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Birth–death processes and associated polynomials

Erik A. van Doorn

Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

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Abstract

We consider birth–death processes on the nonnegative integers and the corresponding sequences of orthogonal polynomials called *birth–death polynomials*. The sequence of *associated polynomials* linked with a sequence of birth–death polynomials and its orthogonalizing measure can be used in the analysis of the underlying birth–death process in several ways. We briefly review the known applications of associated polynomials, which concern transition and first-entrance time probabilities, and establish some new results in this vein. In particular, our findings indicate how the prevalence of recurrence or α -recurrence in a birth–death process can be recognized from certain properties of the orthogonalizing measure for the associated polynomials.

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1. Introduction

We consider a birth–death process $\mathcal{X} \equiv \{X(t), t \geq 0\}$ taking values in $S \equiv \{0, 1, \dots\}$ with birth rates $\{\lambda_n, n \in S\}$ and death rates $\{\mu_n, n \in S\}$, all strictly positive except $\mu_0 \geq 0$. When $\mu_0 > 0$ the process may evanesce by escaping from S , via state 0, to an absorbing state -1 .

Karlin and McGregor [11] have shown that the transition probabilities

$$p_{ij}(t) \equiv \Pr\{X(t) = j | X(0) = i\}, \quad t \geq 0, \quad i, j \in S,$$

can be represented as

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) \psi(dx), \quad t \geq 0, \quad i, j \in S. \tag{1.1}$$

E-mail address: doorn@math.utwente.nl (E.A. van Doorn).

Here $\{\pi_n\}$ are constants given by

$$\pi_0 \equiv 0 \quad \text{and} \quad \pi_n \equiv \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}, \quad n > 0,$$

$\{Q_n(x)\}$ is a sequence of polynomials satisfying the recurrence relation

$$\begin{aligned} \lambda_n Q_{n+1}(x) &= (\lambda_n + \mu_n - x) Q_n(x) - \mu_n Q_{n-1}(x), \quad n > 1, \\ \lambda_0 Q_1(x) &= \lambda_0 + \mu_0 - x, \quad Q_0(x) = 1, \end{aligned} \tag{1.2}$$

and ψ —the *spectral measure* of \mathcal{X} —is a Borel measure of total mass 1 on the interval $[0, \infty)$ with respect to which the *birth–death polynomials* $\{Q_n(x)\}$ are orthogonal.

In what follows we will assume that the Hamburger moment problem associated with the polynomials $\{Q_n(x)\}$ is determined, or, equivalently,

$$\sum_{n=0}^{\infty} \pi_{n+1} \left(\sum_{k=0}^n (\lambda_k \pi_k)^{-1} \right)^2 = \infty \tag{1.3}$$

(see [12] or [6]). As a consequence the spectral measure of \mathcal{X} is the unique measure on the real axis with respect to which the polynomials $\{Q_n(x)\}$ are orthogonal, and hence uniquely determined by the birth and death rates.

Given the birth–death polynomials $\{Q_n(x)\}$ one defines the corresponding sequence of *associated polynomials* $\{\tilde{Q}_n(x)\}$ by replacing λ_n and μ_n by λ_{n+1} and μ_{n+1} , respectively, in the recurrence relation (1.2). That is, the associated polynomials satisfy the recurrence

$$\begin{aligned} \lambda_{n+1} \tilde{Q}_{n+1}(x) &= (\lambda_{n+1} + \mu_{n+1} - x) \tilde{Q}_n(x) - \mu_{n+1} \tilde{Q}_{n-1}(x), \quad n > 1, \\ \lambda_1 \tilde{Q}_1(x) &= \lambda_1 + \mu_1 - x, \quad \tilde{Q}_0(x) = 1. \end{aligned} \tag{1.4}$$

Shohat and Sherman [18] showed that the Hamburger moment problems corresponding to $\{Q_n(x)\}$ and $\{\tilde{Q}_n(x)\}$ are determined simultaneously. So our assumption (1.3) implies that $\tilde{\psi}$ —the orthogonalizing measure for the associated polynomials or *associated measure*—is unique. Moreover, $\tilde{\psi}$ is the spectral measure of the *associated process*, that is, the birth–death process $\tilde{\mathcal{X}}$ with birth rates $\{\tilde{\lambda}_n \equiv \lambda_{n+1}, n \in S\}$ and death rates $\{\tilde{\mu}_n \equiv \mu_{n+1}, n \in S\}$.

