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Research Note

Reasoning by cases in Default Logic

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Abstract

Reiter's Default Logic is one of the most popular formalisms for describing default reasoning. One important defect of Default Logic is, however, the inability to reason by cases. Over the years, several solutions for this problem have been proposed. All these proposals deal with deriving new propositions through reasoning by cases. None, however, discuss the propositions that should no longer be derivable as a result of reasoning by cases. This paper discusses the latter subject. It shows that an intuitively plausible way of dealing with propositions that should no longer be derivable as a result of reasoning by cases, can have far reaching consequences. One of the consequences is that disjunctions must be viewed as describing possible extensions. © 1998 Elsevier Science B.V.

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1. Introduction

Reiter's Default Logic [14] is one of the most popular formalisms for describing default reasoning. Its popularity is the result of the simplicity of its formalism and the fact the default rules possess no contraposition. The contraposition is often undesirable for default rules. We do not wish to conclude, for example, that John is drunk because he may not drive a car. He might not own a driving license. In cases where the contraposition is valid, we can always use a *free default rule*

$$\frac{:\varphi\to\psi}{\varphi\to\psi}.$$

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One important defect of Default Logic is, however, inability to reason by cases. A solution to this problem should enable reasoning by cases but may not result in introducing a contraposition for some of the default rules. Several proposals have been made to extend Default Logic with reasoning by cases [1,4,9,13]. As Moinard shows [13], these approaches all introduce in one way or another some form of a contraposition. Moinard analyzes the problem, and presents a modified definition of a default extension that solves the problem. He also shows that a simple transformation of the default rules, makes it possible to realize reasoning by cases using Reiter's original definition of an extension. Independently, Voorbraak [15] has proposed a similar transformation.

One aspect that has been ignored by all solutions presented so far, are the consequences for the default rules that are applicable [2] if reasoning by cases is not possible. Consider for example the default theory (D, W) where

$$W = \{bird, penguin \vee ostrich\}$$

and

$$D = \left\{ \frac{bird : \neg excep\_bird, can\_fly}{can\_fly}, \frac{penguin : \neg excep\_penguin, \neg can\_fly}{\neg can\_fly}, \frac{penguin : \neg excep\_penguin, excep\_bird}{excep\_bird}, \frac{ostrich : \neg excep\_ostrich, \neg can\_fly}{\neg can\_fly}, \frac{ostrich : \neg excep\_ostrich, excep\_bird}{excep\_bird} \right\}.$$

The reader can convince him/herself that through reasoning by cases we can derive

$$excep\_bird \quad \text{and} \quad \neg can\_fly.$$

Hence the rule

$$\frac{bird : \neg excep\_bird, can\_fly}{can\_fly}$$

is no longer applicable.

The above example shows that the application of a default rule can be blocked [3] through reasoning by cases. This is also intuitively plausible. Now suppose that in the above example we have *penguin* $\vee$ *eagle* instead of *penguin* $\vee$ *ostrich*. Then we can no longer derive *excep_bird* through reasoning by cases. Hence, *can_fly* is derivable. The question is whether this is correct.

[2] A default rule is applicable in an extension if its prerequisite belongs to the extension and its justifications are consistent with the extension.

[3] The application of a default rule is blocked in an extension if its prerequisite belongs to the extension and one of its justifications is inconsistent with the extension.

2. Reasoning by cases

Through reasoning by cases, we can derive new conclusions. Suppose that we have two rules, one stating that penguins cannot fly and one stating that ostriches cannot fly. Then, knowing that the bird Tweety is a penguin or an ostrich, we should be able to conclude that it cannot fly.

To make things even more complicated, default rules can become inapplicable through reasoning by cases. Obviously, in the above example, we may not conclude that the bird Tweety can fly using the rule "birds can fly". Since we only consider exceptional situations with respect to this rule, its application must be blocked.

Reasoning by cases can also be viewed in another way. Every situation described by a disjunction can be considered separately. This is, for example, the normal procedure in natural deduction. In the above example, the application of the rule "birds can fly" will be blocked if we assume that Tweety is a penguin and if we assume that it is an ostrich. Clearly, the application of a default rule should be blocked if it is blocked in every case described by a disjunction.

Should the application of a default rule also be blocked if it is blocked in only one case described by a disjunction? For example, may we conclude that the bird Tweety can fly if we know that it is either a penguin or an eagle? Some people working in the area of nonmonotonic reasoning will answer this question with yes.[4] They argue that Tweety being a penguin is more exceptional than Tweety being an eagle. Therefore, if we only know that the bird Tweety is either a penguin or an eagle, we conclude that it can fly.

Most people, however, will not agree with this line of reasoning. They argue that by stating that the bird Tweety is either a penguin or an eagle, we introduce two situations with no preference between them. Therefore, both situations should be evaluated separately. Furthermore, a conclusion should hold in every situation. Hence, we may not conclude that the bird Tweety can fly because the rule describing that birds can fly is not applicable in case that Tweety is a penguin. So, if we have a default theory (D, W) where

$$W = \{bird(Tweety), penguin(Tweety) \vee eagle(Tweety)\}$$

and

$$D = \left\{ \frac{bird(x) : \neg excep\_bird(x), can\_fly(x)}{can\_fly(x)}, \frac{penguin(x) : \neg excep\_penguin(x), \neg can\_fly(x)}{\neg can\_fly(x)}, \frac{penguin(x) : \neg excep\_penguin(x), excep\_bird(x)}{excep\_bird(x)} \right\},$$

then $can\_fly(Tweety)$ should not be derivable.

[4] Discussions at the *Dutch German workshop on Non-Monotonic Reasoning '95*.

The following example illustrates the reason for evaluating the situations described by a disjunction separately even better. Suppose that we have the following rules:

- A person that injures another person must be punished.

$$\frac{injures(x,y) : \neg ex1(x,y), must\_be\_punished(x)}{must\_be\_punished(x)}.$$

- A person that injures another person in self defense should not be punished.

$$\frac{self\_defence(x) \wedge injures(x,y) : \neg ex2(x,y), \neg must\_be\_punished(x)}{\neg must\_be\_punished(x)},$$

$$\frac{self\_defence(x) \wedge injures(x,y) : \neg ex2(x,y), ex1(x,y)}{ex1(x,y)}.$$

- A person that is dragged into a fight against his/her will, is acting in self defense.

$$\frac{dragged\_into\_fight(x) : \neg ex3(x), self\_defence(x)}{self\_defence(x)}.$$

Now suppose that John has injured Peter:

$$injures(John, Peter)$$

and that a reliable witness testifies that either John or Paul has been dragged into the fight against his will:

$$dragged\_into\_fight(John) \vee dragged\_into\_fight(Paul).$$

If we would not evaluate the situations described by the disjunction separately, we will conclude that John must be punished:

$$must\_be\_punished(John).$$

This would be most unfortunate for John if he was dragged into the fight against his will.

