

Balance laws and centro velocity in dissipative systems

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Starting with a density that is conserved for a dynamical system when dissipation is ignored, a local conservation law is derived for which the total flux (integrated over the spatial domain) is unique. When dissipation is incorporated, the conservation law becomes a balance law. The contribution due to dissipation in this balance law is split in a unique way in a part that is proportional to the density and in a divergence expression that adds to the original (conservative) flux density; the total additional flux is uniquely defined. It is shown that these total fluxes appear in the expression for the centro velocity, i.e., in the velocity of the center of gravity of the density, which shows that this velocity can be defined in a unique way (in contrast to a local velocity). Applications to the Korteweg–de Vries–Burgers equations and to the incompressible Navier–Stokes equations are given.

I. INTRODUCTION

This paper is concerned with some basic observations about balance laws for continuous systems and some consequences. Although the methods and results are rather straightforward, we are not aware of any direct treatment of these matters in the literature.

The starting point is a distinguished density E of a certain continuous system, an expression in the state variable and its derivatives. The system is assumed to be “dissipative” in the sense that the integrated quantity $\langle E \rangle$ (where $\langle \rangle$ denotes integration over the fixed spatial domain) will not be conserved during the evolution. However, it is assumed that the dissipation can be recognized explicitly, and that when it is ignored, $\langle E \rangle$ is a constant of the motion of the resulting “conservative” system. Although this is not essential for the following, it is helpful to think of E as an energylike quantity.

The aim of this paper is to investigate in which way the “dissipation” can be understood in its effect on E . In Sec. II it is shown that when dissipation is ignored and $\langle E \rangle$ is conserved, E satisfies a local conservation law for which the total flux $\langle F \rangle$ can be uniquely defined (the flux itself is unique only in the class of curl-free functions). In the presence of dissipation, E satisfies a local balance law. In Sec. III it is shown that the contribution of dissipation can be split in a unique way in a part that is proportional to E , with the dissipation rate of $\langle E \rangle$ as factor of proportionality, and in a part that changes the original flux density F of E with a certain amount Θ , for which $\langle \Theta \rangle$ is unique. In Sec. IV the resulting formulation of the balance law is interpreted as a conservation law for a modified energy density that depends explicitly on time. Moreover, it is shown that the additional flux due to dissipation will also appear in the expression for the centro velocity of E , i.e., the velocity of its center of gravity. Then, as in the nondissipative case, this centro velocity equals the energy-flux velocity, but now the flux consists of the sum of the flux of the nondissipative system and the flux due to dissipation. Since in general the total flux $\langle \Theta \rangle$ due to dissipation does not vanish, the resulting expression

for the centro velocity is different in the conservative and in the dissipative case. In Sec. V some examples from fluid dynamics are considered; 1-D wave equations like the Korteweg–de Vries–Burgers equation, and the inviscid Navier–Stokes equations. The final section contains some conclusions and remarks.

It may be stressed that the results in this paper are quite general and applicable to nonlinear equations. Most results in the literature about propagation velocity are for linear equations, and then often deal with harmonic waves and relate the velocity to the group or phase velocity. For the energy-flux velocity see Refs. 1–8 for systems without dissipation, and Refs. 9 and 10 for systems with dissipation. For the centro velocity in systems without dissipation see Refs. 11 and 12, the latter in particular also for nonlinear equations, and for dissipative systems see Refs. 13 and 14.

II. UNIQUENESS OF THE TOTAL FLUX IN LOCAL CONSERVATION LAWS

Let the state of a system be described by some vector function $\mathbf{u}(\mathbf{x}, t)$, where \mathbf{x} belongs to a spatial domain $\Omega \subset \mathbb{R}^n$. The domain Ω is assumed to be given here, possibly the whole space. The boundary of Ω is denoted by $\partial\Omega$, and the normal to the boundary by \mathbf{n} . It is assumed, without loss of generality that the evolution is described by an evolution equation (generally a partial differential equation) that is of first order in time. All densities to be considered below are then expressions in \mathbf{u} and its spatial derivatives.