Associated polynomials and their orthogonalizing measures have been employed in the analysis of birth–death processes in two ways. The first application deals with first-entrance times, and will be briefly described in Section 3. The second application concerns the calculation of the spectral measure ψ , and will be sketched in Section 4. A third and new application will be discussed in Section 5, where it is shown that the prevalence of recurrence or α -recurrence in a birth–death process can be recognized from the associated measure. Section 2 contains some preliminary results and we conclude in Section 6 with an example.

2. Preliminaries

The measures ψ and $\tilde{\psi}$ can be studied conveniently through their Stieltjes transforms

$$F(z) \equiv \int_0^\infty \frac{\psi(dx)}{z-x}, \quad z \in \mathbb{C} \setminus \text{supp}(\psi)$$

and

$$\tilde{F}(z) \equiv \int_0^\infty \frac{\tilde{\psi}(dx)}{z-x}, \quad z \in \mathbb{C} \setminus \text{supp}(\tilde{\psi}),$$

respectively, which are analytic in their domains of definition. Indeed, a classical result in the theory of continued fractions [18; 17; 12, Section 8; 3] tells us that the two transforms are related as

$$F(z) = \frac{1}{z - \lambda_0 - \mu_0 - \lambda_0 \mu_1 \tilde{F}(z)}, \quad |\arg z| > 0. \tag{2.1}$$

(Actually, in the more general settings of [3,17,18] the validity of the above relation is restricted to non-real values of z .) So if $\tilde{\psi}$ (and hence $\tilde{F}(z)$) is known, we can use (2.1) to find $F(z)$, and subsequently the Stieltjes inversion formula (see Widder [19, Corollary VIII.7a])

$$\psi([0,x)) + \frac{1}{2} \psi(\{x\}) = -\frac{1}{\pi} \lim_{y \rightarrow 0^+} \int_{-y}^x \Im\{F(\xi + iy)\} d\xi, \quad x \geq 0, \tag{2.2}$$

where $\varepsilon > 0$, to obtain ψ . Relation (2.1) provides the basis for the ensuing analysis.

In Section 5 we shall encounter the moments

$$m_n \equiv \int_0^\infty x^n \psi(dx) \quad \text{and} \quad \tilde{m}_n \equiv \int_0^\infty x^n \tilde{\psi}(dx),$$

in particular for negative values of n . If, for $n < 0$, an integral does not converge, we will say that the corresponding moment equals $+\infty$. With this convention we can write, for $n \geq 1$,

$$m_{-n} = \lim_{\xi \rightarrow 0^+} \int_0^\infty \frac{\psi(dx)}{(x+\xi)^n} \quad \text{and} \quad \tilde{m}_{-n} = \lim_{\xi \rightarrow 0^+} \int_0^\infty \frac{\tilde{\psi}(dx)}{(x+\xi)^n}, \tag{2.3}$$

so that, for $n \geq 0$,

$$m_{-n-1} = -\lim_{\xi \rightarrow 0^-} \frac{1}{n!} \frac{d^n F(\xi)}{d\xi^n} \quad \text{and} \quad \tilde{m}_{-n-1} = -\lim_{\xi \rightarrow 0^-} \frac{1}{n!} \frac{d^n \tilde{F}(\xi)}{d\xi^n}. \tag{2.4}$$

We will have use in Section 5 for Theorem 2.2 below, relating the measure ψ to the moments of orders -1 and -2 of the measure $\tilde{\psi}$. Before we can prove this result, however, we need the following lemma.