Considering cases separately raises an important problem. How do we avoid considering irrelevant cases? A tautology $\eta \vee \neg\eta$ introduces two cases. If we would use it for reasoning by cases, we can defeat any rule. Consider, for example, the default theory (D, W) where

$$W = \{bird(Tweety)\}$$

and

$$D = \left\{\frac{bird(x) : \neg excep\_bird(x), can\_fly(x)}{can\_fly(x)}\right\}.$$

$can\_fly(Tweety)$ is not a theorem of this default theory because of the tautology

$$excep\_bird(Tweety) \vee \neg excep\_bird(Tweety).$$

A similar problem arises when we derive $\varphi \vee \eta$ from φ. If we would use η for reasoning by cases, again we can defeat any rule. From the premise *bird(Tweety)*, we can derive

$$bird(Tweety) \vee excep\_bird(Tweety).$$

Clearly, we should not consider *excep_bird(Tweety)* when reasoning by cases. Only if none of the constituents of a disjunction is derivable, we can be certain that the disjunction describes the alternative situations that are possible.

When reasoning by cases we *assume* that one of the constituents of the disjunction holds. Furthermore, we can replace a conjunction by the set of its constituents. If we do this subsequently with every disjunction and every conjunction, we will end-up with a set of literals that imply all the propositions that we believe. The fact that we may only use a disjunction for reasoning by cases of which the constituents are not derivable and which is not a tautology, implies that the set of literals must be a minimal set. The minimal sets describe exactly the separate situations that we need for reasoning by cases. So, if we view a set of literals as describing an extension in Reiter's definition, reasoning by cases becomes possible. Notice that as a consequence, *a disjunction can be viewed as describing possible extensions.*

Definition 1. Let (D, W) be a default theory.[5] For any set of closed formulas S, let $\Gamma(S) = \{T_1, \ldots, T_n\}$. $T \in \Gamma(S)$ if and only if T is a smallest set, with respect to the subset relation $\subset$, of formulas satisfying following conditions:
(1) $W \subseteq T$;
(2) T is equal to the deductive closure of the set of literals[6] that T contains;
(3) if $\frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D$, $\alpha \in T$ and $\neg\beta_1, \ldots, \neg\beta_m \notin S$, then $\gamma \in T$.
A closed set of formulas E is an extension of the default theory if and only if $E \in \Gamma(E)$.

Notice that we get Reiter's original definition if we replace Condition (2) by $T = Th(T)$. In that case $\Gamma(S)$ will consist of exactly one set T.

To illustrate the modifications made to Reiter's default logic, consider a default theory (D, W) where

$$W = \{\alpha \vee \neg\beta\} \quad \text{and} \quad D = \left\{\frac{\alpha : \delta}{\delta}, \frac{\neg\beta : \delta}{\delta}, \frac{\gamma : \neg\delta}{\neg\delta}\right\}.$$

To satisfy the first two requirements of Definition 1, an extension must contain α or $\neg\beta$. No extension will contain γ or $\neg\delta$, since propositions such as $\delta \vee \neg\delta$ and $\alpha \vee \neg\beta \vee \gamma$ already follow from α and also from $\neg\beta$. Therefore, $\delta \vee \neg\delta$ and $\alpha \vee \neg\beta \vee \gamma$ cannot introduce additional extensions. So, there are only two extensions

$$E_1 = Th(\{\alpha, \delta\}) \quad \text{and} \quad E_2 = Th(\{\neg\beta, \delta\}).$$

[5] For simplicity, we will assume that the default theory is *closed*. *Open* default theories can either be dealt with as described in [14], or by replacing D by the set $\overline{D}$ of ground instances of the rules in D.

[6] In first order logic, the formulas $\exists x\ \varphi$ and $\forall x\ \varphi$ are also viewed as literals.

A side effect of reasoning by cases is that it can result in the absence of extensions. Consider for example a default theory (D, W) where

$$W = \emptyset \quad \text{and} \quad D = \left\{ \frac{: \neg\alpha, \neg\beta}{\alpha \vee \beta} \right\}.$$

In Reiter's Default Logic, the default theory has one extensions: $E = Th(\{\alpha \vee \beta\})$. In the here presented default logic, the default theory has no extension because none of the cases α and β is consistent with the two justifications of the default rule.

The above defined extensions give all the descriptions of the world that we consider possible. Alternative descriptions arise because of disjunctions and because of default rules that block each other application (e.g., the Nixon diamond). We can, of course, consider each description of the world given by an extension separately. We can also look at the information on which all descriptions of the world agree. This information will be called the *belief set* and is often denoted as the skeptical view of the world.

Definition 2. Let (D, W) be a default theory and let $E_1, \ldots, E_n$ be the corresponding extensions. Then the belief set B is defined as

$$B = \bigcap_{i=1}^{n} E_i.$$

We can now apply the above presented results to the two examples given in the beginning of this section and see whether reasoning by cases behaves as is expected. Let us first consider the first example in which we only know that the bird Tweety is a penguin or an eagle. There are two minimal sets of literals that satisfy the premises:

$$\{bird(Tweety), penguin(Tweety)\} \quad \text{and} \quad \{bird(Tweety), eagle(Tweety)\}.$$

In the former case,

$$Th(\{bird(Tweety), penguin(Tweety), \neg can\_fly(Tweety), excep\_bird(Tweety)\})$$

is an extension. In this extension default rule

$$\frac{bird(x) : \neg excep\_bird(x), can\_fly(x)}{can\_fly(x)}$$

is blocked since *excep_bird(Tweety)* is derivable. In the latter case, however, it is the only applicable default rule. Hence,

$$Th(\{bird(Tweety), eagle(Tweety), can\_fly(Tweety)\})$$

is an extension. Since we have one situation in which Tweety can and one in which it cannot fly, we may not conclude that it can fly.

