-module. Locally, a graded vector field V is represented as

$$V = \sum_{i=1}^{m} f_i \frac{\partial}{\partial r_i} + \sum_{j=1}^{n} g_j \frac{\partial}{\partial s_j}, \tag{1.14}$$

where f_i, $g_j \in C^\infty(U) \otimes \Lambda(n)$, r_i $(i = 1, \ldots, m)$ is a local coordinate system on $U \subset \mathbb{R}^m$.

The derivations $\partial/\partial r_i$ $(i = 1, \ldots, m)$ are even, while the derivations $\partial/\partial s_j$ are odd, they satisfy the relations

$$\frac{\partial}{\partial r_i} r_k = \delta_{ik}; \qquad \frac{\partial}{\partial r_i} s_j = 0; \qquad \frac{\partial}{\partial s_j} r_i = 0; \qquad \frac{\partial}{\partial s_j} s_l = \delta_{jl}$$

$$(i, k = 1, \ldots, m, \ j, l = 1, \ldots, n) \tag{1.15}$$

10. *A graded differential (k-)form* is introduced as a k-linear map β on $\mathrm{Der}(C^\infty(U) \otimes \Lambda(n))$ which has to satisfy

$$\langle \zeta_1, \ldots, f\zeta_l, \ldots, \zeta_k \,|\, \beta \rangle = (-1)^{|f| \sum_{k=1}^{l-1} |\zeta_i|} f \langle \zeta_1, \ldots, \zeta_k \,|\, \beta \rangle \tag{1.16a}$$

and

$$\langle \zeta_1, \ldots, \zeta_j, \zeta_{j+1}, \ldots, \zeta_k \,|\, \beta \rangle = (-1)^{1 + |\zeta_j||\zeta_{j+1}|} \langle \zeta_1, \ldots, \zeta_{j+1}, \zeta_j, \ldots, \zeta_k \,|\, \beta \rangle \tag{1.16b}$$

for all

$$\zeta_i \in \mathrm{Der}(C^\infty(U) \otimes \Lambda(n)) \quad \textit{and} \quad f \in C^\infty(U) \otimes \Lambda(n).$$

The set of k-forms is denoted by $\Omega^k(U)$.

Note: Actually we had to write $\Omega^k(U, C^\infty(U) \otimes \Lambda(n))$, but we have made our choice for the abbreviated notation $\Omega^k(U)$.

$\Omega^k(U)$ has the structure of a (right) $C^\infty(U) \otimes \Lambda(n)$-module by

$$\langle \zeta_1, \ldots, \zeta_k \,|\, \beta f \rangle = \langle \zeta_1, \ldots, \zeta_k \,|\, \beta \rangle f. \tag{1.17}$$

One puts

$$\Omega^0(U) = U^\infty(U) \otimes \Lambda(n) \quad \text{and} \quad \Omega(U) = \bigoplus_{k=0}^{\infty} \Omega^k(U).$$

Moreover, $\Omega(U)$ can be given the structure of a *bigraded* $(\mathbb{Z}_+, \mathbb{Z}_2)$ commutative algebra, that is if $\beta_i \in \Omega^{k_i}(U)_{j_i}$ $(i = 1, 2)$ then

$$\beta_1 \beta_2 \in \Omega^{k_1 + k_2}(U)_{j_1 + j_2} \tag{1.18a}$$

and

$$\beta_1 \beta_2 = (-1)^{k_1 k_2 + j_1 j_2} \beta_2 \beta_1. \tag{1.18b}$$

(For a general definition of $\beta_1 \beta_2$, see [8].)

One defines the *exterior derivative*

$$d: \Omega^0(U) \to \Omega^1(U) \qquad f \to \mathrm{d}f \tag{1.19}$$

by the condition

$$\langle \zeta \mid \mathrm{d}f \rangle = \zeta f \tag{1.19a}$$

for $\zeta \in \mathrm{Der}(C^\infty(U) \otimes \Lambda(n))$ and $f \in \Omega^0(U) = C^\infty(U) \otimes \Lambda(n)$.

By (1.18), (1.19) and the definitions of $\beta_1\beta_2$ (see [8], pp. 245, 246)

$$\mathrm{d}r_i \quad (i = 1, \ldots, m), \qquad \mathrm{d}s_j \quad (j = 1, \ldots, n) \tag{1.20}$$

defined by

$$\left\langle \frac{\partial}{\partial r_k} \middle| \mathrm{d}r_i \right\rangle = \delta_{ik}, \qquad \left\langle \frac{\partial}{\partial s_j} \middle| \mathrm{d}r_i \right\rangle = 0 \tag{1.21}$$

$$\left\langle \frac{\partial}{\partial r_k} \middle| \mathrm{d}s_l \right\rangle = 0, \qquad \left\langle \frac{\partial}{\partial s_j} \middle| \mathrm{d}s_l \right\rangle = \delta_{jl}$$

generates $\Omega(U)$ and $\beta \in \Omega(U)$ can be uniquely written as

$$\beta = \sum_{\mu,\nu} \mathrm{d}r_\mu \, \mathrm{d}s^\nu f_{\mu,\nu}, \tag{1.22}$$

where

$$\mu = (\mu_1, \ldots, \mu_k) \quad 1 \leqslant \mu_1 < \cdots < \mu_k \leqslant n,\ l(\mu) = k,$$

$$\nu = (\nu_1, \ldots, \nu_n) \quad \nu_i \in N = \mathbb{Z}_+ \backslash \{0\}, \qquad |\nu| = \sum_{i=1}^{n} \nu_i, \tag{1.22a}$$

$$f_{\mu,\nu} \in C^\infty(U) \otimes \Lambda(n).$$

Note, in particular, that by (1.18)

$$\mathrm{d}s_j \, \mathrm{d}s_k = \mathrm{d}s_k \, \mathrm{d}s_j \tag{1.23}$$

and by consequence

$$(\mathrm{d}s_j)^{\nu_j} = \mathrm{d}s_j \cdots \mathrm{d}s_j \neq 0, \quad \nu_j \text{ times}. \tag{1.23a}$$