Lemma 2.1. *The mass at zero of the measure ψ can be represented as*

$$\psi(\{0\}) = \lim_{z \rightarrow 0} zF(z), \quad |\arg z| \geq \theta > 0.$$

Proof. Letting

$$\phi(t) \equiv \int_0^\infty e^{-xt} \psi(dx), \quad t \geq 0, \tag{2.5}$$

we obviously have $\psi(\{0\}) = \lim_{t \rightarrow \infty} \phi(t)$. Moreover, by the Abelian theorem for Laplace transforms,

$$\lim_{t \rightarrow \infty} \phi(t) = -\lim_{z \rightarrow 0} z \int_0^\infty e^{zt} \phi(t) dt, \quad |\arg z| \geq \theta > 0.$$

The lemma follows upon substituting (2.5) and interchanging the integrals. \square

Theorem 2.2. *The moment of order -1 of the associated measure $\tilde{\psi}$ satisfies*

$$\tilde{m}_{-1} \equiv \int_0^\infty \frac{\tilde{\psi}(dx)}{x} \leq \frac{\lambda_0 + \mu_0}{\lambda_0 \mu_1}. \tag{2.6}$$

Moreover, the following statements hold true.

- (i) *If $\tilde{m}_{-1} = (\lambda_0 + \mu_0)/\lambda_0 \mu_1$ and $\tilde{m}_{-2} < \infty$, then $\psi(\{0\}) = 1/(1 + \lambda_0 \mu_1 \tilde{m}_{-2}) > 0$.*
- (ii) *If $\tilde{m}_{-1} = (\lambda_0 + \mu_0)/\lambda_0 \mu_1$ and $\tilde{m}_{-2} = \infty$, then $\psi(\{0\}) = 0$ and $m_{-1} = \infty$.*
- (iii) *If $\tilde{m}_{-1} < (\lambda_0 + \mu_0)/\lambda_0 \mu_1$, then $\psi(\{0\}) = 0$,*

$$m_{-1} = \frac{1}{\lambda_0 + \mu_0 - \lambda_0 \mu_1 \tilde{m}_{-1}} < \infty \quad \text{and} \quad \frac{m_{-2}}{m_{-1}^2} = 1 + \lambda_0 \mu_1 \tilde{m}_{-2} \leq \infty.$$

Proof. From (2.1) we obtain

$$\lambda_0 \mu_1 \int_0^\infty \frac{\tilde{\psi}(dx)}{x + \xi} = \lambda_0 + \mu_0 + \xi - \left(\int_0^\infty \frac{\psi(dx)}{x + \xi} \right)^{-1}, \quad \xi > 0. \tag{2.7}$$

Hence,

$$\int_0^\infty \frac{\tilde{\psi}(dx)}{x + \xi} < \frac{\lambda_0 + \mu_0 + \xi}{\lambda_0 \mu_1}, \quad \xi > 0,$$

which, in view of (2.3), yields the bound (2.6) upon letting $\xi \rightarrow 0+$.

The statements about m_{-1} in (ii) and (iii) follow directly from (2.3) and (2.7). The formula for m_{-2} in (iii) follows from (2.3) and (2.7) by letting $\xi \rightarrow 0+$ after taking derivatives in (2.7) with respect to ξ . Finally, combining Lemma 2.1 with (2.1) gives us

$$\psi(\{0\}) = \lim_{\xi \rightarrow 0-} \frac{\xi}{\xi - \lambda_0 - \mu_0 - \lambda_0 \mu_1 \tilde{F}(\xi)}. \tag{2.8}$$

Upon taking the limit (and applying l'Hôpital's rule and (2.4), if necessary) we obtain the statements about $\psi(\{0\})$. \square