To motivate the following, assume that the evolution equation for \mathbf{u} can be thought of to consist of a conservative part \mathbf{K}_1 and a part that may account for dissipation (or production) \mathbf{K}_2 . To recognize the effect of dissipation in the following, a parameter ν is introduced and the equation for \mathbf{u} is written as

$$\partial_t \mathbf{u} = \mathbf{K}_1(\mathbf{u}) + \nu \mathbf{K}_2(\mathbf{u}). \quad (2.1a)$$

Unfortunately, both notions of “conservative” and “dissipa-

tive" are difficult to define in general. What is meant here is that Eq. (2.1a) with $v = 0$,

$$\partial_t \mathbf{u} = \mathbf{K}_1(\mathbf{u}), \quad (2.1b)$$

has some conserved density E , while E is generally not conserved for (2.1a) when $v \neq 0$. In this section we consider the conservative case before dealing with the dissipative equation in the next section.

To be more specific, let E be this density and assume that it is positive definite in the following sense:

$$\langle E(\mathbf{u}) \rangle \equiv \int_{\Omega} E(\mathbf{u}) \geq 0, \quad \text{if } \mathbf{u} \neq 0.$$

Here, E will be referred to as the energy density and $\langle E \rangle$ as the total energy. The statement that E is a conserved density of (2.1b) is defined to mean that the total energy is conserved:

$$\partial_t \langle E(\mathbf{u}) \rangle = 0. \quad (2.2)$$

The first result states that then E satisfies a local conservation law.

Proposition 2.1: For the density E satisfying (2.2), there is a local conservation law of the form

$$\partial_t E(\mathbf{u}) + \operatorname{div} \mathbf{F}(\mathbf{u}) = 0, \quad (2.3)$$

for some flux density \mathbf{F} that can be chosen to satisfy the boundary condition

$$\mathbf{F}(\mathbf{u}) \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega. \quad (2.4)$$

With this boundary condition, the flux is unique, possibly up to the addition of a function \mathbf{f} satisfying

$$\operatorname{div} \mathbf{f} = 0, \quad \text{in } \Omega \text{ and } \mathbf{f} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega. \quad (2.5)$$

For all \mathbf{F} satisfying the boundary condition (2.4) the total flux is uniquely defined and is given by

$$\langle \mathbf{F} \rangle = \langle \mathbf{x} \cdot \partial_t E \rangle, \quad (2.6)$$

where $\partial_t E$ is calculated for solutions of (2.1b).

Before proving this result, some remarks are in order.

Remark 2.1: Concerning uniqueness it can be said that \mathbf{F} satisfying Eqs. (2.3) and (2.4) is unique up to elements from the kernel of the divergence operation satisfying the homogeneous boundary conditions. Since any function can be written as the sum of a solenoidal function and some gradient function, \mathbf{F} could be made unique by requiring $\operatorname{curl} \mathbf{F} = 0$. The actual construction of \mathbf{F} will be performed in this way in the proof below. Note that if $n = 1$, the only function that satisfies (2.5) is the zero function, so that \mathbf{F} is unique then. If $n = 2$, for any scalar function ψ that is constant on $\partial\Omega$, the function $\mathbf{f} = (-\psi_y, \psi_x)$ satisfies (2.5). In the same way, if $n = 3$, any function $\mathbf{f} = \operatorname{curl} \mathbf{a}$, with $\operatorname{curl} \mathbf{a} \cdot \mathbf{n} = 0$ on $\partial\Omega$, satisfies (2.5).

Remark 2.2: If one just starts with a local conservation law like (2.3), say

$$\partial_t E(\mathbf{u}) + \operatorname{div} \mathbf{F}^*(\mathbf{u}) = 0, \quad (2.7)$$

for some flux density \mathbf{F}^* , then upon integrating over the domain Ω and using Gauss' theorem, one obtains

$$\partial_t \langle E(\mathbf{u}) \rangle + \int_{\partial\Omega} \mathbf{F}^*(\mathbf{u}) \cdot \mathbf{n} = 0. \quad (2.8)$$

Conservation of total energy, i.e., (2.2), is recovered if

$\int_{\partial\Omega} \mathbf{F}^*(\mathbf{u}) \cdot \mathbf{n} = 0$. Note, however, that unless the flux density \mathbf{F}^* satisfies the pointwise condition (2.4), the total flux $\langle \mathbf{F}^* \rangle$ will differ in general from that given by (2.6) [see formula (2.10) below].