In the second example, we also have two extensions. One for each case introduced by the disjunction

$$dragged\_into\_fight(John) \vee dragged\_into\_fight(Paul).$$

$$E_1 = Th(\{injures(John, Peter), dragged\_into\_fight(John), self\_defence(John), ex1(John, Peter), \neg must\_be\_punished(John)\}),$$

$$E_2 = Th(\{injures(John, Peter), dragged\_into\_fight(Paul), self\_defence(Paul), must\_be\_punished(John)\}).$$

Since in only one of the two situations John must be punished, we do not know whether John must be punished. Additional information should be collected to enable us to make a choice between the two situations that are represented by the two extensions.

3. Discussion

To enable reasoning by cases in Default Logic, one important modification was required; the view that disjunctions describe possible extensions. This raises the question concerning the consequences of this modification.

Viewing a disjunction as describing possible extensions is an important deviation from the "normal" interpretation of a disjunction. In Default Logic multiple extensions arise because the application of one default rule blocks the application of another default rule and vice versa; e.g., the Nixon diamond. This can be interpreted as a disjunction stating that one of the default rules is applicable. For each case described by this disjunction we create an extension describing that case.

For real disjunctions we can do the same. We can introduce an extension for each case described by a disjunction. Such an extension is equal to the deductive closure of the literals that it contains. It can therefore be viewed as a *partial model* in the logical sense. Hence, a disjunction describes possible partial models.

A proposition is *true* with respect to the set of a partial model (the extensions) if it is *true* in each of the partial models. Therefore, the belief set represents the propositions that are true in all partial models.

Another consequence of viewing disjunctions as describing possible extensions is that we can no longer represent a material implication $\alpha \to \beta$ by the disjunction $\neg\alpha \vee \beta$. Representing an implication be a disjunction would enable reasoning by cases using the implication. Clearly, $\alpha \to \beta$ does not represent two possible extension; one in which $\neg\alpha$ holds and one in which β holds. Instead, it represents that an extension satisfying α must also satisfy β and an extension satisfying $\neg\beta$ must also satisfy $\neg\alpha$. Therefore, we need a new representation of a material implication. Default rules can be used for this purpose. We can represent a material implication $\alpha \to \beta$ by the two default rules:

$$\frac{\alpha :}{\beta} \quad \text{and} \quad \frac{\neg\beta :}{\neg\alpha}.$$

In the literature, several variants of Default Logic have appeared. Refer, for example, to [3,6,11]. These variants modify Default Logic in order to gain some desired property. They usually realize this by changing the third condition in Reiter's definition [14] of the operator Γ. These changes are usually not affected by the view that disjunctions describe possible extensions. Therefore, the proposed modifications can also be applied to the default logic presented here.

4. Reiter's Default Logic

Can we reformulate the here presented default logic in terms of Reiter's Default Logic? If it is possible, we must ensure that an extension is equal to the deductive closure of the literals that it contains. To do this, a special set of default rules, called the hypotheses, can be used. These hypotheses ensure that one of the cases describe by a disjunction will hold. We have seen in Section 2 that not every disjunction may be used for reasoning by cases. Therefore, a restriction must be placed on the disjunctions for which we introduce hypotheses.

$$H = \left\{ \frac{\alpha \vee \beta : \neg\beta}{\alpha}, \frac{\alpha \vee \beta : \neg\alpha}{\beta} \;\middle|\; \alpha \vee \beta \in \Xi \right\}.$$

Here, Ξ contains every disjunction $\alpha \vee \beta$ such that for any set of literals Λ; $\Lambda \mid\!\!\sim \alpha \vee \beta$ if and only if $\Lambda \mid\!\!\sim \alpha$ or $\Lambda \mid\!\!\sim \beta$. So, a disjunction in Ξ does not contain literals of which the truth values are irrelevant for the meaning of the disjunction. For example, $(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta) \notin \Xi$.

Since the set of hypotheses consists of non-normal defaults rules, we may wonder whether this can result in the inexistence of extensions. Unfortunately, the answer is yes. This is illustrated by the following default theory (D, W) where

$$D = \left\{ \frac{\alpha : \beta}{\beta}, \frac{\beta : \alpha}{\alpha} \right\} \quad \text{and} \quad W = \{\alpha \vee \beta\}.$$

Definition 1 gives us the extension $Th(\{\alpha, \beta\})$. Reiter's Default Logic, however, gives us no extension for the default theory $(D \cup H, W)$. The reason is that β is derivable after applying the hypothesis

$$\frac{\alpha \vee \beta : \neg\beta}{\alpha}$$

and α is derivable after applying the hypothesis

$$\frac{\alpha \vee \beta : \neg\alpha}{\beta}.$$

Since the derivation of β (α), depends on the hypothesis, it should not block the application of the hypothesis.

Although the addition of the set of hypotheses H to the set of default rules can result in the inexistence of an extension in Reiter's Default Logic, if extensions do exist, then they are also extensions according to the above presented new default logic.

Theorem 3. *Let (D, W) be a default theory. Then E is an extension of (D, W) according to Definition* 1 *if E is a Reiter-extension of the default theory $(D \cup H, W)$.*

Proof. To prove the theorem, we must prove that for a fixed point E of the operator Γ_{R} defined by Reiter, $E \in \Gamma(E)$ holds.

Let E be a Reiter extension of the default theory $(D \cup H, W)$. Clearly, E satisfies the first and the third condition of Definition 1. Furthermore, we can prove for every $\varphi \in E$ that E contains a set of literals that imply φ by induction to the length of φ.

- Let φ be a literal. Then the set of literals in E contains φ.
- Let $\varphi = \alpha \wedge \beta$. Since $\Gamma_{\mathrm{R}}(E)$ is deductively closed, $\{\alpha, \beta\} \subseteq E$.
- Let $\varphi = \neg(\alpha \vee \beta)$. Since $\Gamma_{\mathrm{R}}(E)$ is deductively closed, $\{\neg\alpha, \neg\beta\} \subseteq E$.
- Let $\varphi = \alpha \vee \beta$. If $\alpha \vee \beta \in \Xi$, then, since E is a Reiter extension, there is a hypothesis adding either α or β to $\Gamma_{\mathrm{R}}(E)$. Hence, $\alpha \in E$ or $\beta \in E$. If $\alpha \vee \beta \notin \Xi$, then, since $\Gamma_{\mathrm{R}}(E)$ is deductively closed, there is a $\xi \in \Gamma_{\mathrm{R}}(E)$ such that $\xi \vdash \alpha \vee \beta$ and ξ contains less atoms. Hence, $\xi \in E$.
- Let $\varphi = \neg(\alpha \wedge \beta)$. If $\neg\alpha \vee \neg\beta \in \Xi$, then, since E is a Reiter extension, there is a hypothesis adding either $\neg\alpha$ or $\neg\beta$ to $\Gamma_{\mathrm{R}}(E)$. Hence, $\neg\alpha \in E$ or $\neg\beta \in E$. If $\neg\alpha \vee \neg\beta \notin \Xi$, then, since $\Gamma_{\mathrm{R}}(E)$ is deductively closed, there is a $\xi \in \Gamma_{\mathrm{R}}(E)$ such that $\xi \vdash \neg\alpha \vee \neg\beta$ and ξ contains less atoms. Hence, $\xi \in E$.