3. First-entrance times

Associated polynomials and their orthogonalizing measure have been employed in the analysis of first-entrance times of birth–death processes. Indeed, a simple probabilistic argument reveals that the

first-entrance time distribution

$$F_{i0}(t) \equiv \Pr\{X(\tau) = 0 \text{ for some } \tau, 0 < \tau \leq t \mid X(0) = i\}, \quad i > 0,$$

can be expressed as

$$F_{i0}(t) = \mu_1 \int_0^t \tilde{p}_{i-1,0}(\tau) \, d\tau, \quad i > 0, \tag{3.1}$$

with $\{\tilde{p}_{ij}(t)\}$ denoting the transition probabilities of the associated process $\tilde{\mathcal{X}}$. The representation formula (1.1) applied to $\tilde{\mathcal{X}}$ subsequently tells us that

$$F_{i0}(t) = \mu_1 \int_0^t \int_0^\infty e^{-x\tau} \tilde{Q}_{i-1}(x) \tilde{\psi}(dx) \, d\tau, \quad i > 0,$$

which, by interchanging the integrals, can be rewritten as

$$F_{i0}(t) = \mu_1 \int_0^\infty \frac{1 - e^{-xt}}{x} \tilde{Q}_{i-1}(x) \tilde{\psi}(dx), \quad i > 0. \tag{3.2}$$

This observation has been the starting point for first-entrance time analysis for a general birth–death process in [12], and for a number of specific processes in [13–15]. Relation (3.2) together with some properties of associated polynomials may also be used to relate the exponential decay rates of transition and first-entrance time probabilities of a birth–death process. Results in this vein are reported in [9].

We finally note that by taking derivatives in (3.1) or (3.2) the representation

$$f_{i0}(t) = \mu_1 \tilde{P}_{i-1,0}(t) = \mu_1 \int_0^\infty e^{-xt} \tilde{Q}_{i-1}(x) \tilde{\psi}(dx), \quad i > 0, \tag{3.3}$$

for the first-entrance time density $f_{i0}(t) = F'_{i0}(t)$ emerges. (The discrete-time version of this result was recently observed in [7]).

4. Computing the spectral measure

Besides their use in the analysis of first-entrance times, associated polynomials can play a role in the calculation of the spectral measure ψ . The general idea of the approach is described in [13, Section 2], and is briefly reproduced here in the current notation.

So let $F(z)$ and $\tilde{F}(z)$ be the Stieltjes transforms of the measures ψ and $\tilde{\psi}$, that is, of the spectral measures of the birth–death processes \mathcal{X} and $\tilde{\mathcal{X}}$, respectively. As we have seen the two transforms are related by (2.1). By iterating (2.1) a relation will be obtained between ψ and the spectral measure $\psi^{(k)}$ of the birth–death process $\mathcal{X}^{(k)}$ with birth rates $\{\lambda_n^{(k)} \equiv \lambda_{n+k}, n \in S\}$ and death rates $\{\mu_n^{(k)} \equiv \mu_{n+1}, n \in S\}$. Indeed, letting

$$F_k(z) \equiv \int_0^\infty \frac{\psi^{(k)}(dx)}{z - x}, \quad z \in \mathbb{C} \setminus \text{supp}(\psi^{(k)}), \quad k > 0,$$

(so that $F_1(z) \equiv \tilde{F}(z)$) one obtains after some algebra

$$F(z) = -\frac{1}{\lambda_0} \left(\frac{\mu_k \tilde{Q}_{k-2}(z) F_k(z) + \tilde{Q}_{k-1}(z)}{\mu_k \tilde{Q}_{k-1}(z) F_k(z) + \tilde{Q}_k(z)} \right), \quad k > 0, \quad (4.1)$$

where $\tilde{Q}_{-1}(x) = 0$. Thus, if $F_k(z)$ is known for some $k > 0$, one can use (4.1) to obtain $F(z)$, and then apply the Stieltjes inversion formula (2.2) to obtain ψ . Concrete examples of this method of calculation can be found in [13] (see also [1]). Karlin and McGregor [14] suggest that the procedure may be applied to analyse certain linear growth birth–death processes. (As an aside we remark that these processes have been analysed later in [10] using generating functions.)

Another opportunity to apply relation (4.1), suggested in [15], arises when the birth rates λ_n and death rates μ_n are, from some state k onwards, periodic functions of n with period p , say. In that case $F_k(z) = F_{k+p}(z)$, so an appropriately adapted version of (4.1) yields an equation which can be solved for $F_k(z)$, after which we can use (4.1) to calculate $F(z)$.