By means of (1.21) $\mathrm{d}: \Omega^0(U) \to \Omega^1(U)$ has the following explicit representation

$$\mathrm{d}f = \sum_{i=1}^{m} \mathrm{d}r_i \frac{\partial f}{\partial r_i} + \sum_{j=1}^{n} \mathrm{d}s_j \frac{\partial f}{\partial s_j}. \tag{1.24}$$

11. Since $\Omega(U)$ is a bigraded commutative algebra $(\mathbb{Z}_+, \mathbb{Z}_2)$, $\mathrm{End}(\Omega(U))$ is bigraded too and if $u \in \mathrm{End}\,\Omega(U)$ is of bidegree $(b, j) \in (\mathbb{Z}_+, \mathbb{Z}_2)$ then

$$u(\Omega^a(U)_i) \in \Omega^{a+b}(U)_{i+j}. \tag{1.25}$$

Now $u \in \mathrm{End}\,\Omega(U)$ of *bidegree* (b, j) is a (bigraded) *derivation of* $\Omega(U)$ if for any

$\alpha \in \Omega^a(U)_i k$, $\beta \in \Omega(U)$

$$u(\alpha\beta) = u(\alpha)\beta + (-1)^{ab+ij}\alpha u(\beta) \tag{1.26}$$

(Leibniz rule).

There exists a *unique* derivation (*exterior differentiation*)

$$\mathrm{d}: \Omega(U) \to \Omega(U)$$

of *bidegree* $(1, 0)$, such that

(1) $\mathrm{d}|_{\Omega^0(U)}$ is defined by (1.19), (1.24),
(2) $\mathrm{d}^2 = 0$.

If $\beta \in \Omega(U)$

$$\beta = \sum_{\mu,\nu} \mathrm{d}r_\mu \, \mathrm{d}s^\nu f_{\mu,\nu}$$

then

$$\mathrm{d}\beta = \sum_{u,\nu} (-1)^{l(\mu)+|n|} \mathrm{d}r_\mu \, \mathrm{d}s^\nu \, \mathrm{d}f_{\mu,\nu}. \tag{1.27}$$

Other familiar operations on ordinary manifolds have their counter parts in the graded case.

$\zeta \in \mathrm{Der}(C^\infty(U) \otimes \Lambda(n))$ is an *interior differentiation by* ζ (or *contraction by* ζ), $i(\zeta)$ defined by

$$\langle \zeta_1, \ldots, \zeta_b \,|\, i(\zeta)\beta \rangle = (-1)^{|\zeta|\sum_{i=1}^{b}|\zeta_i|} \langle \zeta, \zeta_1, \ldots, \zeta_b | \beta \rangle \tag{1.28}$$

for $\zeta, \zeta_1, \ldots, \zeta_b \in \mathrm{Der}(C^\infty(U) \otimes \Lambda(n))$, and $\beta \in \Omega^{b+1}(U)$.

Moreover

$$i(\zeta): \Omega(U) \to \Omega(U) \quad (\beta \in \Omega^{b+1}(U), \quad i(\zeta)\beta \in \Omega^b(U))$$

is a *derivation of bidegree* $(-1, |\zeta|)$.

The bigraded derivations on $\Omega(U)$ can be shown to constitute a *bigraded Lie algebra* $\mathrm{Der}\,\Omega(U)$ by the following Lie bracket if $u_1, u_2 \in \mathrm{Der}\,\Omega(U)$ of bidegree (b_i, b_j) $(i = 1, 2)$ then

$$[u_1, u_2] = u_1 u_2 - (-1)^{b_1 b_2 + j_1 j_2} u_2 u_1 \in \mathrm{Der}\,\Omega(U) \tag{1.29}$$

(Lemma 4.3.2 of [8]).

From (1.29), we have that *Lie differentiation by the vector field* ζ,

$$L_\zeta = \mathrm{d}i(\zeta) + i(\zeta)\mathrm{d} \tag{1.30}$$

is a derivation of $\Omega(U)$ of bidegree $(0, |\zeta|)$.

The fact that *exterior differentiation* (d), *interior differentiation by* ζ ($i(\zeta)$), and *Lie differentiation by* $\zeta(L_\zeta)$ are *derivations*, will be used to implement them on the computer starting from the representation of vector fields and differential forms (1.14), (1.22).

2. General Description of the Procedures

A general description of the procedures, constituting the graded differential geometry package is given. The source code of the procedures is given in [9]. We list below some global data required by the package

```
D!@DIF,DIMODDV-DIMEVENV
OPERATOR VNAT , EVENLVN , ODDLVN , SUPMULNAME , !@VECVAR
OPERATOR ALT, WEDGE.
```

Note. Since the construction is effectively based on an extension of the ordinary differential geometry package described in [1], we shall often refer to the procedures described in that paper.

We shall first discuss the representations of graded functions, graded vector fields, and graded differential forms as algebraic objects.

2.1. REPRESENTATION OF GRADED FUNCTIONS

Since a graded function f is an element of the graded commutative algebra $C^\infty(\mathbb{R}^m) \otimes \Lambda(n)$

$$f \in C^\infty(\mathbb{R}^m) \otimes \Lambda(n) \tag{2.1a}$$

and ordinary differential forms can be considered as such with a slight modification, we take the following representation (see [1]).

$$f = \sum_\mu f_\mu s_\mu \quad (1.13) \leftrightarrow f = \sum F_\mu {}^*\mathrm{ALT}(\mu_1, \ldots, \mu_k) \tag{2.1b}$$

where μ is a multi-index

$$\mu \in M_n = \{\mu = (\mu_1, \ldots, \mu_k) \mid \mu_i \in \mathbb{N}, \quad 1 \leqslant \mu_1 < \cdots < \mu_k \leqslant n\}.$$

Note that the global variables DIMEVENV , DIMODDV are just

DIMEVENV := m

DIMODDV := n.