Hence, there is a $T \in \Gamma(E)$ such that $T \subseteq E$.

To show that E is a smallest set satisfying the three requirements of an element of $\Gamma(E)$, we will show that T satisfies the three requirements of $\Gamma_{\mathrm{R}}(E)$

- Since $W \subseteq T$, T satisfies the first condition of Reiter's definition of an extension.
- Since T is deductively closed, T satisfies the second condition of Reiter's definition of an extension.
- For each rule

  $$\frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D$$

  such that $\alpha \in T$ and $\neg\beta_1, \ldots, \neg\beta_m \notin E$, $\gamma \in T$. Let

  $$\frac{\alpha \vee \beta : \neg\beta}{\alpha} \in H$$

  be a hypothesis with $\alpha \vee \beta \in T$ and $\beta \notin E$. According to Definition 1, there is a possibly empty set of literals $\Lambda \subset T$ such that $\Lambda \vdash \alpha \vee \beta$. Since $\alpha \vee \beta \in \Xi$, $\Lambda \vdash \alpha$. Hence, $\alpha \in T$.

Hence, $E = \Gamma_{\mathrm{R}}(E) \in \Gamma(E)$. □

5. Closure properties

Gabbay [7] has initiated the study of the closure properties of the nonmonotonic derivability relation ($\mid\!\sim$) [7, 10, 12]. Here, the nonmonotonic derivability relation is defined as

$W \mid\!\sim_D \varphi$ if and only if B is the belief set of (D, W) and $\varphi \in B$.

Gabbay [7] argues that there are three axioms that must be satisfied by all nonmonotonic logics.

Reflexivity. If $\varphi \in W$, then $W \mathrel{|\!\sim}_D \varphi$.

Cautious Monotonicity. If $W \mathrel{|\!\sim}_D \varphi$ and $W \mathrel{|\!\sim}_D \psi$, then $W \cup \{\varphi\} \mathrel{|\!\sim}_D \psi$.

Cut. If $W \mathrel{|\!\sim}_D \varphi$ and $W \cup \{\varphi\} \mathrel{|\!\sim}_D \psi$, then $W \mathrel{|\!\sim}_D \psi$.

These axioms characterize the property called *cumulativity*.

We wish, of course, that all logical consequences of the set of premises are also derivable.

Deduction. If $W \vdash \varphi$, then $W \mathrel{|\!\sim}_D \varphi$.

This axiom implies *Reflexivity*, it implies together with *Cut* the axiom *Right Weakening*, and it implies together with *Cautious Monotonicity* and *Cut* the axiom *Left Logical Equivalence*. The latter two axioms have been proposed by Kraus, Lehmann and Magidor [10]. They also proposed an axiom characterizing reasoning by cases.

Or. If $W \cup \{\varphi\} \mathrel{|\!\sim}_D \eta$ and $W \cup \{\psi\} \mathrel{|\!\sim}_D \eta$, then $W \cup \{\varphi \vee \psi\} \mathrel{|\!\sim}_D \eta$.

Nonmonotonic logics satisfying *Deduction*, *Cautious Monotonicity*, *Cut* and *Or* are said to belong to system **P**.

Like Reiter's default logic, the here presented Default Logic is not cumulative [12]. It inherits the absence of cumulativity from Reiter's Default Logic. So, *Cautious Monotonicity* does not hold. To illustration this, consider a default theory (D, W) with the set of rules

$$D = \left\{ \frac{\alpha : \beta}{\beta}, \frac{\beta : \gamma}{\gamma}, \frac{\gamma : \neg\beta}{\neg\beta} \right\}.$$

If $W = \{\alpha\}$, then we have one extension resulting in the belief set $B = Th(\{\alpha, \beta, \gamma\})$. If, however, $W' = \{\alpha, \gamma\}$, then we have two extensions resulting in the belief set $B = Th(\{\alpha, \gamma\})$. So, *Cautious Monotonicity* does not hold for the here presented default logic.

The absence of cumulativity is often seen as a defect of a logic. The underlying intuition is that there should be no difference between deriving that a proposition holds and observing that it holds. There is, however, an important difference between the two. We believe a derived proposition for some specific reason. An observed proposition, however, may be believed for the same reason as well as other reasons. So, an observed proposition need not represent the same information as a derived proposition (refer also to [2]).

The axiom *Or*, characterizes reasoning by cases. Nevertheless, it does not hold for the here presented default logic. The reason is that we may not use any disjunction. In Section 2, we have seen that we can block the application of any default rules if we may use tautologies for reasoning by cases. The axiom *Or* does not exclude the use of tautologies for reasoning by cases. Hence, we need a more restricted axiom than *Or*.

Restricted Or.

If $W \cup \{\varphi\} \mathrel{|\!\!\sim}_D \eta$, $W \cup \{\psi\} \mathrel{|\!\!\sim}_D \eta$ and $\varphi \vee \psi \in \Xi$, then $W \cup \{\varphi \vee \psi\} \mathrel{|\!\!\sim}_D \eta$.

As already defined, Ξ contains every disjunction $\alpha \vee \beta$ such that for any set of literals Λ; $\Lambda \vdash \alpha \vee \beta$ if and only if $\Lambda \vdash \alpha$ or $\Lambda \vdash \beta$.