Remark 2.3: In order to specify the phrase in (2.6), suppose that E is differentiable and let $D_u E(\mathbf{u})$ denote its Frechet derivative. That is, for any function \mathbf{v} , and ϵ real,

$$D_u E(\mathbf{u}) \cdot \mathbf{v} = \left. \frac{d}{d\epsilon} E(\mathbf{u} + \epsilon \mathbf{v}) \right|_{\epsilon=0},$$

or, equivalently, a first-order Taylor expansion reads

$$E(\mathbf{u} + \epsilon \mathbf{v}) = E(\mathbf{u}) + \epsilon D_u E(\mathbf{u}) \cdot \mathbf{v} + O(\epsilon^2).$$

Then $\partial_t E(\mathbf{u})$ in (2.6) can be written like $\partial_t E(\mathbf{u}) = D_u E(\mathbf{u}) \cdot \partial_t \mathbf{u}$, and inserting the evolution equation (2.1b), we obtain

$$\partial_t E(\mathbf{u}) = D_u E(\mathbf{u}) \cdot \mathbf{K}_1(\mathbf{u}). \quad (2.9)$$

This is the expression that is meant in (2.6).

Proof of Proposition 2.1: The proof of this proposition uses the following standard result from potential theory.

Lemma: For a given function g on Ω , consider the following Poisson equation for a scalar function θ satisfying homogeneous Neumann boundary conditions:

$$-\nabla^2 \theta = g(x), \quad \text{in } \Omega \text{ and } \nabla \theta \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

This problem has a solution θ if and only if $\langle g \rangle = \int_{\Omega} g \, dx = 0$, and this solution θ is uniquely determined up to an additive constant.

Using this lemma, the proof of the proposition is immediate: Take for the function g the expression $\partial_t E(\mathbf{u})$, i.e., $D_u E(\mathbf{u}) \cdot \mathbf{K}_1(\mathbf{u})$ according to (2.9). Because of the requirement (2.2), the solvability condition is satisfied and a solution θ is obtained. Then define \mathbf{F} to be $\mathbf{F} = -\nabla \theta$. Since θ is defined uniquely up to a constant, \mathbf{F} is uniquely defined from θ and satisfies $\operatorname{curl} \mathbf{F} = 0$. Moreover, the boundary conditions for θ imply that $\mathbf{F} \cdot \mathbf{n} = 0$. The uniqueness up to functions \mathbf{f} that satisfy (2.5) is clear. The expression for the total flux follows easily from the following formula that is obtained by applying Gauss' theorem:

$$\int_{\Omega} \mathbf{x} \operatorname{div} \mathbf{F} = - \int_{\Omega} \mathbf{F} + \int_{\partial\Omega} \mathbf{x} (\mathbf{F} \cdot \mathbf{n}). \quad (2.10)$$

This completes the proof of the proposition.

III. UNIQUE DECOMPOSITION OF THE BALANCE LAW

In this section we will take a density E and flux \mathbf{F} that satisfy (2.3) if u evolves according to (2.1b). Then if u evolves according to (2.1a), the density E will satisfy an expression like

$$\partial_t E(\mathbf{u}) + \operatorname{div} \mathbf{F}(\mathbf{u}) = -\nu S(\mathbf{u}), \quad (3.1)$$

where S is some scalar density due to the addition of the term $\nu \mathbf{K}_2(\mathbf{u})$ to the equation. (It can be interpreted as a "loss" or "production" term, but no sign restrictions will be imposed here.) Using the Frechet derivative as in Remark 2.4, $S(\mathbf{u})$ reads

$$S(\mathbf{u}) = -D_u E(\mathbf{u}) \cdot \mathbf{K}_2(\mathbf{u}).$$

We will refer to (3.1) as a local balance law for E . The aim is now to rewrite (3.1) in such a way that the contribution

from S that is responsible for a direct decrease (or increase) in E is separated from the contribution that adds to the flux F . To that end, we start with the *global* expression corresponding to (3.1). Integrating (3.1) over Ω one obtains

$$\partial_t \langle E(\mathbf{u}) \rangle = -\nu \langle S(\mathbf{u}) \rangle. \quad (3.2)$$