5. Recurrence and α -recurrence

A third application of associated polynomials and their orthogonalizing measure $\tilde{\psi}$ in the analysis of birth–death processes will be discussed in this section, where we show that the prevalence of certain properties of the process can be recognized from $\tilde{\psi}$.

The birth–death process \mathcal{X} is *recurrent* if for some state $i \in S$ (and then for all states $i \in S$) return to state i is certain, which is equivalent to

$$\int_0^\infty p_{ii}(t) dt = \infty,$$

\mathcal{X} is called *transient* otherwise. If \mathcal{X} is recurrent it is *positive recurrent* (or *ergodic*) if for some state $i \in S$ (and then for all states $i \in S$)

$$\lim_{t \rightarrow \infty} p_{ii}(t) > 0$$

and *null-recurrent* otherwise. When $\mu_0 > 0$ the process \mathcal{X} is always transient. Criteria for recurrence and positive recurrence when $\mu_0 = 0$ are given in the next theorem.

Theorem 5.1. *If $\mu_0 = 0$, then the following are equivalent:*

- (i) \mathcal{X} is recurrent,
- (ii) $\sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} = \infty$,
- (iii) $\int_0^\infty \psi(dx)/x = \infty$,
- (iv) $\int_0^\infty \tilde{\psi}(dx)/x = 1/\mu_1$,

and the following are equivalent:

- (v) \mathcal{X} is positive recurrent,
- (vi) $\sum_{n=0}^{\infty} \pi_n < \infty$,
- (vii) $\psi(\{0\}) > 0$,
- (viii) $\int_0^{\infty} \tilde{\psi}(dx)/x = 1/\mu_1$ and $\int_0^{\infty} \tilde{\psi}(dx)/x^2 < \infty$.

Proof. Equivalence of (i)–(iii), and (v)–(vii) are well known (see [12]). The equivalence of (iii) and (iv), as well as (vii) and (viii), follow immediately from Theorem 2.2. \square

From the general theory of continuous-time Markov chains (see, for example, [2]) we know that there exists a number $\alpha \geq 0$, called the *decay parameter* of \mathcal{X} , such that for each pair $i, j \in S$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log p_{ij}(t) = -\alpha.$$

Clearly, if $\alpha > 0$ the process must be transient. The birth–death process \mathcal{X} is said to be α -recurrent if for some state $i \in S$ (and then for all states $i \in S$)

$$\int_0^{\infty} e^{\alpha t} p_{ii}(t) dt = \infty$$

and α -transient otherwise. An α -recurrent birth–death process is said to be α -positive if for some state $i \in S$ (and then for all states $i \in S$)

$$\lim_{t \rightarrow \infty} e^{\alpha t} p_{ii}(t) > 0$$

and α -null otherwise.

Representation formula (1.1) is easily seen to imply that

$$\alpha = \xi \equiv \min \text{supp}(\psi), \tag{5.1}$$

the smallest point in the support of the spectral measure ψ (see [4,8]). Moreover, it is clear that α -recurrence, α -transience and α -positivity reduce to recurrence, transience and positive recurrence, respectively, when $\alpha = 0$.

Our main goal in this section is to obtain criteria for the birth–death process \mathcal{X} to be α -recurrent and α -positive. To this end we define $\mathcal{X}^{(\alpha)}$ to be the birth–death process with birth rates $\{\lambda_n^{(\alpha)}, n \in S\}$ and death rates $\{\mu_n^{(\alpha)}, n \in S\}$, satisfying $\mu_0^{(\alpha)} = 0$ and

$$\lambda_n^{(\alpha)} \equiv \lambda_n \frac{Q_{n+1}(\alpha)}{Q_n(\alpha)} \quad \text{and} \quad \mu_{n+1}^{(\alpha)} \equiv \mu_{n+1} \frac{Q_n(\alpha)}{Q_{n+1}(\alpha)}, \quad n \geq 0, \tag{5.2}$$