Theorem 4. *Let (D, W) be a default theory. Then the belief set satisfies the following axioms:*

- *Deduction*

  if $W \vdash \varphi$, then $W \mathrel{|\!\!\sim}_D \varphi$

- *Cut*

  if $W \cup \{\psi\} \mathrel{|\!\!\sim}_D \eta$ and $W \mathrel{|\!\!\sim}_D \psi$, then $W \mathrel{|\!\!\sim}_D \eta$

- *Left Logical Equivalence*

  if $W' \equiv W$ and $W \mathrel{|\!\!\sim}_D \eta$, then $W' \mathrel{|\!\!\sim}_D \eta$

- *Right Weakening*

  if $\models \eta \rightarrow \mu$ and $W \mathrel{|\!\!\sim}_D \eta$, then $W \mathrel{|\!\!\sim}_D \mu$

- *And*

  if $W \mathrel{|\!\!\sim}_D \eta$ and $W \mathrel{|\!\!\sim}_D \mu$, then $W \mathrel{|\!\!\sim}_D \eta \wedge \mu$

- *Restricted Or*

  if $W \cup \{\varphi\} \mathrel{|\!\!\sim}_D \eta$, $W \cup \{\psi\} \mathrel{|\!\!\sim}_D \eta$ and $\varphi \vee \psi \in \Xi$, then $W \cup \{\varphi \vee \psi\} \mathrel{|\!\!\sim}_D \eta$

Proof. *Deduction.* The axiom: "if $W \vdash \varphi$, then $W \mathrel{|\!\!\sim}_D \varphi$" immediately follows from requirements (1) and (2) of Definition 1.

Cut. We will show that every extension E of the default theory (D, W) is also an extension of the default theory $(D, W \cup \{\psi\})$. Since $B = \bigcap_i E_i$, for every extension E of the default theory $(D, W \cup \{\psi\})$, $\eta \in E$ holds. Hence, $W \mathrel{|\!\!\sim}_D \eta$ will hold.

Let E be an extension of the default theory (D, W). Since $W \mathrel{|\!\!\sim}_D \psi$, $\psi \in E$. One can easily verify that E satisfies the three requirements of $\Gamma_{(D, W \cup \{\psi\})}(E)$ that Definition 1 gives for the default theory $(D, W \cup \{\psi\})$.

Now suppose that $E \notin \Gamma_{(D, W \cup \{\psi\})}(E)$. So, for some $T \in \Gamma_{(D, W \cup \{\psi\})}(E)$, $T \subset E$. Since $W \cup \{\psi\} \subseteq T$, $W \subseteq T$. One can easily verify that T also satisfies the other two requirements of $\Gamma_{(D,W)}(E)$ that Definition 1 gives for the default theory (D, W). Since $T \subset E$, $E \notin \Gamma_{(D,W)}(E)$. Contradiction.

Hence, $E \in \Gamma_{(D, W \cup \{\psi\})}(E)$.

Left Logical Equivalence. Let E be an extension of (D, W). Since E is a deductively closed set, $W' \subseteq E$. Therefore, E satisfies the three requirements of $\Gamma_{(D,W')}(E)$ that Definition 1 gives for the default theory (D, W').

Now suppose that $E \notin \Gamma_{(D,W')}(E)$. So, for some $T \in \Gamma_{(D,W')}(E)$, $T \subset E$. Since T is a deductively closed set, $W \subseteq T$. One can easily verify that T also satisfies the other two requirements of $\Gamma_{(D,W)}(E)$ that Definition 1 gives for the default theory (D, W). Since $T \subset E$, $E \notin \Gamma_{(D,W)}(E)$. Contradiction.

Hence, $E \in \Gamma_{(D,W')}(E)$.

Right Weakening. Since $W \mathrel{|\!\sim}_D \eta$, for every extension E, $\eta \in E$. Since $\models \eta \to \mu$ and since E is deductively closed, $\mu \in E$. Hence, $W \mathrel{|\!\sim}_D \mu$.

And. Since $W \mathrel{|\!\sim}_D \eta$, for every extension E, $\eta \in E$. Furthermore, since $W \mathrel{|\!\sim}_D \mu$, for every extension E, $\mu \in E$. Therefore, for every extension E, $\{\eta, \mu\} \subseteq E$. Since E is deductively closed, $\eta \wedge \mu \in E$. Hence, $W \mathrel{|\!\sim}_D \eta \wedge \mu$

Restricted Or. We will show that every extension E of the default theory $(D, W \cup \{\varphi \vee \psi\})$ is also an extension of the default theory $(D, W \cup \{\varphi\})$ or the default theory $(D, W \cup \{\psi\})$. Since $B = \bigcap_i E_i$, for every extension E of the default theory $(D, W \cup \{\varphi\})$ and for every extension E of the default theory $(D, W \cup \{\psi\})$, $\eta \in E$ holds. Hence, $W \cup \{\varphi \vee \psi\} \mathrel{|\!\sim}_D \eta$ will hold.

Let E be an extension of the default theory $(D, W \cup \{\varphi \vee \psi\})$. Since $\varphi \vee \psi \in \Xi$ and since E is a deductively closed set with respect to the literals that it contains, either $\varphi \in E$ or $\psi \in E$.

Suppose that $\varphi \in E$. Then E satisfies the three requirements of $\Gamma_{(D,W\cup\{\varphi\})}(E)$ that Definition 1 gives for the default theory $(D, W \cup \{\varphi\})$.

Now suppose that $E \notin \Gamma_{(D,W\cup\{\varphi\})}(E)$. So, for some $T \in \Gamma_{(D,W\cup\{\varphi\})}(E)$, $T \subset E$. Since $W \cup \{\varphi\} \subseteq T$, $W \cup \{\varphi \vee \psi\} \subseteq T$. One can easily verify that T also satisfies the other two requirements of $\Gamma_{(D,W\cup\{\varphi\vee\psi\})}(E)$ that Definition 1 gives for the default theory $(D, W \cup \{\varphi \vee \psi\})$. Since $T \subset E$, $E \notin \Gamma_{(D,W\cup\{\varphi\vee\psi\})}(E)$. Contradiction.

Hence, $E \in \Gamma_{(D,W\cup\{\varphi\})}(E)$.

In a similar way, we can prove that $E \in \Gamma_{(D,W\cup\{\psi\})}(E)$ if $\psi \in E$. □

It is not difficult to verify that *Deduction* implies *Reflexivity*. Furthermore, *And* and *Right Weakening* imply *Modus Ponens in the Consequent* [10].

