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(Pre)kernel catchers for cooperative games *

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Abstract. The paper provides a new (pre)kernel catcher in that the relevant set always contains the (pre)kernel. This new (pre)kernel catcher gives rise to a better lower bound ε_{***} such that the kernel is included in strong ε-cores for all real numbers ε not smaller than the relevant bound ε_{***}.

Zusammenfassung. Es wird eine äußere Approximation des (Prä-)Kerns vorgestellt. Diese liefert eine bessere untere Schranke ε_{***} derart, daß der Kern in jedem starken ε-Mark mit $\varepsilon \geq \varepsilon_{***}$ enthalten ist.

Key words: Cooperative game, (pre)kernel, strong ε-core, upper and lower bounds

Schlüsselwörter: Kooperative Spiele, (Prä-)Kern, starkes ε-Mark, untere und obere Schranken

1. Introduction

The solution part of cooperative game theory aims to solve the problem of how to allocate the overall profits of a joint enterprise to the participants. The various solution concepts prescribe somehow equitable divisions of the overall profits among the players in the game (which is the mathematical model of the joint enterprise). The core is considered to be one of the most important solution concepts for cooperative games. An allocation of the overall profits is said to belong to the core of a given game if no coalition (subset of players) has any incentive to object against the proposed allocation, that is the sum of all the allocations to the members of any coalition majorizes the worth of the coalition in the given game. Unfortunately, core-allocations need not to exist. However, if a cost of ε is imposed on the formation of any non-trivial coalition in the game, then the notion of core-allocations can be replaced by so-called strong ε-core-allocations which are guaranteed to exist for large real numbers ε.

Another widely used approach in cooperative game theory is the concept called prekernel. An allocation of the overall profits is said to belong to the prekernel of a given game if each pair of players is in equilibrium concerning their mutual threats arising from the proposed allocation. A threat of one player against another player with respect to an allocation is determined by the maximal gain that some coalition, containing one player but not the other, may achieve by withdrawing from the proposed allocation in favour of the worth of the coalition in the given game. It is known that prekernel allocations always exist. The kernel can be regarded as a generalized version of the prekernel.

The intersection of the prekernel and the core was thoroughly studied in the context of assignment games (cf. [9]), bankruptcy games (cf. [1]), as well as some network flow games (cf. [5]).

Maschler, Peleg and Shapley (1979) established that the part of the (pre)kernel inside any strong ε-core depends only on the latter's geometrical shape. That is, a strong ε-core element belongs to the (pre)kernel if and only if it possesses the so-called bisection property, i.e., it is always the midpoint of a certain bargaining range between each two players where each endpoint of the bargaining range is in the boundary of the strong ε-core. By choosing the real number ε large enough, the entire kernel is included in the strong ε-core and the above geometrical result can be regarded as a geometrical characterization of the entire kernel. Therefore it is worthwhile to provide lower bounds on ε to guarantee that the kernel is included in strong ε-cores for all ε not smaller than the relevant lower bound. Maschler et al. (1979) provided such a lower bound ε_* on ε, but they were still interested in better lower bounds. Chang and Kan (1992) provided a better lower bound ε_{**} than ε_*. The goal of this paper is to provide

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a better lower bound ε_{***} than ε_{**}. Our lower bound ε_{***} on ε arises from a new (pre)kernel catcher which can be interpreted as an improvement of the so-called reasonable set being the (pre)kernel catcher used by Maschler et al. (1979).

2. Definitions and notions

This paper deals with cooperative games (N, v) in coalitional form where the player set $N = \{1, 2, \ldots, n\}$, $n \geq 3$, and so-called characteristic function $v: 2^N \to \mathbb{R}$ satisfies $v(\emptyset) := 0$. Any non-empty subset of N is called a coalition. Usually, the cooperative game (N, v) is identified with its characteristic function v. For the sake of notation, we write $x(S)$ instead of $\sum_{i \in S} x_i$ for any $x \in \mathbb{R}^n$, any $S \in 2^N \backslash \{\emptyset\}$, whereas $x(\emptyset) := 0$.