EXAMPLE 2.1. $f = f_1 s_1 + f_2 s_1 s_2$ is represented as

f := F(1)*ALT(1) + F(2)*ALT(1, 2)$

whereas the dependency of F(*) is given on the DEPendency List DEPL!*

DEPL!*:=(((f 1) R1 R2 ... RM) ((F 2) R1 R2 ... RM))$ □

2.2. REPRESENTATION OF GRADED VECTOR FIELDS

A graded vector field or a derivation of the graded commutative algebra

$C^\infty(\mathbb{R}^m) \otimes \Lambda(n)$ has a local coordinate presentation

$$V = \sum_{i=1}^{m} a_i \frac{\partial}{\partial r_i} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial s_j}, \tag{2.2}$$

where a_i, $b_j \in C^\infty(\mathbb{R}^m) \otimes \Lambda(n)$ ($i = 1, \dots, m$; $j = 1, \dots, n$).

Note. In any case, the left module structure for the graded vector fields will be assumed and used.

For *ordinary* vector fields [2] we took the polynomial representation

$$V = \sum_{i=1}^{m} f_i \frac{\partial}{\partial r_i} \leftrightarrow V := \text{FOR I} := \text{1:D!@DIF SUM (F(I)*(D}\uparrow\text{I))\$} \tag{2.3}$$

where the global variable D!@DIF has the value of the dimension of the manifold; or

$$\frac{\partial}{\partial r_i} \leftrightarrow D\uparrow I \quad (i, I = 1, \dots, m).$$

For graded vector fields, we take a similar representation taking care of the even and odd variables, i.e.

$$V = \sum_{i=1}^{m} a_i \frac{\partial}{\partial r_i} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial s_j} \leftrightarrow$$

$$V := \text{(FOR I} := \text{1:DIMEVENV SUM (A(I)*DR}\uparrow\text{I))} +$$

$$\text{(FOR J} := \text{1:DIMODDV SUM (B(J)*DS}\uparrow\text{J))\$} \tag{2.4a}$$

where $A(I)$ ($I := 1, \dots$, DIMEVENV), $B(J)$ ($J := 1, \dots$, DIMODDV) are elements of $C^\infty(\mathbb{R}^m) \otimes \Lambda(n)$ (2.1.6) or

$$\frac{\partial}{\partial r_i} \leftrightarrow DR\uparrow I \quad (i, I = 1, \dots, m), \tag{2.4b}$$

$$\frac{\partial}{\partial s_j} \leftrightarrow DS\uparrow J \quad (j, J = 1, \dots, n).$$

EXAMPLE 2.2

$$v = f_1 \frac{\partial}{\partial r_1} + (f_2 s_2 + f_3 s_1 s_3) \frac{\partial}{\partial s_1} + f_4 \frac{\partial}{\partial s_2}.$$

V:= F(1)*DR + F(2)*ALT(2)*DS + F(3)*ALT(1, 3)*DS + F(4)*DS**2$

where F(1), ..., $F(4)$ are functions dependent of R1, ..., RM. □

The choice for a bivariate polynomial was made to get a quick access to the components of the vector field by means of the procedure COEFF, and the sparse representation of polynomials.

2.3. REPRESENTATION OF GRADED DIFFERENTIAL FORMS

A graded differential form $\beta \in \Omega(\mathbb{R}^m)$ has the representation

$$\beta = \sum_{\mu,\nu} \mathrm{d}r_\mu \, \mathrm{d}s^\nu f_{\mu,\nu}, \tag{2.5}$$

where

$$\mu \in M_n,\ \nu = (\nu_1, \ldots, \nu_n) \in N^n, \qquad f_{\mu,\nu} \in C^\infty(\mathbb{R}^m) \otimes \Lambda(n).$$

Note. In any case, the right module structure of $\Omega(\mathbb{R}^m)$ will be assumed and used. A basic differential form $\mathrm{d}r_\mu \, \mathrm{d}s^\nu$ ($\mu \in M_n$; $\nu \in N^n$) is represented as

$$\mathrm{d}r_\mu \, \mathrm{d}s^\nu \leftrightarrow \mathrm{WEDGE}(\mu_1, \ldots, \mu_k, s_1, \nu_1, s_2, \nu_2, \ldots, s_n, \nu_n) \tag{2.6}$$

under the following condition:

if $\nu_i = 0$ then s_i, ν_i are omitted from the WEDGE representation;
if $\nu_i = 1$ then ν_i is omitted.

EXAMPLE 2.3

$$\mathrm{d}r_1 \, \mathrm{d}r_3 \, \mathrm{d}s_1^2 \, \mathrm{d}s_3^3 \leftrightarrow \mathrm{WEDGE}(1, 3, s_1, 2, s_3, 3).$$ □

We now start the description of the procedures.

DER(FIELD,FUNCTION)
The procedure constructs the action of a graded vector field (2.4) on a graded function (2.1.b), a derivation

parameters:
FIELD: the graded vector field
FUNCTION: the graded function
procedure calls:
OPCOEFF, MULFORM, B!@IP
result:
The result is just the derivation of the function with respect to the field.

EXAMPLE 2.4

```
VEC:= DR + (F(3)*ALT(2) + F(4)*ALT(1,3))*DS
        + DS↑2
FUN:= F(1)*ALT(1) + F(2)*ALT(1,2)

DER(VEC,FUN);
DF(F(1),R1)*ALT(1) + DF(F(2)),R1)*ALT(1,2)
+ F(1)*F(3)*ALT(2) + F(1)*F(4)*ALT(1,3)
- F(2)*F(4)*ALT(1,2,3) - F(2)*ALT(1).
```
□

EVENFP(FU)
The procedure checks whether an element of $\Lambda(n)$ is even or odd.

parameters:
FU: an element of $\Lambda(n)$; i.e. s_μ: ALT$(\mu_1, \ldots, \mu_k)$
result:
if s_μ even then 1 (k even)
else NIL.