6. Semantics

The semantics for the here presented default logic is based on Etherington's semantics [5] for Reiter's Default Logic. The semantics proposed by Etherington uses sets of interpretations. The default rules are used to define a preference relation on the sets. The reason for defining a preference relation on sets of interpretations instead of a preference relation on interpretations is because we must be able to represent that a proposition is unknown; i.e., the proposition is *true* in one interpretation and *false* in another interpretation. We need this information for handling the justifications of a default rule.

We do not know the truth value of a proposition if it is *true* in one interpretation and *false* in another. But can we really say that α is *unknown* if we know $\alpha \vee \beta$ to hold? No, α is *unknown* only if β is *known* to hold. If β holds, the truth value of α does not matter any more. In other words, any interpretation in which β is *true*, can be replaced by an interpretation that is identical except for the truth value of α.

Each case that we consider while reasoning by cases, can be interpreted as describing a situation in which a proposition either has a *known* truth value or is *really unknown*. So, in the context of the known propositions, there are no restrictions on the truth values that we assign to the unknown proposition.

In Kleene's strong three-valued semantics, the truth value *unknown* denotes that a proposition can either be *true* or *false*. Unfortunately, we can not use this semantics here because tautologies can have the truth value *unknown*. In the here defined default logic, tautologies hold. Therefore, we will use sets of two-valued interpretations where each set mimics one three-valued interpretation.

Definition 5. A partial two-valued interpretation $\mathcal{M}$ is a set of two-valued interpretations such that for some consistent set of literals Λ:

$$\mathcal{M} = \{I \mid \forall \ell \in \Lambda(I \models \ell)\}.$$

Definition 6. Let $\mathcal{M}$ be a partial two-valued interpretation and let φ be a closed formula. Then $\mathcal{M} \models \varphi$ if and only if for each $I \in \mathcal{M}$: $I \models \varphi$.

Definition 7. Let S be a set of closed formulas. Then the models of S are defined as the largest partial interpretations satisfying S.

$$Mod(S) = \{\mathcal{M} \mid \forall \varphi \in S(\mathcal{M} \models \varphi), \forall \mathcal{N} \supset \mathcal{M}(\exists \varphi \in S(\mathcal{N} \not\models \varphi))\}.$$

Using the partial interpretations we can modify Etherington's definitions.

Definition 8. A default rule

$$\delta = \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D$$

prefers $\mathcal{N}$ to $\mathcal{M}$, $\mathcal{M} \leqslant_\delta \mathcal{N}$, if and only if

- $\mathcal{M} \models \alpha$;
- $\mathcal{M} \not\models \neg\beta_i$ for $1 \leqslant i \leqslant m$;
- $\mathcal{N}$ is the largest, with respect to the subset relation $\subset$, partial interpretation such that $\mathcal{N} \subseteq \mathcal{M}$ and $\mathcal{N} \models \gamma$.

If $\mathcal{M} \not\models \gamma$; i.e., $\mathcal{N} \subset \mathcal{M}$, then δ *strictly* prefers $\mathcal{N}$ to $\mathcal{M}$, $\mathcal{M} <_\delta \mathcal{N}$.

A set of default rules D prefers $\mathcal{N}$ to $\mathcal{M}$, $\mathcal{M} \leqslant_D \mathcal{N}$ if and only if for some $\delta \in D$: $\mathcal{M} \leqslant_\delta \mathcal{N}$. $\leqslant_D^*$ will be used to denote the transitive closure of $\leqslant_D$.

For a normal default theory (D, W) it suffices to consider the $\leqslant_D$ maximal partial interpretations $\mathcal{N}$ such that $\mathcal{M} \leqslant_D^* \mathcal{N}$ and $\mathcal{M} \in Mod(W)$. $\mathcal{N}$ is a $\leqslant_D$ maximal partial interpretation if and only if for no partial interpretation $\mathcal{P}$, $\mathcal{N} <_D \mathcal{P}$ holds.

The requirement $\mathcal{M} \leqslant_D^* \mathcal{N}$ and $\mathcal{M} \in Mod(W)$ guarantees that $\mathcal{N}$ is grounded in the premises. This restricts the $\leqslant_D$ maximal partial interpretation $\mathcal{N}$ that we consider.

For non-normal default theories we must account for the fact that a rule may require the continued consistency of its justification without ensuring this itself. To handle this, Etherington introduces a *stability* condition.

Here, we extend this stability condition to handle a problem specific to the here presented default logic. To illustrate the problem, consider the default theory

$$\left(\left\{\frac{:b}{b}\right\}, \{a \vee b\}\right).$$

This default theory has a $\leqslant_D$ maximal partial interpretation satisfying a and b. However, the corresponding set of propositions $Th(\{a,b\})$ is not an extension. The consequent of the default rule also ensures that the disjunction $a \vee b$ holds. Therefore, there is no need for a to hold.

Definition 9. Let (D,W) be a default theory. Furthermore, let $\mathcal{N}$ be a $\leqslant_D$ maximal partial interpretation with respect to some $\mathcal{M} \in Mod(W)$; hence, for some partial interpretation $\mathcal{M}$, $\mathcal{M} \in Mod(W)$ and $\mathcal{M} \leqslant_D^* \mathcal{N}$.

$\mathcal{N}$ is stable for (D,W) if and only if there is a $D' \subseteq D$ such that:

- $\mathcal{M} \leqslant_{D'}^* \mathcal{N}$;
- for each $\frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D'$: $\mathcal{N} \not\models \neg\beta_i$ for $1 \leqslant i \leqslant m$; and
- for no partial interpretation $\mathcal{L}$ with $\mathcal{N} \subset \mathcal{L}$: $\mathcal{L} \models \varphi$ for each

$$\varphi \in W \cup \left\{\gamma \;\middle|\; \frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D'\right\}.$$

Given the stability requirement, we can now prove *soundness* and *completeness*.

Theorem 10 (Soundness). *Let (D,W) be a default theory and let E be an extension of (D,W). Let Λ be the set of literals in E. According to Definition 5, Λ determines a unique partial interpretation $\mathcal{N}$; i.e., $E = \{\varphi \mid \mathcal{N} \models \varphi\}$.*

Then, $\mathcal{N}$ is stable for (D,W).

Proof. Firstly, we will prove that there is a partial interpretation $\mathcal{M} \in Mod(W)$ such that $\mathcal{N} \subseteq \mathcal{M}$.