Let's recall several known notions and solution concepts. The sets of all pre-imputations and imputations respectively in the n-person game v are given by

$$\mathscr{T}^*(v) := \{x \in \mathbb{R}^n \mid x(N) = v(N)\},$$

$$\mathscr{T}(v) := \{x \in \mathscr{T}^*(v) \mid x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

The core of the game v and its interior are defined by

$$\mathscr{C}(v) := \{x \in \mathscr{T}^*(v) \mid x(S) \geq v(S) \text{ for all } S \in 2^N \backslash \{\emptyset, N\}\},$$

$$\mathscr{C}^0(v) := \{x \in \mathscr{T}^*(v) \mid x(S) > v(S) \text{ for all } S \in 2^N \backslash \{\emptyset, N\}\}.$$

The excess of coalition $S \in 2^N$ with respect to the vector $x \in \mathbb{R}^n$ in the n-person game v is defined to be

$$e^v(S, x) := v(S) - x(S) \quad \text{whereas} \quad e^v(\emptyset, x) := 0.$$

For any $\varepsilon \in \mathbb{R}$, the strong ε-core of the game v is given by

$$\mathscr{C}_\varepsilon(v) := \{x \in \mathscr{T}^*(v) \mid e^v(S, x) \leq \varepsilon \text{ for all } S \in 2^N \backslash \{\emptyset, N\}\}.$$

Obviously, if $\varepsilon = 0$, then the notion of strong ε-core agrees with the core. The maximum surplus of player i over another player j with respect to the pre-imputation x in the game v is defined to be

$$s_{ij}^v(x) := \max\, [e^v(S, x) \mid S \in 2^N, i \in S, j \notin S].$$

We say player i outweighs player j with respect to the imputation $x \in \mathscr{T}(v)$ if both $x_j > v(\{j\})$ and $s_{ij}^v(x) > s_{ji}^v(x)$. The kernel $\mathscr{K}(v)$ of the game v is defined as the set of all imputations for which no player outweighs another player. Besides, the prekernel of the game v is given by

$$\mathscr{K}^*(v) := \{x \in \mathscr{T}^*(v) \mid s_{ij}^v(x) = s_{ji}^v(x) \text{ for all } i, j \in N, i \neq j\}.$$

Maschler et al. (1979) proved that the parts of the kernel and the prekernel inside the core always coincide. To be precise, $\mathscr{K}(v) \cap \mathscr{C}_\varepsilon(v) = \mathscr{K}^*(v) \cap \mathscr{C}_\varepsilon(v)$ for all $\varepsilon \leq 0$. For a detailed treatment of all these notions and solution concepts, we refer to Driessen (1988, Chap. II). For any player i in the n-person game v, the notions of his largest contribution, his almost largest contribution and his almost smallest contribution, respectively, are defined to be

$$r_i^v := \max\, [v(S \cup \{i\}) - v(S) \mid S \in 2^N, i \notin S],$$

$$\bar{r}_i^v := \max\, [v(S \cup \{i\}) - v(S) \mid S \in 2^N, i \notin S, S \neq N \backslash \{i\}],$$

$$m_i^v := \min\, [v(S \cup \{i\}) - v(S) \mid S \in 2^N, i \notin S, S \neq \emptyset].$$

For the sake of notation, write $r^v = (r_1^v, r_2^v, \ldots, r_n^v) \in \mathbb{R}^n$, $\bar{r}^v = (\bar{r}_1^v, \bar{r}_2^v, \ldots, \bar{r}_n^v) \in \mathbb{R}^n$, and $m^v = (m_1^v, m_2^v, \ldots, m_n^v) \in \mathbb{R}^n$. The corresponding sets of all (pre-)imputations that give no player more (or less) than the relevant largest (smallest) contribution are given by

$$\mathscr{R}(v) := \{x \in \mathscr{T}^*(v) \mid x_i \leq r_i^v \text{ for all } i \in N\},$$

$$\bar{\mathscr{R}}(v) := \{x \in \mathscr{T}^*(v) \mid x_i \leq \bar{r}_i^v \text{ for all } i \in N\},$$

$$\mathscr{L}(v) := \{x \in \mathscr{T}(v) \mid x_i \geq m_i^v \text{ for all } i \in N\}.$$

The so-called reasonable set $\mathscr{R}(v)$ was first introduced by Milnor (1952) and studied by Maschler et al. (1979) who proved that the reasonable set is a (pre)kernel catcher. That is, $\mathscr{K}(v) \subset \mathscr{R}(v)$ and $\mathscr{K}^*(v) \subset \mathscr{R}(v)$ for any game v. Moreover, it is clear that $\mathscr{C}(v) \subset \mathscr{R}(v)$ because $x_i = v(N) - x(N \backslash \{i\}) \leq v(N) - v(N \backslash \{i\}) \leq r_i^v$ for all $i \in N$, all $x \in \mathscr{C}(v)$. The set $\mathscr{L}(v)$ was studied by Kikuta (1976), Funaki (1986) and Chang and Kan (1992) who proved the inclusion $\mathscr{K}(v) \subset \mathscr{L}(v) \cup \mathscr{C}^0(v)$ for games v satisfying certain conditions.