EVENODDPART(CO.ARCO)
The procedure splits a graded function ($\in C^\infty(\mathbb{R}^m) \otimes \Lambda(n)$) in its even and odd part.

parameters:
CO: a graded function
ARCO: an array of size 2
procedure calls:
OPCOEFF, EVENFP
effect:
The even part of the graded function is stored as ARCO(0), while the odd part is stored as ARCO(1).

SUPCOM(F,G)
The procedure constructs the graded Lie bracket (1.8) (or super commutator) of two graded vector fields F and G; $F, G \in \mathrm{Der}(C^\infty(U) \otimes \Lambda(n))$

$$[F, G] = FG - (-1)^{|F||G|}GF$$

parameters:
F: the first vector field
G: the second vector field.
procedure calls:
DER, EVENODDPART
result:
The result is the graded Lie bracket of the vector fields F and G.

EVENWEDGEP(L)
The procedure EVENWEDGEP calculates whether a basic differential form $dr_\mu ds^\nu \in \Omega(\mathbb{R}^m, C^\infty(\mathbb{R}^m) \otimes \Lambda(n))$ is even or odd with respect to the first part of the bidegree $(b, j) = (l(\mu) + |\nu|, |\nu|)$.

parameters:
L: a basic differential form, WEDGE(*).
result:
The result of the procedure is
T: in case $l(\mu) + |\nu|$ is even (0 mod 2)
NIL: in case $l(\mu) + |\nu|$ is odd (1 mod 2).

CHANGESIGNP(I,L1,L2)
The procedure CHANGESIGNP establishes a sign, which has to be taken care of in the construction of the multiplication in the bigraded algebra of differential forms $\Omega(\mathbb{R}^m)$.

parameters:
I: any number (in application ± 1)
L1: T or NIL
L2: T or NIL.
result:

i	L1	L2	result	i	L1	L2	result
1	T	NIL	NIL	-1	T	NIL	T
1	T	T	NIL	-1	T	T	T
1	NIL	NIL	T	-1	NIL	NIL	NIL
1	NIL	T	NIL	-1	NIL	T	T

DERADALL(L1,L2)
The procedure DERADALL constructs the s-part of the product of two basic differential forms i.e.,

$$\pm s^{\nu_1+\nu_2} = s^{\nu_1}\cdot s^{\nu_2} \ (\nu_1, \nu_2 \in N^n).$$

parameters:
L1: the first list, associated to s^{ν_1}
L2: the second list, associated to s^{ν_2}.
result:
the list associated to $s^{\nu_1+\nu_2}$.

Note: the procedure DERADALL is an extension of the procedure DERAD, available in REDUCE, and used in the representation of higher-order derivatives of functions.

EXAMPLE 2.5

```
LISP;L1:='(s1 2 s2 s4)$
     L1:='(s2 s3 s4 2)$
```

DERADALL(L1, L2)
(S1 2 S2 2 S3 S4 3). □

SPLITWEDGE(A)
The procedure SPLITWEDGE splits a basic differential form $dr_\mu ds^\nu$ ($\mu \in M_n$, $\nu \in N^n$) represented as WEDGE (*) in the 'dr_μ-' (even) and 'ds^ν-' (odd) parts.

parameters:
A: the basic differential form: WEDGE(*).
result:
the result of the procedure SPLITWEDGE is a list, which CAR is the list associated to dr_μ, the CDR being the list associated to ds^ν.

EXAMPLE 2.6

```
lisp;l1:='(wedge 1 3 5 s1 s2 3 s3)$
        l1a:=splitwedge(l1);
```

```
((1 3 5) S1 S2 3 S3)
car 11a;
(1 3 5)
cdr 11a;
(S1 S2 3 S3).
```

BASICSUPMUL(W1, W2, FAKW1, FAKW2)
The procedure BASICSUPMUL constructs the multiplication of two graded differential forms W1*FAKW1 and W2*FAKW2 in the bigraded algebra of differential forms $\Omega(\mathbb{R}^m, C^\infty(\mathbb{R}^m) \otimes \Lambda(n))$, whereas each differential form consists of only one term.

parameters:
W1: the basic differential form associated to W1*FAKW1.
W2: the basic differential form associated to W2*FAKW2.
FAKW1: the coefficient of W1*FAKW1 (FAKW1 $\in C^\infty(\mathbb{R}^m) \otimes \Lambda(n)$).
FAKW2: the coefficient of W2*FAKW2 (FAKW2 $\in C^\infty(\mathbb{R}^m) \otimes \Lambda(n)$).
procedure calls:
SPLITWEDGE, MERGESTRICT [1], EVENWEDGEP, MULFORM [1]
EVENODDPART, CHANGESIGNP, DERADALL
result:
the result is the graded (super) multiplication of two oneterm differential forms in the right $C^\infty(\mathbb{R}^m) \otimes \Lambda(n)$-module representation of $\Omega(\mathbb{R}^m, C^\infty(\mathbb{R}^m) \otimes \Lambda(n))$.

EXAMPLE 2.7

```
l1  := wedge(1, 2, s1, 2)$
l11 := alt(1)$
l2  := wedge(s2)$
l12 := 3*r1**2(alt(1) + alt(2))$
basicsupmul(l1, l2, l11, l12);
WEDGE(1, 2, S1, 2, S2)*(-3*R1**2*ALT(1, 2)).
```
□

SUPMUL(A,B)
The procedure SUPMUL constructs the graded multiplication of two general differential forms, and is actually a repeated application of the procedure BASICSUPMUL.

parameters:
A: the first differential form.
B: the second differential form.
procedure call:
OPCOEFF, BASICSUPMUL.
result:
the graded multiplication of two graded differential forms.