Let

$$D' = \left\{\delta' \in D \;\middle|\; \delta' = \frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma}, \mathcal{N} \not\models \neg\beta_i, 1 \leqslant i \leqslant m\right\}.$$

Then we will prove that for every partial interpretation $\mathcal{L}$ with $\mathcal{N} \subset \mathcal{L} \subseteq \mathcal{M}$ there is a default rule $\delta' \in D'$ and a partial interpretation $\mathcal{P}$ such that $\mathcal{L} <_{\delta'} \mathcal{P}$, $\mathcal{N} \subseteq \mathcal{P} \subset \mathcal{L}$. Therefore, we may conclude that $\mathcal{M} \leqslant_{D'}^* \mathcal{N}$.

Finally, we will prove that for no $\delta \in D$ there is a partial interpretation $\mathcal{P}$ such that $\mathcal{N} <_\delta \mathcal{P}$. Hence, $\mathcal{N}$ is a $\leqslant_D$ maximal partial interpretation. $\mathcal{N}$ is also stable since for each $\delta' \in D'$: $\mathcal{N} \not\models \neg\beta$ for $1 \leqslant i \leqslant m$ and since $E = \{\varphi \mid \mathcal{N} \models \varphi\}$ is the smallest set satisfying the requirements of $\Gamma(E)$.

Since $W \subseteq E$, there is a smallest set of literals $\Lambda^0 \subseteq \Lambda$ such that $\forall \varphi \in W(\Lambda^0 \vdash \varphi)$. According to Definition 5, Λ^0 determines a unique partial interpretation $\mathcal{M}$. Hence, $\mathcal{M} \in Mod(W)$.

Let $\mathcal{L}$ with $\mathcal{N} \subset \mathcal{L} \subseteq \mathcal{M}$ be a partial interpretation. Furthermore, let Λ' be a set of literals that determines the partial interpretation $\mathcal{L}$. Finally, let E' be the deductive closure of Λ'. Clearly, $\Lambda' \subseteq \Lambda$ and $E' = \{\varphi \mid \mathcal{L} \models \varphi\}$.

Suppose that for no rule

$$\delta' = \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D',$$

$\mathcal{L} \models \alpha$. Since $\mathcal{L} \subseteq \mathcal{M}$, $W \subseteq E'$. Furthermore, for every

$$\delta = \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D - D',$$

$\mathcal{N} \models \neg\beta_i$ for some $1 \leqslant i \leqslant m$. So, for some $1 \leqslant i \leqslant m$, $\neg\beta_i \in E$. Hence, E' satisfies the three requirements Definition 1 gives for $\Gamma(E)$. Therefore, for some $T \in \Gamma(E)$, $T \subseteq E'$. Since $T \subseteq E' \subset E$, $E \notin \Gamma(E)$. Contradiction.

Hence, for some rule

$$\delta' = \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D',$$

$\mathcal{L} \models \alpha$.

Suppose that for no rule

$$\delta' = \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D'$$

such that $\alpha \in E'$, $\gamma \in E - E'$ holds. Since $\alpha \in E'$, $\alpha \in E$. Therefore, since $E \in \Gamma(E)$, $\gamma \in E$. Hence, since $\gamma \notin E - E'$, $\gamma \in E'$. Furthermore, since $\mathcal{L} \subseteq \mathcal{M}$, $W \subseteq E'$. Hence, E' satisfies the three requirements Definition 1 gives for $\Gamma(E)$. Therefore, for some $T \in \Gamma(E)$, $T \subseteq E'$. Since $T \subseteq E' \subset E$, $E \notin \Gamma(E)$. Contradiction.

Hence there is a rule

$$\delta' = \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D'$$

such that $\mathcal{L} \models \alpha$, $\mathcal{L} \not\models \neg\beta_i$ for $1 \leqslant i \leqslant m$ and $\gamma \in E - E'$. Let $\mathcal{P}$ be the largest partial interpretation such that $\mathcal{N} \subseteq \mathcal{P} \subseteq \mathcal{L}$ and $\mathcal{P} \models \gamma$. Clearly, δ', $\mathcal{L}$ and $\mathcal{P}$ satisfy the conditions of Definition 8. Hence, $\mathcal{L} <_{\delta'} \mathcal{P}$.

Hence, for some $\mathcal{M} \in Mod(W)$, $\mathcal{M} \leqslant^*_{D'} \mathcal{N}$.

Since E is a fixed point of Γ, for no rule

$$\delta = \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D,$$

if $\mathcal{N} \models \alpha$ and $\mathcal{N} \not\models \beta_i$ for $1 \leqslant i \leqslant m$, then $\mathcal{N} \not\models \gamma$. Hence, $\mathcal{N}$ is a $\leqslant_D$ maximal partial interpretation.

Since for each

$$\frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D' :$$

$\mathcal{N} \not\models \neg\beta_i$ for $1 \leqslant i \leqslant m$ and since $E = \{\varphi \mid \mathcal{N} \models \varphi\}$ is the smallest set satisfying the requirements of $\Gamma(E)$, $\mathcal{N}$ is stable for (D, W). □

Theorem 11 (Completeness). *Let (D, W) be a default theory and let a $\leqslant_D$ maximal partial interpretation $\mathcal{N}$ be stable for (D, W). Furthermore, let $E = \{\varphi \mid \mathcal{N} \models \varphi\}$. Then, E is an extension of (D, W).*

Proof. Since $\mathcal{N}$ is stable for (D, W), there is a (possibly infinitely long) sequence of rules $\delta'_1, \ldots, \delta'_n \in D$ such that

$$\mathcal{M}_0 \leqslant_{\delta'_1} \cdots \leqslant_{\delta'_n} \mathcal{M}_n = \mathcal{N} \quad \text{and} \quad \mathcal{M}_0 \in Mod(W).$$

Since $\mathcal{M}_0 \in Mod(W)$ and $\mathcal{N} \subseteq \mathcal{M}_0$, $W \subseteq E$.

From the definition of a partial interpretation, it follows that E is deductively closed with respect to the set of literals that it contains.

Finally, for each

$$\delta = \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D$$

if $\alpha \in E$ and $\neg\beta_i \notin E$ for each $1 \leqslant i \leqslant m$, then, since $\mathcal{N}$ is a $\leqslant_D$ maximal partial interpretation, $\gamma \in E$. Hence, E satisfies the three requirement of $\Gamma(E)$ Definition 1 gives for the default theory (D, W). Therefore, for some $T \in \Gamma(E)$, $T \subseteq E$.