The goal of this paper is twofold. The first purpose is to provide a (pre)kernel catcher that is included in the reasonable set. In Sect. 3 it is established that the set $\bar{\mathscr{R}}(v) \cup \mathscr{C}^0(v)$ is a (pre)kernel catcher. Evidently, $\bar{\mathscr{R}}(v) \cup \mathscr{C}^0(v) \subset \mathscr{R}(v)$ because of $\mathscr{C}^0(v) \subset \mathscr{C}(v) \subset \mathscr{R}(v)$ as well as $\bar{\mathscr{R}}(v) \subset \mathscr{R}(v)$ which is due to the fact that $\bar{r}_i^v \leq r_i^v$ for all $i \in N$. The second purpose is to provide a lower bound ε_{***} on ε such that the kernel is included in strong ε-cores for all $\varepsilon \geq \varepsilon_{***}$. In Sect. 4 we present such a lower bound for a subclass of games by studying the chain of inclusions $\mathscr{K}(v) \subset (\mathscr{L}(v) \cap \bar{\mathscr{R}}(v)) \cup \mathscr{C}^0(v) \subset \mathscr{C}_\varepsilon(v)$.

3. A new (pre)kernel catcher

The next theorem states that any (pre-)imputation of the (pre)kernel belongs to the interior of the core or the set $\bar{\mathscr{R}}(v)$.

Theorem 3.1. *$\mathscr{K}(v) \subset \bar{\mathscr{R}}(v) \cup \mathscr{C}^0(v)$ and $\mathscr{K}^*(v) \subset \bar{\mathscr{R}}(v) \cup \mathscr{C}^0(v)$ for any game v.*

Proof. Let $x \in \mathscr{K}(v) \cup \mathscr{K}^*(v)$. If $x \in \mathscr{C}^0(v)$, then the relevant inclusion trivially holds. So, let $x \notin \mathscr{C}^0(v)$. In order to prove that $x \in \bar{\mathscr{R}}(v)$, suppose on the contrary that $x \notin \bar{\mathscr{R}}(v)$. Then $x_i > \bar{r}_i^v$ for some $i \in N$ by definition of the set $\bar{\mathscr{R}}(v)$. Now the definition of the real number $\bar{r}_i^v$ yields that

$$x_i > v(S \cup \{i\}) - v(S) \text{ for all } S \in 2^N \text{ with } i \notin S,\ S \neq N \backslash \{i\}$$

or equivalently,

$$e^v(S \cup \{i\}, x) < e^v(S, x) \quad \text{for all } S \in 2^N \text{ with } i \notin S,\ S \neq N \backslash \{i\}. \tag{3.1}$$

In particular, $x_i > v(\{i\})$ by choosing $S = \emptyset$ and together with $x \in \mathscr{K}(v) \cup \mathscr{K}^*(v)$, this implies that $s_{ji}^v(x) \leq s_{ij}^v(x)$ for all $j \in N \backslash \{i\}$. Choose player $k \in N \backslash \{i\}$ such that $s_{ki}^v(x) = \max\, [s_{ji}^v(x) \mid j \in N \backslash \{i\}]$ and next, choose any coalition $T \in 2^N$ satisfying $i \in T$, $k \notin T$, $e^v(T, x) = s_{ik}^v(x)$. In case $T \neq \{i\}$, it follows from (3.1) that

$$s_{ji}^v(x) \geq e^v(T \backslash \{i\}, x) > e^v(T, x) = s_{ik}^v(x) \geq s_{ki}^v(x) \geq s_{ji}^v(x) \quad \text{for all } j \in T \backslash \{i\}.$$