SUPDF(W)
In general, a graded differential form is a sum of terms, where each of them is a

product of a basic graded form with a graded function. Therefore, the procedure SUPDF splits a general form into its constituent parts and constructs the result by summing up the various graded exterior derivatives of all terms.

This procedure returns the graded exterior derivative of W.

parameter:
W: a graded differential form.
procedure calls:
BASICSUPDF, EMPTYWEDGE, OPCOEFF
result:
The graded exterior derivative of W is returned as the result.

BASICSUPDF(W,F)

This procedure computes the exterior derivative of one term, which has already been split into a form and function factor.

parameter:
W: a basic graded differential form
F: a graded function.
procedure calls:
EVENWEDGEP, GETFIRSTOP, SPLITWEDGE
result:
The graded exterior derivative of $W * F$.

Note: The order of the parameters W and F are according to the convention, that we consider the forms as a right module.

EMPTYWEDGE( )

This procedure returns the identity as a graded form, depending on the value of SUPMULNAME.

result:
One as a zeroform. For example WEDGE( ), assuming that SUPMULNAME holds the value 'WEDGE'.

EXAMPLE 2.8

```
w := wedge (1, s1, s2, 2, s3, 4)$
f1 := r1$
f2 := sin(r2)$
f3 := alt(1,2)*aa $
basicsupdf(w,f1);
basicsupdf(w,f2);
- WEDGE(1,2,S1,S2,2,S3,4)*COS(R2)
basicsupdf(w,f3);
- WEDGE(1,S1,S2,3,S3,4)*ALT(1)*AA + WEDGE(1,S1,2,S2,2,S3,4)*ALT(2)
*AA
supdf( (w + wedge(2) * (f1 + f2 + f3) );
```

− WEDGE(2,S1)*ALT(2)*AA + WEDGE(2,S2)*ALT(1)*AA − WEDGE(1,S1, S2,3,S3,4)*ALT(1)*AA + WEDGE(1,S1,2,S2,2,S3,4)*ALT(2)*AA − WEDGE(1, 2,S1,S2,2,S3,4)*COS(RS) + WEDGE(1,2)

SUPIP(F,SW)

To implement the graded contraction, the 'even and odd parts' are done separately by two procedures. Furthermore, some help functions are introduced. This is the general procedure for computing the inner product of graded field with a graded differential form.

parameters:
F: the vector field
SW: the differential form
procedure calls:
BASICSUPIPEVEN, BASICSUPIPODD, OPCOEFF
result:
The inner product of the vector field and the differential form.

BSUPIP(I,L)

This is a function, which is used in computing the contribution of even basic vector fields.

parameters:
I: an integer ($1 \leqslant I \leqslant$ DIMEVEN).
L: a list of integers (representing the even part of a differential form).
result:
Either NIL is returned, if I does not occur in L or a list of integers. The first element is either +1 or −1, followed by the elements of L, where I is deleted. The sign of the first element depends on the location of I in L, an odd place gives a +1, an even place gives a −1.

CHANGEFORSUPIP(NAME)

This is a function to get a better performance of SUPIP. The argument of a basic form consists, in general, of an even part, a list of positive integers, and an odd part, a list of odd variable – names possibly followed immediately by a positive integer, combined into one list. This list has to be split into its odd and even part for each term of an inner product. These lists are, in fact, stored under the PROPERTY !@OPELEMENT of the parameter NAME, where NAME has been a third parameter of OPCOEFF, namely to decompose the differential form of an inner product operation.

If this splitting is done once and for all, the procedure will have a better performance.

parameter:
NAME: an identifier, which should have been used at least once by OPCOEFF as the third parameter.
result:

This function is called for its side effect, each element of the first element of the list stored under the PROPERTY !@OPELEMENT of NAME will be replaced by the result of the action of the function SPLITWEDGE.

EXAMPLE 2.9

```
opcoeff(wedge(1)+wedge(s2)+wedge(1,2,s1,s2,4), wedge,cw)$
lisp mapcar(car get('cw,'!@opelement),'print)$
(WEDGE 1)
(WEDGE 1 2 S1 2 S2 4)
(WEDGE S2)
changeforsupip(cw)$
input 3;

((1 2) S1 2 S2 4)
        (NIL S2)
```

□

BASICSUPIPEVEN(I,F1,J,F2)
This function computes the contribution of even basic vector fields from an inner product, it works 'only' in conjunction with SUPIP.

parameters:
I : an integer, the I-th even basic vector field.
F1: the coefficient of the I-th even basic vector field.
J : an integer, which points to J-th basic differential form from the input of SUPIP.
F2: the coefficient of the J-th basic differential form from the input of SUPIP.
result:
The inner product of one fieldterm with one formterm.
procedure calls:
BSUPIP, EVENWEDGEP, EVENODDPART, MULFORM

BASICSUPIPODD(I,F1,J,F2)
The description is analogous to BASICSUPIPEVEN, the only difference is that odd basic vector fields are used, coming from the input of SUPIP.

INVDERADD(VAR,L)
This function is used to compute the shortened list of a basic differential form, if a partial contribution unequal to ∅ has to be computed, reflecting the fact that the order of the differential form after an inner product is reduced by one.

parameters:
VAR: an identifier, the name of an odd variable.
L : a list, where VAR must be present (precisely) once.
result:
A 'shortened' list L.