which are all positive (see, for example [16]). The polynomials $\{Q_n^{(\alpha)}(x)\}$, constants $\{\pi_n^{(\alpha)}\}$ and measure $\psi^{(\alpha)}$ corresponding to $\mathcal{X}^{(\alpha)}$ are easily seen to satisfy

$$Q_n^{(\alpha)}(x) = \frac{Q_n(x + \alpha)}{Q_n(\alpha)}, \quad n \geq 0,$$

$$\pi_n^{(\alpha)} = \pi_n Q_n^2(\alpha), \quad n \geq 0$$

and

$$\psi^{(\alpha)}([0, x]) = \psi([\alpha, x + \alpha]), \quad x \geq 0.$$

In view of representation formula (1.1) we can now express the transition probabilities $p_{ij}^{(\alpha)}(t)$ of $\mathcal{X}^{(\alpha)}$ in terms of the transition probabilities of \mathcal{X} , namely,

$$p_{ij}^{(\alpha)}(t) = e^{\alpha t} \frac{Q_j(\alpha)}{Q_i(\alpha)} p_{ij}(t), \quad i, j \in S, \quad t \geq 0. \tag{5.3}$$

It follows that the process \mathcal{X} is α -recurrent (α -positive) if and only if the process $\mathcal{X}^{(\alpha)}$ is recurrent (positive recurrent). Hence, we can apply Theorem 5.1 to $\mathcal{X}^{(\alpha)}$ and translate the results in terms of \mathcal{X} to obtain criteria for α -recurrence and α -positivity of \mathcal{X} . Thus proceeding, we obtain the next theorem.

Theorem 5.2. *Let the birth–death process \mathcal{X} have decay parameter α . Then the following are equivalent:*

- (i) \mathcal{X} is α -recurrent,
- (ii) $\sum_{n=0}^{\infty} (\lambda_n \pi_n Q_n(\alpha) Q_{n+1}(\alpha))^{-1} = \infty$,
- (iii) $\int_{\alpha}^{\infty} \psi(dx)/(x - \alpha) = \infty$,
- (iv) $\int_{\alpha}^{\infty} \tilde{\psi}(dx)/(x - \alpha) = (\lambda_0 - \alpha)/\lambda_0 \mu_1$

and the following are equivalent:

- (v) \mathcal{X} is α -positive;
- (vi) $\sum_{n=0}^{\infty} \pi_n Q_n^2(\alpha) < \infty$,
- (vii) $\psi(\{\alpha\}) > 0$,
- (viii) $\int_{\alpha}^{\infty} \tilde{\psi}(dx)/(x - \alpha) = (\lambda_0 - \alpha)/\lambda_0 \mu_1$ and $\int_{\alpha}^{\infty} \tilde{\psi}(dx)/(x - \alpha)^2 < \infty$.

6. An example

We consider a birth–death process \mathcal{X} with unspecified values of λ_0 and μ_0 , but constant rates $\lambda_n = \lambda$ and $\mu_n = \mu$ for $n \geq 1$. The coefficients in the recurrence relation (1.4) for the associated polynomials $\{\tilde{Q}_n(x)\}$ are therefore constant, and it follows that these polynomials can be represented as

$$\tilde{Q}_n(x) = \left(\frac{\mu}{\lambda}\right)^{n/2} U_n\left(\frac{\lambda + \mu - x}{2\sqrt{\lambda\mu}}\right), \quad n \geq 0, \tag{6.1}$$

where $\{U_n(x)\}$ are the Chebysev polynomials of the second kind. Moreover, the corresponding measure $\tilde{\psi}$ satisfies

$$\tilde{\psi}(dx) = \frac{1}{2\pi\lambda\mu} \sqrt{4\lambda\mu - (\lambda + \mu - x)^2} dx$$

in the interval $|\lambda + \mu - x| < 2\sqrt{\lambda\mu}$, and is zero outside this interval. It follows in particular that

$$\tilde{\xi} \equiv \min \text{supp}(\tilde{\psi}) = \lambda + \mu - 2\sqrt{\lambda\mu}. \tag{6.2}$$

Finally, the Stieltjes transform of $\tilde{\psi}$ is given by

$$\tilde{F}(z) = \frac{1}{2\lambda\mu}(z - \lambda - \mu + \sqrt{(z - \lambda - \mu)^2 - 4\lambda\mu}), \tag{6.3}$$

for values of z outside the interval $|\lambda + \mu - z| < 2\sqrt{\lambda\mu}$. (See, for example [13, Eqs. (5.6)–(5.8)] for the above results.)