That is, we arrive at a contradiction whenever $T \neq \{i\}$. It remains to consider the case $T = \{i\}$. Now we obtain that

$$s_{ji}^v(x) \leq s_{ki}^v(x) \leq s_{ik}^v(x) = e^v(\{i\}, x) = v(\{i\}) - x_i < 0 \quad \text{for all } j \in N \backslash \{i\}.$$

From $s_{ji}^v(x) < 0$ for all $j \in N\backslash\{i\}$, we deduce that

$$v(S) - x(S) < 0 \quad \text{for all } j \in N \backslash \{i\}, \text{ all } S \in 2^N \text{ with } j \in S,\ i \notin S.$$

This yields that $v(S) - x(S) < 0$ for all $S \in 2^N$ satisfying $i \notin S$, $S \neq \emptyset$. In addition, this result together with (3.1) implies that

$$v(S) - x(S) = e^v(S, x) < e^v(S\backslash\{i\}, x) < 0 \quad \text{for all } S \in 2^N,\ i \in S,\ S \neq N,\ S \neq \{i\}.$$

Remark that also $v(\{i\}) - x_i < 0$. We conclude that $v(S) < x(S)$ for all $S \in 2^N \backslash \{\emptyset, N\}$. This, however, contradicts the assumption $x \notin \mathscr{C}^0(v)$. In any case we arrive at a contradiction which completes the proof of the theorem. □

The (pre)kernel catcher $\bar{\mathscr{R}}(v) \cup \mathscr{C}^0(v)$ mentioned in the above theorem should be considered as an improvement of the (pre)kernel catcher $\mathscr{R}(v)$ presented by Maschler et al. (1979). In order to illustrate the relationship $\bar{\mathscr{R}}(v) \cup \mathscr{C}^0(v) \subset \mathscr{R}(v)$ between both (pre)kernel catchers, let's treat the example involving the 3-person game v defined by $v(N) = 1$ and $v(S) = 0$ for all $S \in 2^N \backslash \{N\}$. Then $r_i^v = 1$, $\bar{r}_i^v = 0$ for all $i \in N$, so $\bar{\mathscr{R}}(v) = \emptyset$ and

$$\mathscr{R}(v) = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1,\ x_1 \leq 1,\ x_2 \leq 1,\ x_3 \leq 1\} = \operatorname{conv}\{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\},$$

$$\mathscr{C}(v) = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1,\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\} = \operatorname{conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

Notice that the set $\bar{\mathscr{R}}(v)$ may be empty, whereas the reasonable set $\mathscr{R}(v)$ is always nonempty (without proof).

4. Kernel catchers in terms of strong ε-cores

This section is devoted to conditions on the real number ε that guarantee that the strong ε-core contains the kernel. The approach to include the kernel in some strong ε-core is strongly based on Chang and Kan's result that, under certain circumstances, any imputation of the kernel belongs to the interior of the core or the set $\mathscr{L}(v)$.

Theorem 4.1. *(cf. [2], page 90) Let v be a zero-normalized game (i.e., $v(\{i\}) = 0$ for all $i \in N$) satisfying $v(S) \leq v(N)$ for all $S \in 2^N$. Then*

$$\mathscr{K}(v) \subset \mathscr{L}(v) \cup \mathscr{C}^0(v).$$

In the final stage of their research on the kernel of such games, Chang and Kan (1992) provided some critical number $\varepsilon_{**}(v)$ in such a way that the inclusions

$$\mathscr{K}(v) \subset (\mathscr{L}(v) \cup \mathscr{C}^0(v)) \cap \mathscr{R}(v) \subset \mathscr{C}_\varepsilon(v) \quad \text{hold for all } \varepsilon \geq \varepsilon_{**}(v).$$

See also Remark 4.4. Our final goal is to provide a better lower bound on ε for such games that ensures that the strong ε-core contains the kernel. The improvement of the lower bound on ε will be obtained from the replacement of the reasonable set $\mathscr{R}(v)$ by the new kernel catcher $\bar{\mathscr{R}}(v) \cup \mathscr{C}^0(v)$. That is, we examine the inclusion

$$(\mathscr{L}(v) \cup \mathscr{C}^0(v)) \cap (\bar{\mathscr{R}}(v) \cup \mathscr{C}^0(v)) \subset \mathscr{C}_\varepsilon(v).$$