Introducing the instantaneous dissipation rate α as the time-dependent functional

$$\alpha(\mathbf{u}) = \langle S(\mathbf{u}) \rangle / \langle E(\mathbf{u}) \rangle, \quad (3.3)$$

(3.2) can be written like

$$\partial_t \langle E(\mathbf{u}) \rangle = -\nu \alpha(\mathbf{u}) \langle E(\mathbf{u}) \rangle. \quad (3.4)$$

The definition of α implies that

$$\langle S - \alpha E \rangle = 0, \quad (3.5)$$

which makes it possible to split S in the desired way.

Proposition 3.1: The density S can be decomposed like

$$S = \alpha E + \text{div } \Theta, \quad (3.6)$$

where $\Theta(\mathbf{u})$ is a loss flux density that can be chosen to satisfy the boundary condition

$$\Theta \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega. \quad (3.7)$$

Just as in proposition 2.1, Θ is unique up to the addition of a function \mathbf{f} that satisfies (2.5); the total loss flux is unique and given by

$$\langle \Theta \rangle = \langle \mathbf{x} \cdot (S - \alpha E) \rangle. \quad (3.8)$$

Proof: This proposition follows in the same way as proposition 2.1:

$\Theta = -\nabla\theta$, where θ satisfies $-\nabla^2\theta = S - \alpha E$, in Ω and $\nabla\theta \cdot \mathbf{n} = 0$ on $\partial\Omega$.

Substitution of (3.6) into (3.1) leads to the following proposition.

Proposition 3.2: The balance law (3.1) can be formulated as,

$$\partial_t E(\mathbf{u}) + \text{div}[\mathbf{F}(\mathbf{u}) + \nu\Theta(\mathbf{u})] = -\nu\alpha(\mathbf{u})E(\mathbf{u}), \quad (3.9)$$

with

$$[\mathbf{F}(\mathbf{u}) + \nu\Theta(\mathbf{u})] \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \quad (3.10)$$

where \mathbf{F} , Θ , and α are defined as before, and where the total flux $\langle \mathbf{F}(\mathbf{u}) + \nu\Theta(\mathbf{u}) \rangle$ is uniquely defined.

This result shows in an explicit way that the "loss density" S is split in a part that takes account for the change in $\langle E \rangle$ according to (3.4) and a part that is added to the flux \mathbf{F} of the conservative system.

IV. INTERPRETATION AND CONSEQUENCE FOR THE CENTRO VELOCITY

The dissipation rate α as it appears in the balance law (3.9) acts like a uniform damping factor for E . This can be seen quite clearly in the following way. Introduce the primitive of α that will be a functional β that depends on the complete evolution of \mathbf{u} from the initial time up to time t :

$$\beta(\mathbf{u}, t) := \int_0^t \alpha(\mathbf{u}(s)) ds. \quad (4.1)$$

Multiplying (3.9) by $e^{\nu\beta}$ the result can be written like a local

conservation law for a density E^* that depends explicitly on time:

$$\partial_t E^*(\mathbf{u}) + \text{div } \mathbf{G}^*(\mathbf{u}) = 0, \quad (4.2)$$

where

$$E^* = e^{\nu\beta} E, \quad \text{and } \mathbf{G}^* = e^{\nu\beta} [\mathbf{F} + \nu\Theta]. \quad (4.3)$$