We first wish to find out under which conditions the process \mathcal{X} is transient, null-recurrent and positive recurrent. To resolve this problem we recall that \mathcal{X} is transient if $\mu_0 > 0$, and observe from (6.3) that

$$\tilde{F}(0) = -\min \left\{ \frac{1}{\lambda}, \frac{1}{\mu} \right\} \quad \text{and} \quad \tilde{F}'(0) = \frac{\tilde{F}(0)}{|\lambda - \mu|}.$$

Considering (2.4), conditions (iv) and (viii) of Theorem 5.1 therefore imply that the process \mathcal{X} is

$$\text{positive recurrent} \quad \text{if } \mu_0 = 0 \text{ and } \lambda < \mu, \tag{6.4}$$

$$\text{null recurrent} \quad \text{if } \mu_0 = 0 \text{ and } \lambda = \mu \tag{6.5}$$

and transient otherwise. Of course, these conclusions can also be drawn directly from the rates of the process by applying conditions (ii) and (vi) of Theorem 5.1.

Next, we wish to establish the α -classification of the process. Recalling that $\alpha = \xi \equiv \min \text{supp}(\psi)$, we know from [5, Theorem III.4.2] that $\alpha \leq \tilde{\xi}$. Moreover, since $\tilde{\xi} \equiv \min \text{supp}(\tilde{\psi})$ is a limit point of $\text{supp}(\tilde{\psi})$, we have from [9, Theorem 3.3] that $\xi < \tilde{\xi}$ if and only if $\lambda_0 + \mu_0 + \lambda_0\mu\tilde{F}(\tilde{\xi}) < \tilde{\xi}$. It follows by (6.2) and (6.3) that

$$\alpha < \tilde{\xi} \Leftrightarrow \lambda_0 + \mu_0 - \lambda_0\sqrt{\mu/\lambda} < (\sqrt{\lambda} - \sqrt{\mu})^2. \tag{6.6}$$

If $\alpha < \tilde{\xi}$ then [5, Theorem II.4.4] tells us that ψ must have an atom at α . In this case the process is α -positive by condition (vii) of Theorem 5.2. If $\alpha = \tilde{\xi}$, however, then, by (6.2) and (6.3) again, we have

$$\int_{\alpha}^{\infty} \frac{\tilde{\psi}(dx)}{x - \alpha} = -\tilde{F}(\tilde{\xi}) = \frac{1}{\sqrt{\lambda\mu}}$$

and

$$\int_{\alpha}^{\infty} \frac{\tilde{\psi}(dx)}{(x - \alpha)^2} = -\tilde{F}'(\tilde{\xi}-) = \infty,$$

so that, by conditions (iv) and (viii) of Theorem 5.2, the process cannot be α -positive, but will be α -null if $(\lambda_0 - \xi)/\lambda_0\mu = 1/\sqrt{\lambda\mu}$. With (6.2) the latter condition translates into $\lambda = \mu$, or, $\lambda > \mu$ and $\lambda_0 = \lambda - \sqrt{\lambda\mu}$. Collecting our results we find that the process \mathcal{X} is

$$\alpha\text{-positive} \quad \text{if } \lambda_0 + \mu_0 - \lambda_0\sqrt{\mu/\lambda} < (\sqrt{\lambda} - \sqrt{\mu})^2, \tag{6.7}$$

$$\alpha\text{-null} \quad \text{if } \lambda = \mu \quad \text{or } (\lambda > \mu \text{ and } \lambda_0 = \lambda - \sqrt{\lambda\mu}) \tag{6.8}$$

and α -transient otherwise.

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