EXAMPLE 2.10

```
invderad('s1,'(1 2 3 s1));
(1 2 3)
invderad('s1,'(1 s1 4));
(1 S1 3).
```

□

3. Nonlocal Symmetries and Hierarchies of (Super)Symmetries of the Supersymmetric mKdV Equation

As an application of the developed software described in the preceding sections, we discuss the Lie algebraic structure of the supersymmetric modified KdV equation

$$v_t = -v_3 + 6v^2 v_1 - 3\varphi(v\varphi_1)_1, \qquad \varphi_t = -\varphi_3 + 3v(v\varphi)_1, \tag{3.1}$$

whereas in (3.1) the integer indices refer to differentiation with respect to x, i.e. $v_1 = v_x$, $v_3 = v_{xxx}$; x, t, v are even, while φ is an odd variable [11].

Putting $\varphi = 0$ in (3.1), we return to the classical case. Classical higher-order symmetries are vector fields defined on the infinite jet bundle $J^\infty(x, t; v, \varphi)$ and which satisfy the symmetry condition

$$\mathcal{L}_V(D^\infty I) \subset D^\infty I, \tag{3.2}$$

where $D^\infty I$ denotes the infinite prolongation of the exterior differential system I describing the partial differential equation at hand by means of the action of the total partial derivative vector field D_x, D_t, defined by

$$D_x = \partial_x + u_x\partial_u + \varphi_x\partial_\varphi + \cdots, \qquad D_t = \partial_t + u_t\partial_u + \varphi_t\partial_\varphi + \cdots \tag{3.3}$$

In (3.2), $\mathcal{L}_V$ denotes the Lie derivative by the vector field V. Due to the fact that (3.3) satisfies (3.2) in an obvious way, we restrict our search for symmetries to vertical vector fields, i.e. the components ∂_x, ∂_t are taken to be zero.

Vertical vector fields have been proved to have the following representation

$$V = f\partial_v + g\partial_\varphi + (D_x f)\partial_{v_1} + (D_x g)\partial_{\varphi_1} + \cdots,$$

so we are only interested in the generating functions [15], f, g of the vector field. The functions f, g are assumed to depend on a finite number of variables of $J^\infty(x, t; v, \varphi)$.

In the graded case at hand, we proceed in a similar way, keeping in mind the left module structure of vector fields.

We restrict our search for higher-order symmetries at this moment to even vector fields.

Equation (3.1) is graded in the classical sense, i.e.

$$\begin{aligned} &\deg(x) = -1, \qquad \deg(v) = 1, \qquad \deg(v_1) = 2, \\ &\deg(t) = -3, \qquad \deg(\varphi) = \tfrac{1}{2}, \qquad \deg(\varphi_1) = 1\tfrac{1}{2}, \end{aligned} \tag{3.5}$$

so each term in (3.1) is of degree 4 and $3\frac{1}{2}$, respectively. Our search is for even vector fields whose defining functions f, g depend on

$$\varphi, v, \varphi_1, v_1, \ldots, \varphi_5, v_5,$$

more specifically,

$$f = f_1 + f_2\varphi\varphi_5 + f_3\varphi\varphi_4 + f_4\varphi_2\varphi_3 + f_5\varphi_1\varphi_3 + f_6\varphi\varphi_3 + f_7\varphi_1\varphi_2 + f_8\varphi\varphi_2, \tag{3.6a}$$

$$g = g_1\varphi_5 + g_2\varphi_4 + g_3\varphi_3 + g_4\varphi\varphi_1\varphi_3 + g_5\varphi_2 + g_6\varphi\varphi_1\varphi_2 + g_7\varphi_1 + g_8\varphi, \tag{3.6b}$$

whereas in (2.6) $f_1, \ldots, f_8, g_1, \ldots, g_8$ are dependent on the even variables $v, \ldots, v_5$.

The vector field V (3.4), (3.6) has to satisfy the symmetry condition (3.2) which is equivalent to

$$\mathscr{L}_V(v_t + v_2 - 6v^2v_1 + 3\varphi(v\varphi)_1) \doteq 0, \qquad \mathscr{L}_V(\varphi_t + \varphi_2 - 3v(v\varphi)_1) \doteq 0, \tag{3.7}$$

where '$\doteq 0$' is to be understood as equal to zero on the submanifold of the infinite jet bundle $J^\infty(x, t; v, \varphi)$ defined by (3.1) and its differential consequences.

Conditions (3.7) lead to an overdetermined system of partial differential equations for the functions $f_1, \ldots, f_8, g_1, \ldots, g_8$, using the described software for graded differential geometry.

Solving the resulting system of equations [2], we arrive at the following result.

THEOREM 1. *The supersymmetric modified KdV equation* (3.1) *admits the following even symmetries arising from* (3.6)

$$X_1 = v_1\partial_v + \varphi_1\partial_\varphi, \tag{2.8a}$$

$$X_3 = (-v_3 + 6v^2v_1 - 3v\varphi\varphi_2 - 3v_1\varphi\varphi_1)\partial_v + (-\varphi_3 + 3vv_1\varphi + 3v^2\varphi_1)\partial_\varphi, \tag{2.8b}$$

$$\begin{aligned} X_5 = {} & (v_5 - 10v_3v^2 - 40v_2v_1v - 10v_1^3 + 30v_1v^4 + \\ & + 5v\varphi\varphi_4 + 10v_1\varphi\varphi_3 + 5v\varphi_1\varphi_3 + 5v_1\varphi_1\varphi_2 - 20v^3\varphi\varphi_2 + \\ & + 10v_2\varphi\varphi_2 + 5v_3\varphi\varphi_1 - 60v_1v^2\varphi\varphi_1)\partial_v + \\ & + (\varphi_5 - 5v^2\varphi_3 - 15v_1v\varphi_2 - 15v_2v\varphi_1 - 10v_1^2\varphi_1 + 10v^4\varphi_1 - \\ & - 5v_3v\varphi - 10v_2v_1\varphi + 20v_1v^3\varphi)\partial_\varphi. \end{aligned} \tag{2.8c}$$

Note. Of course, (3.1) admits the scaling symmetry which is equivalent to

$$-x\partial_x - 3t\partial_t + v\partial_v + \tfrac{1}{2}\varphi\partial_\varphi \tag{2.8d}$$

due to (3.5).