[To see this, note that $\alpha(\mathbf{u})$ is a *functional* of \mathbf{u} , so the term $e^{\nu\beta}$ is at each t just a number, not a function of x , and can be interchanged with the divergence operation.] Since \mathbf{G}^* satisfies the boundary condition $\mathbf{G}^* \cdot \mathbf{n} = 0$ on $\partial\Omega$, according to (3.10), it follows that E^* is really a conserved density:

$$\partial_t \langle E^*(\mathbf{u}) \rangle = 0. \quad (4.4)$$

An important consequence of the decomposition (3.9) concerns the centro velocity of the density E . This is defined as the velocity \mathbf{V} of the center of gravity of E :

$$\mathbf{V} = \partial_t \mathbf{X}, \quad \text{where } \langle (\mathbf{x} - \mathbf{X}) E \rangle = 0. \quad (4.5)$$

To calculate \mathbf{V} , start with

$$0 = \partial_t \langle (\mathbf{x} - \mathbf{X}) E \rangle = \langle (\mathbf{x} - \mathbf{X}) \partial_t E \rangle - \mathbf{V} \langle E \rangle.$$

Then substitute the expression (3.9) for $\partial_t E$. Write $\mathbf{G} = [\mathbf{F} + \nu\Theta]$ and apply (3.10) to see that $\mathbf{G} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Consequently, the boundary integral in the following partial integration vanishes [cf. (2.10)]:

$$\int_{\Omega} (\mathbf{x} - \mathbf{X}) \text{div } \mathbf{G} = - \int_{\Omega} \mathbf{G} + \int_{\partial\Omega} (\mathbf{x} - \mathbf{X}) \mathbf{G} \cdot \mathbf{n}. \quad (4.6)$$

Using all these properties we may deduce that

$$\begin{aligned} \langle (\mathbf{x} - \mathbf{X}) \partial_t E \rangle &= \langle (\mathbf{x} - \mathbf{X}) \{ -\text{div}[\mathbf{F} + \nu\Theta] - \nu\alpha E \} \rangle \\ &= \langle \mathbf{F} + \nu\Theta \rangle - \nu\alpha \langle (\mathbf{x} - \mathbf{X}) E \rangle \\ &= \langle \mathbf{F} + \nu\Theta \rangle. \end{aligned}$$

Thus it follows that

$$\mathbf{V} = \frac{\langle \mathbf{F} + \nu\Theta \rangle}{\langle E \rangle} = \frac{\langle \mathbf{F} \rangle}{\langle E \rangle} + \nu \frac{\langle \Theta \rangle}{\langle E \rangle}. \quad (4.7)$$

This expression for \mathbf{V} shows that, just as in the case when there is no dissipation ($\nu = 0$), the expression for the centro velocity is the energy-flux velocity, but now the total flux incorporates a term $\langle \Theta \rangle$ due to dissipation.

The expression (4.7) with $\nu = 0$;

$$\mathbf{V} = \langle \mathbf{F} \rangle / \langle E \rangle, \quad (4.8)$$

differs from (4.7) since $\langle \Theta \rangle \neq 0$ in general. Only when the dissipation is *uniform*, i.e., when

$$S(\mathbf{u}) = \alpha(\mathbf{u})E(\mathbf{u}), \quad (4.9)$$

for some functional α , Θ vanishes identically and (4.7) and (4.8) coincide.

In the literature it is customary to take (4.8) as the energy velocity even when dissipation is present (see Refs. 9 and 10).

V. EXAMPLES FROM FLUID DYNAMICS

Consider as a first example the *Korteweg-de Vries* (KdV) equation to which some dissipation is added. This equation for a scalar function $u(x, t)$ of one space variable x reads

$$\partial_t u - \partial_x [u_{xx} - 3u^2] = -\nu D(u). \quad (5.1)$$

For the equation with $\nu = 0$, a conserved density is $E = \frac{1}{2}u^2$ so that

$$\partial_t E + \partial_x F = 0, \quad \text{with } F = \frac{1}{2}u_x^2 - uu_{xx} - 2u^3. \quad (5.2)$$

In this case of one-space dimension, let Ω be the whole real line. Functions u are considered that vanish, together with all their derivatives sufficiently fast at infinity. Then the flux density F as given in (5.2) satisfies the pointwise boundary condition (2.4), and so this flux density is unique, according to Remark 2.1. With this F the centro velocity is given by (4.8). Uniform damping is encountered for e.g.,

$$D(u) = \frac{1}{2}\alpha(u)u, \quad \text{with some functional } \alpha. \quad (5.3)$$

Then the loss-density S in the balance law (3.1) reads $S(u) = \frac{1}{2}\alpha(u)u^2 = \alpha(u)E(u)$, so α is the dissipation rate. A simple example is the case $\alpha(u) = \alpha$, with α a constant.