In this context, we introduce for an arbitrarily zero-normalized game v the critical number $\varepsilon_{***}(v)$ as follows. Write α_+ instead of $\max[0, \alpha]$ for any $\alpha \in \mathbb{R}$.

$$\varepsilon_{***}(v) = \begin{cases} \max\limits_{S \neq \emptyset, N} \min[v(S), v(S) - m^v(S), v(S) - v(N) + \bar{r}^v(N\backslash S)] & \text{if } \mathscr{C}(v) = \emptyset, \\ \max\limits_{S \neq \emptyset, N} \min[[\min[v(S), v(S) - m^v(S)]]_+, [v(S) - v(N) + \bar{r}^v(N\backslash S)]_+] & \text{if } \mathscr{C}(v) \neq \emptyset,\ \mathscr{L}(v) \neq \emptyset \text{ and } \bar{\mathscr{R}}(v) \neq \emptyset, \\ 0 & \text{if } \mathscr{C}(v) \neq \emptyset \text{ and } (\mathscr{L}(v) = \emptyset \text{ or } \bar{\mathscr{R}}(v) = \emptyset). \end{cases}$$

Theorem 4.2. *Let v be a zero-normalized game. Then*

$$(\mathscr{L}(v) \cap \bar{\mathscr{R}}(v)) \cup \mathscr{C}(v) \subset \mathscr{C}_\varepsilon(v) \quad \textit{for all } \varepsilon \geq \varepsilon_{***}(v).$$

Proof. Let $x \in (\mathscr{L}(v) \cap \bar{\mathscr{R}}(v)) \cup \mathscr{C}(v)$ and $S \in 2^N \backslash \{\emptyset, N\}$. If $x \in \mathscr{C}(v)$, then $x(S) \geq v(S)$. If $x \in \mathscr{L}(v)$, then $x_i \geq m_i^v$ for all $i \in N$, so $x(S) \geq m^v(S)$ and moreover, $x(S) \geq 0$ since $x_i \geq v(\{i\}) = 0$ for all $i \in N$. If $x \in \bar{\mathscr{R}}(v)$, then $x_i \leq \bar{r}_i^v$ for all $i \in N$, so $x(S) = v(N) - x(N\backslash S) \geq v(N) - \bar{r}^v(N\backslash S)$. In order to show that $v(S) - x(S) \leq \varepsilon$ for all $\varepsilon \geq \varepsilon_{***}(v)$, we distinguish three cases involving the sets $\mathscr{C}(v)$, $\mathscr{L}(v)$ and $\bar{\mathscr{R}}(v)$.

Case one. Suppose that $\mathscr{C}(v) = \emptyset$. Then $x \in \mathscr{L}(v) \cap \bar{\mathscr{R}}(v)$ implies $x(S) \geq 0$, $x(S) \geq m^v(S)$ as well as $x(S) \geq v(N) - \bar{r}^v(N\backslash S)$. Hence $x(S) \geq \max[0, m^v(S), v(N) - \bar{r}^v(N\backslash S)]$ and consequently,

$$v(S) - x(S) \leq \min[v(S), v(S) - m^v(S), v(S) - v(N) + \bar{r}^v(N\backslash S)] \leq \varepsilon_{***}(v).$$

Case two. Suppose that $\mathscr{C}(v) \neq \emptyset$, $\mathscr{L}(v) \neq \emptyset$ and $\bar{\mathscr{R}}(v) \neq \emptyset$. Then $x \in \mathscr{L}(v) \cup \mathscr{C}(v)$ implies

$$x(S) \geq \min[v(S), \max[0, m^v(S)]].$$

Furthermore, $x(S) \geq \min[v(S), v(N) - \bar{r}^v(N\backslash S)]$ because of $x \in \bar{\mathscr{R}}(v) \cup \mathscr{C}(v)$. Now it follows that

$$x(S) \geq \max[\min[v(S), \max[0, m^v(S)]], \min[v(S), v(N) - \bar{r}^v(N\backslash S)]]$$

and consequently,

$$v(S) - x(S) \leq \min[[\min[v(S), v(S) - m^v(S)]]_+, [v(S) - v(N) + \bar{r}^v(N\backslash S)]_+] \leq \varepsilon_{***}(v).$$