In a similar way as in the even case, we searched for odd symmetries and arrived at the following odd or supersymmetry

$$Y_{1/2} = \varphi_1\partial_v + v\partial_\varphi \tag{3.9}$$

and there exist no other (local) higher-order odd symmetries.

Motivated by results obtained for the super KdV equation [12] we introduce nonlocal variables in order to construct a nonlocal symmetry of (3.1).

First note that

$$q_{1/2} = \int_{-\infty}^{x} v\varphi \, dx \tag{3.10}$$

is a potential of (3.1), i.e.

$$(q_{1/2})_x = v\varphi, \tag{3.11a}$$

$$(q_{1/2})_t = \int_{-\infty}^{x} v_t\varphi + v\varphi_t \, dx \tag{2.11b}$$

$$= \int_{-\infty}^{x} (-v_3+6v^2v_1-3v\varphi\varphi_2-3v_1\varphi\varphi_1)\varphi+v(-\varphi_3+3v(v\varphi)_1)$$

$$= -v_2\varphi+v_1\varphi_1-v\varphi_2+3v^3\varphi$$

and

$$Q_{1/2} = \int_{-\infty}^{\infty} v\varphi \, dx \tag{3.12}$$

is a conserved quantity of (3.1).

We make the following observation.

THEOREM 2. *The vector field* Z_1, *defined by*

$$q_{1/2}\varphi_1\partial_v + (q_{1/2}v-\varphi_1)\partial_\varphi \tag{3.13}$$

is a nonlocal symmetry of (3.1).

Proof. In order to prove that a vector field V

$$V = V_v\partial_v + V_\varphi\partial_\varphi \tag{3.14}$$

is a nonlocal symmetry of (3.1), we have to take the prolongation of the total partial derivative vector fields D_x, D_t towards to nonlocal variables, i.e.

$$\tilde{D}_x = D_x + (v\varphi)\partial_{q_{1/2}}, \qquad \tilde{D}_t = (-v_2\varphi+v_1\varphi_1-v\varphi_2+3v^3\varphi)\partial_{q_{1/2}} \tag{3.15}$$

then the symmetry condition

$$\mathscr{L}_V(-v_t-v_3+6v^2v_1-3v\varphi\varphi_2-3v_1\varphi\varphi_1)=0, \tag{3.16}$$
$$\mathscr{L}_V(-\varphi_t-\varphi_3+3v^2\varphi_1+3vv_1\varphi)=0$$

can be written as [15]

$$-\tilde{D}_t(V_v)-(\tilde{D}_x)^3(V_v)+12v(V_v)v_1+6v^2\tilde{D}_x(V_v)-3V_v\varphi\varphi_2- \tag{3.17a}$$
$$-3vV_\varphi\varphi_2-3v\varphi(\tilde{D}_x)^2(V_\varphi)-3\tilde{D}_x(V_v)\varphi\varphi_1-3v_1V_\varphi\varphi_1-3v_1\varphi\tilde{D}_x(V_\varphi)=0$$

and

$$-\tilde{D}_t(V_\varphi)-(\tilde{D}_x)^3(V_\varphi)+6vV_v\varphi_1+3v^2\tilde{D}_x(V_\varphi)+3V_v\varphi+ \\ +3v\tilde{D}_x(V_v)\varphi+3vv_1\tilde{D}_x(V_\varphi)=0. \tag{3.17b}$$

Substitution of $V_v=q_{1/2}\varphi_1$ and $V_\varphi=q_{1/2}v-\varphi_1$ into (3.17a) and (3.17b) yields, after a tedious calculation, that (3.16) is satisfied by (3.13).

In a way similar to the super KdV equation [13], we construct a hierarchy of nonlocal odd symmetries $(Y_{n+(1/2)})_{n\in\mathbb{N}}$ by defining

$$Y_{n+(3/2)}=[Z_1,\ Y_{n+(1/2)}]\quad (n\in\mathbb{N}), \tag{3.18}$$

whereas the even hierarchy X_1, X_3, $X_5,\ldots,(X_{n+1})$ results from

$$X_{2n+1}=2[Y_{n+(1/2)},\ Y_{n+(1/2)}]\quad (n\in\mathbb{N}). \tag{3.19}$$

Moreover, the even and odd potentials p_1, $p_3,\ldots,q_{1/2}$, $q_{3/2}$, $q_{5/2},\ldots$ leading to even and odd conserved quantities, result from the construction of the graded Lie algebra (3.18), (3.19) by *prolongation of the vector fields towards nonlocal variables.*

We now present the even potentials p_1, p_3 and the nonlocal superpotentials $q_{1/2}$, $q_{3/2}$, $q_{5/2}$, the quantities being P_1, P_3, $Q_{1/2}$, $Q_{3/2}$, $Q_{5/2}$.

$$p_1=\int_{-\infty}^{x}(v^2-\varphi\varphi_1)\,\mathrm{d}x,\qquad p_3=\int_{-\infty}^{x}(-v^4-v_1^2+3\varphi\varphi_1v^2+\varphi_1\varphi_2)\,\mathrm{d}x, \tag{3.20}$$

$$q_{1/2}=\int_{-\infty}^{x}(v\varphi)\,\mathrm{d}x,\qquad q_{3/2}=\int_{-\infty}^{x}(p_1v\varphi+v\varphi_1)\,\mathrm{d}x, \tag{3.21}$$

$$q_{5/2}=\int_{-\infty}^{x}(-(p_1)v^2\varphi-2p_1v\varphi_1+v^3\varphi-2v\varphi_2)\,\mathrm{d}x.$$