Let $D' = \{\delta'_1, \ldots, \delta'_n\}$ be the set stated by the stability condition. Since $\mathcal{M}_0 \leqslant^*_{D'} \mathcal{M}_{j-1} \leqslant_{\delta'_j} \mathcal{M}_j \leqslant^*_{D'} \mathcal{N}$, for each

$$\delta' = \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D',$$

$\alpha \in E$. Furthermore, since $\mathcal{N}$ is stable, for each

$$\delta' = \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D',$$

$\beta_i \notin E$ for $1 \leqslant i \leqslant m$. Finally, since $\mathcal{N}$ is stable, there is no partial interpretation $\mathcal{L}$ such that $\mathcal{N} \subset \mathcal{L}$ and $\mathcal{L} \models \varphi$ for each

$$\varphi \in W \cup \left\{\gamma \;\middle|\; \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D'\right\}.$$

Hence, E is a smallest set closed with respect to the set of literal it contains such that

$$W \cup \left\{\gamma \;\middle|\; \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma} \in D'\right\} \subseteq E.$$

Therefore, $E \subseteq T$.

Hence, E is an extension of (D, W). □

7. Related work

Since Reiter presented his Default Logic, several proposals have been made to enable reasoning by cases. The oldest proposal, by Besnard et al. [1], is to use *free default rules*

$$\frac{:\alpha \rightarrow \beta}{\alpha \rightarrow \beta}.$$

In a free default rule, the rule is actually described by a material implication in the consequent of the default rule. The default rule itself guarantees that the material implication is applied whenever this is consistently possible. Since reasoning by cases is not a problem using material implications, it is neither problem when using free default rules.

Delgrande and Jackson [4] propose free default rules with additional justifications. They point out that *semi-normal default rules* are used to specify preferences between the default rules. To describe the same preferences using free default rules, they propose to use *semi-normal free default rules*

$$\frac{:\alpha \rightarrow \beta, \gamma}{\alpha \rightarrow \beta}.$$

Since a material implication possesses a contraposition, so does a rule described by a (semi-normal) free default rule. As we saw in the introduction this is not always desirable. When we wish to describe a rule that possesses no contraposition, a (semi-normal) free default rule is not a good candidate.

Konolige [9] proposes a slightly modified *free default rule*

$$\frac{:\beta}{\alpha \rightarrow \beta}$$

to enable reasoning by cases. This default rule seems to enable reasoning by cases while avoiding the contraposition. The justification β blocks the introduction of the material implication whenever it can be used in a contraposition. Unfortunately, as is shown by Moinard [13], we get some "shadow contraposition". The set of theorems of the default theory

$$\left(\left\{\frac{:b}{a \rightarrow b}, \frac{:c}{a \rightarrow c}\right\}, \{\neg b \vee \neg c\}\right)$$

contains the proposition $\neg a$.

Moinard [13] proposes a modified *free default rule*. To enable reasoning by cases, he translates a default rules

$$\frac{\alpha : \beta}{\gamma} \quad \text{to} \quad \frac{:\alpha \wedge \beta \wedge \gamma}{\alpha \rightarrow \gamma}.$$

Voorbraak [15] proposes a slightly different but equivalent translation. He translates

$$\frac{\alpha : \beta}{\gamma} \quad \text{to} \quad \frac{:\alpha \wedge \beta}{\alpha \rightarrow \gamma}.$$

These translations enable reasoning by cases without contraposition or shadow contraposition.

Moinard derives the translation from another approach to enable reasoning by cases. In this approach, sets of default rules are used in the definition of an extension. Suppose that $\alpha_1 \vee \cdots \vee \alpha_m \vee \delta_1 \vee \cdots \vee \delta_n \in E_i$ and suppose that

$$\frac{\alpha_j : \beta_j}{\gamma_j} \in D.$$

Then $E_{i+1} = Th(E_i \cup \{\gamma_1 \vee \cdots \vee \gamma_m \vee \delta_1 \vee \cdots \vee \delta_n\})$, provided that certain conditions are met. Moinard shows that the conditions are: $E \not\vdash \neg(\alpha_j \wedge \beta_j \wedge \gamma_j)$ for $1 \leqslant j \leqslant m$. Here $E_0 = Th(W)$ and $E = \bigcup_i E_i$.

Although Moinard's and Voorbraak's solution to the problem of reasoning by cases avoids the problems with contraposition and shadow contraposition, it does not consider the cases described by a disjunction separately. As a consequence, a default rule may be applicable though it is not applicable in one of the cases described by a disjunction. As we have seen, this can result in counter intuitive conclusions.

The ability to reason by cases also solves a related problem, applying a default rule

$$\frac{\alpha : \beta}{\gamma}$$

in the context of disjunctive information $\neg\beta \vee \delta$. The intuition is that the default rule should not be applicable. Several solutions have been proposed for this specific problem [2,6,8]. The solution proposed by Gelfond et al. [8] also enables reasoning by cases under certain conditions.

Gelfond et al. [8] introduce a new type of default rule with a special kind of disjunctive consequent

$$\frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma_1 \mid \ldots \mid \gamma_n}.$$

If such a default rule is applicable in an extension, then the extension must contain one of the consequences γ_i. As a result, the application of this default rule leads to multiple extensions

To reason by cases and to solve the problem of applying a default rule in the context of disjunctive information, disjunctive information must be described by disjunctive default rules. So, instead of the proposition $\varphi \vee \psi$, we must use the disjunctive default rule

$$\frac{:}{\varphi \mid \psi}.$$

There is one major objection against this approach. We must replace every disjunction by a disjunctive default rule. This even holds for implicit disjunctions such as $\neg(\varphi \wedge \psi)$. Clearly, this is undesirable. Another objection against this approach is that default rules are normally considered describing background knowledge. In this approach, however, we must also use default rules for describing contingent facts describing implicit and explicit disjunctions.

8. Conclusions

We have evaluated different proposals that have been made to enable reasoning by cases in Default Logic. These approaches all enable the derivation of new conclusions through reasoning by cases. None of the proposals, however, take into account the propositions that should no longer be derivable because of reasoning by cases. We have seen that, to avoid deriving intuitively implausible conclusions, an applicable default rule must also be applicable when reasoning by cases. To ensure this, the cases described by a disjunction must be considered separately. This forces us to view disjunctions as describing possible extensions. Reiter's definition has been modified according to this view. This modification can also be applied to several variants of Default Logic that have been proposed in the literature.

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