Case three. Suppose that $\mathscr{C}(v) \neq \emptyset$ and ($\mathscr{L}(v) = \emptyset$ or $\bar{\mathscr{R}}(v) = \emptyset$). Then $x \in \mathscr{C}(v)$ implies $v(S) - x(S) \leq 0 \leq \varepsilon_{***}(v)$. In any case, $v(S) - x(S) \leq \varepsilon_{***}(v)$ for all $S \in 2^N \backslash \{\emptyset, N\}$ and therefore, $x \in \mathscr{C}_\varepsilon(v)$ for all $\varepsilon \geq \varepsilon_{***}(v)$. □

Corollary 4.3. *Let v be a zero-normalized game satisfying $v(S) \leq v(N)$ for all $S \in 2^N$. Then*

$$\mathscr{K}(v) \subset \mathscr{C}_\varepsilon(v) \quad \text{for all } \varepsilon \geq \varepsilon_{***}(v).$$

Proof. Combine Chang and Kan's result with Theorem 3.1 and apply Theorem 4.2. That is, we have

$$\begin{aligned}\mathscr{K}(v) &\subset (\mathscr{L}(v) \cup \mathscr{C}^0(v)) \cap (\bar{\mathscr{R}}(v) \cup \mathscr{C}^0(v)) \\ &= (\mathscr{L}(v) \cap \bar{\mathscr{R}}(v)) \cup \mathscr{C}^0(v) \subset (\mathscr{L}(v) \cap \bar{\mathscr{R}}(v)) \cup \mathscr{C}(v) \subset \mathscr{C}_\varepsilon(v) \\ &\qquad \text{for all } \varepsilon \geq \varepsilon_{***}(v). \quad \square\end{aligned}$$

Remark 4.4. For zero-normalized games v satisfying $v(S) \leq v(N)$ for all $S \in 2^N$, Chang and Kan (1992) established in a similar way that $\mathscr{K}(v) \subset (\mathscr{L}(v) \cup \mathscr{C}^0(v)) \cap \mathscr{R}(v) \subset \mathscr{C}_\varepsilon(v)$ for all $\varepsilon \geq \varepsilon_{**}(v)$ where the critical number $\varepsilon_{**}(v)$ is defined by

$$\varepsilon_{**}(v) = \begin{cases} \max\limits_{S \neq \emptyset, N} \min\,[v(S), v(S) - m^v(S), v(S) - v(N) + r^v(N \setminus S)] & \text{if } \mathscr{C}(v) = \emptyset, \\ \max\limits_{S \neq \emptyset, N} \min\,[v(S), [v(S) - m^v(S)]_+, v(S) - v(N) + r^v(N \setminus S)] & \text{if } \mathscr{C}(v) \neq \emptyset \text{ and } \mathscr{L}(v) \neq \emptyset, \\ \max\limits_{S \neq \emptyset, N} \min\,[0, v(S), v(S) - v(N) + r^v(N \setminus S)] & \text{if } \mathscr{L}(v) = \emptyset. \end{cases}$$

Let's clarify, without going into technical details, why the new lower bound $\varepsilon_{***}(v)$ is better than the above lower bound $\varepsilon_{**}(v)$. The inequality $\varepsilon_{***}(v) \leq \varepsilon_{**}(v)$ is mainly due to the facts that $\bar{r}_i^v \leq r_i^v$ for all $i \in N$ as well as $v(S) - v(N) + r^v(N \setminus S) \geq 0$ for all $S \in 2^N \setminus \{\emptyset, N\}$. In fact, if $\mathscr{L}(v) = \emptyset$, then $\varepsilon_{**}(v)$ is simply equal to zero (e.g., choose $S = \{i\}$ where $v(\{i\}) = 0$). In addition, be aware that $\mathscr{C}(v) \neq \emptyset$ whenever $\mathscr{L}(v) = \emptyset$ or $\bar{\mathscr{R}}(v) = \emptyset$. Involving the proofs of some of these statements, see the appendix.

A. Appendix

Lemma A.1. *Let $S \in 2^N \setminus \{\emptyset, N\}$. Then we have*

(i) $\bar{r}^v(S) \geq v(S)$ *and* $r^v(N \setminus S) \geq v(N) - v(S) \geq m^v(N \setminus S)$.
(ii) $\bar{r}^v(N) \geq v(N)$ *iff* $\bar{\mathscr{R}}(v) \neq \emptyset$.
$r^v(N) \geq v(N)$ *iff* $\mathscr{R}(v) \neq \emptyset$. *In fact,* $\mathscr{R}(v) \neq \emptyset$ *always.*
(iii) $\bar{\mathscr{R}}(v) = \emptyset$ *implies* $\mathscr{C}^0(v) \neq \emptyset$ *and* $m^v(N) > v(N)$ *implies* $\mathscr{C}^0(v) \neq \emptyset$.