The x-derivatives are just the integrands in (3.20), (3.21) while the t-derivatives are given by

$$(p_1)_t=3v^4+v_1^2-2vv_2-9v^2\varphi\varphi_1+\varphi\varphi_3-2\varphi_1\varphi_2,$$

$$(p_3)_t=-4v^6+4v_2v^3-v_2^2+2v_1v_3-12v^2v_1^2+21v^4\varphi\varphi_1- \\ -9vv_2\varphi\varphi_1+3v_1^2\varphi\varphi_1+12vv_1\varphi\varphi_2-3v^2\varphi\varphi_3+ \\ +9v^2\varphi_1\varphi_2-\varphi_1\varphi_4+2\varphi_2\varphi_3, \tag{3.22}$$

$$(q_{3/2})_t=p_1(-v_2\varphi+v_1\varphi_1-v\varphi_2+3v^3\varphi)+2v^2v_1\varphi- \\ -v_2\varphi_1+4v^3\varphi_1+v_1\varphi_2-v\varphi_3,$$

$$(q_{5/2})_t=-p_1^2(-v_2\varphi+v_1\varphi_1-v\varphi_2+3v^3\varphi)+ \\ +p_1(2v\varphi_3-2v_1\varphi_2-8v^3\varphi_1+2v_2\varphi_1-4v^2v_1\varphi)+ \\ +2v\varphi_4-2v_1\varphi_3-9v^3\varphi_2+2v_2\varphi_2-13v^2v_1\varphi_1+ \\ +4v\varphi\varphi_1\varphi_2+5v^5\varphi-9v^2v_2\varphi. \tag{3.23}$$

The symmetries obtained are given by

$$Z_1 = (q_{1/2}\varphi_1)\partial_v + (q_{1/2}v - \varphi_1)\partial_\varphi + (q_{1/2}v\varphi + \varphi\varphi_1)\partial_{p_1} + \\ + (q_{1/2}p_1 - q_{3/2})\partial_{q_{1/2}}, \tag{3.24}$$

$$Y_{1/2} = \varphi_1\partial_v + v\partial_\varphi + v\varphi\partial_{p_1} + p_1\partial_{q_{1/2}}, \tag{3.25}$$

$$Y_{3/2} = (2q_{1/2}v_1 - p_1\varphi_1 + v^2\varphi - \varphi_2)\partial_v + (2q_{1/2}\varphi_1 - p_1 v + v_1)\partial_\varphi + \\ + (2q_{1/2}(v^2 - \varphi\varphi_1) - p_1 v\varphi - 2v\varphi_1 + v_1\varphi)\partial_{p_1} + \\ + (2q_{1/2}v\varphi - \tfrac{1}{2}(p_1)^2 + \tfrac{1}{2}v^2 + \varphi\varphi_1)\partial_{q_{1/2}}, \tag{3.26}$$

$$Y_{5/2} = \{\tfrac{1}{2}(p_1)^2\varphi_1 + p_1(\varphi_2 - v^2\varphi) - 2q_{3/2}v_1 + \varphi_3 - \tfrac{5}{2}v^2\varphi_1 - 3vv_1\varphi\}\partial_v + \\ + \{\tfrac{1}{2}(p_1)^2 v + p_1(-v_1) - 2q_{3/2}\varphi_1 + v_2 - \tfrac{3}{2}v^3 + 2v\varphi\varphi_1\}\partial_\varphi + \\ + \{\tfrac{1}{2}(p_1)v^2\varphi + p_1(2v\varphi_1 - v_1\varphi) - 2q_{3/2}(v^2 - \varphi\varphi_1) + 2v\varphi_2 - 2v_1\varphi_1 + \\ + v_2\varphi - \tfrac{7}{2}v^3\varphi\}\partial_{p_1} + \\ + \{\tfrac{1}{8}(p_1)^4 - \tfrac{1}{4}(p_1)^2(+v^2 + 4\varphi\varphi_1) - 2p_1 q_{3/2}v\varphi - p_1\varphi\varphi_2 - \\ - 2q_{3/2}v\varphi_1 - \varphi_1\varphi_2 + v^2\varphi\varphi_1 - \tfrac{11}{8}v^4 + vv_2 - \tfrac{1}{2}v_1^2\}\partial_{q_{3/2}}, \tag{3.27}$$

$$X_1 = v_1\partial_v + \varphi_1\partial_\varphi, \tag{3.28}$$

$$X_3 = (-v_3 + 6v^2v_1 - 3v\varphi\varphi_2 - 3v_1\varphi\varphi_1)\partial_v + \\ + (-\varphi_3 + 3v^2\varphi_1 + 3vv_1\varphi)\partial_\varphi, \tag{3.29}$$

$$X_5 = (v_5 - 10v_3v^2 - 40v_2v_1v - 10v_1^3 + 30v_1v^4 + \\ + 5v\varphi\varphi_4 + 10v_1\varphi\varphi_3 + 5v\varphi_1\varphi_3 + 5v_1\varphi_1\varphi_2 - 20v^3\varphi\varphi_2 + \\ + 10v_2\varphi\varphi_2 + 5v_3\varphi\varphi_1 - 60v_1v^2\varphi\varphi_1)\partial_v + \\ + (\varphi_5 - 5v^2\varphi_3 - 15v_1v\varphi_2 - 15v_2v\varphi_1 - 10v_1^2\varphi_1 + 10v^4\varphi_1 - \\ - 5v_3v\varphi - 10v_2v_1\varphi + 20v_1v^3\varphi)\partial_\varphi. \tag{3.30}$$

The graded Lie algebra structure is similar to the structure obtained for the KdV equation [13].

4. Conclusion

We discussed the construction of a graded differential geometry package which has already turned out to be very useful in the study of supersymmetric equations in mathematical physics. As an application, we studied here symmetry structures of the Manin–Radul supersymmetric extension of the modified KdV equation.

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