Proof. (i) Let $S \in 2^N \setminus \{\emptyset, N\}$. Write $S = \{i_1, i_2, \ldots, i_s\}$ where $i_1 < i_2 < \ldots < i_s$ and $N \setminus S = \{j_1, j_2, \ldots, j_{n-s}\}$ where $j_1 < j_2 < \ldots < j_{n-s}$. Then $\bar{r}_{i_1}^v \geq v(\{i_1\})$ and $\bar{r}_{i_k}^v \geq v(\{i_1, i_2, \ldots, i_k\}) - v(\{i_1, i_2, \ldots, i_{k-1}\})$ for all $2 \leq k \leq s$. From this it follows that

$$\bar{r}^v(S) = \sum_{k=1}^{s} \bar{r}_{i_k}^v \geq v(\{i_1, i_2, \ldots, i_s\}) = v(S).$$

We have that $r_{j_1}^v \geq v(N) - v(N \setminus \{j_1\}) \geq m_{j_1}^v$ as well as $r_{j_k}^v \geq v(N \setminus \{j_1, j_2, \ldots, j_{k-1}\}) - v(N \setminus \{j_1, j_2, \ldots, j_k\}) \geq m_{j_k}^v$ for all $2 \leq k \leq n - s$. From this it follows that

$$r^v(N \setminus S) = \sum_{k=1}^{n-s} r_{j_k}^v \geq v(N) - v(S) \geq \sum_{k=1}^{n-s} m_{j_k}^v = m^v(N \setminus S).$$

(ii) Obviously, if $\bar{\mathscr{R}}(v) \neq \emptyset$, choose any $x \in \bar{\mathscr{R}}(v)$, so $x_i \leq \bar{r}_i^v$ for all $i \in N$ and hence, $v(N) = x(N) \leq \bar{r}^v(N)$. Conversely, if $\bar{r}^v(N) \geq v(N)$, then it is evident that $\bar{r}^v + [v(N) - \bar{r}^v(N)]\, e^i \in \bar{\mathscr{R}}(v)$ for all $i \in N$. Here $e^i \in \mathbb{R}^n$ denotes the i-th unit vector in $\mathbb{R}^n$ defined by $e_i^i := 1$ and $e_j^i := 0$ for all $j \in N \setminus \{i\}$. In a similar way, one shows that $\mathscr{R}(v) \neq \emptyset$ iff $r^v(N) \geq v(N)$. It turns out, however, that the inequality $r^v(N) \geq v(N)$ always holds because the proof of part (i) concerning the vector r^v also applies with $S = \emptyset$.

(iii) Suppose that $\bar{\mathscr{R}}(v) = \emptyset$ or equivalently, $\bar{r}^v(N) < v(N)$. Define the vector $y \in \mathbb{R}^n$ by $y_i := \bar{r}_i^v + n^{-1}[v(N) - \bar{r}^v(N)]$ for all $i \in N$. Then we have $y(N) = v(N)$ as well as $y(S) > \bar{r}^v(S) \geq v(S)$ for all $S \in 2^N \setminus \{\emptyset, N\}$ where the latter inequality follows from part (i). Hence, $y \in \mathscr{C}^0(v)$ and thus, $\mathscr{C}^0(v) \neq \emptyset$ whenever $\bar{\mathscr{R}}(v) = \emptyset$.

Suppose now that $m^v(N) > v(N)$. Define the vector $z \in \mathbb{R}^n$ by $z_i := m_i^v + n^{-1}[v(N) - m^v(N)]$ for all $i \in N$. Then we have $z(N) = v(N)$ as well as $z(S) > m^v(S) + v(N) - m^v(N) = v(N) - m^v(N \setminus S) \geq v(S)$ for all $S \in 2^N \setminus \{\emptyset, N\}$ where the latter inequality follows from part (i). Hence, $z \in \mathscr{C}^0(v)$ and thus, $\mathscr{C}^0(v) \neq \emptyset$ whenever $m^v(N) > v(N)$. $\square$

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