The Korteweg-de Vries-Burgers equation is of the form (5.1) with the viscous dissipation given by

$$D(u) = -u_{xx}. \quad (5.4)$$

Then the loss density in (1) is $S(u) = -uu_{xx}$. Since $\langle S(u) \rangle = \langle -uu_{xx} \rangle = \langle u_x^2 \rangle$, the dissipation rate becomes

$$\alpha(u) = 2\langle u_x^2 \rangle / \langle u^2 \rangle. \quad (5.5)$$

Therefore, since $S(u) = -uu_{xx} = \alpha E + \partial_x \theta$, θ must satisfy the equation

$$\partial_x \theta = -uu_{xx} - \langle u_x^2 \rangle / \langle u^2 \rangle \cdot u^2. \quad (5.6)$$

The total flux $\langle \theta \rangle$, of interest for the centro velocity can be written as

$$\langle \theta \rangle = \langle -x\partial_x \theta \rangle. \quad (5.7)$$

From this expression, which follows from an integration by parts of its right-hand side upon use of the boundary conditions on θ , it is seen that $\langle \theta \rangle$ does not vanish in general since arbitrary solutions will not be symmetric. This shows the necessity of the additional term in (4.7) compared to (4.8). See also the remarks in the next section.

In Ref. 14 the result (4.7) is related to an expression derived by Vainshtein, and the velocity is investigated in great detail for linear wave equations.

As another example we consider the incompressible Navier-Stokes equations. Incompressible fluid flow is described by a velocity field \mathbf{v} satisfying $\text{div } \mathbf{v} = 0$, and pressure p by

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = \nu \nabla^2 \mathbf{v}, \quad (5.8)$$

or, equivalently,

$$\partial_t \mathbf{v} + (\text{curl } \mathbf{v}) \times \mathbf{v} + \nabla [p + \frac{1}{2}|\mathbf{v}|^2] = \nu \nabla^2 \mathbf{v},$$

where ν is the kinematic viscosity. For the kinetic energy density $E = \frac{1}{2}|\mathbf{v}|^2$ the balance law reads

$$\partial_t E + \text{div}[(p + E)\mathbf{v}] = \nu \mathbf{v} \cdot \nabla^2 \mathbf{v}. \quad (5.9)$$

The flux $\mathbf{F} = (p + E)\mathbf{v}$ satisfies the pointwise no-flux condition whenever the velocity field satisfies $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$. From (5.9) it follows that

$$\partial_t \langle E \rangle = -\nu \alpha \langle E \rangle, \quad \text{with } \alpha = 2(\langle |\nabla \mathbf{v}|^2 \rangle / \langle |\mathbf{v}|^2 \rangle). \quad (5.10)$$

For inviscid fluids the centro velocity of E is given by the expression

$$\mathbf{V}_E = \langle (p + E)\mathbf{v} \rangle / \langle E \rangle. \quad (5.11)$$

For viscous fluids an additional flux $\langle \Theta \rangle$ has to be added; it is the three-dimensional analog of the viscous contribution in the KdV-Burgers equation.

For plane flows there is another interesting density: the *enstrophy density*. Expressed in terms of the component ω of the vorticity vector $\text{curl } \mathbf{v}$ perpendicular to the plane, it is given by $W = \frac{1}{2}\omega^2$, and it is conserved for inviscid fluids. In terms of ω Eq. (5.8) takes the form known as the vorticity balance equation,

$$\partial_t \omega + \mathbf{v} \cdot \nabla \omega = \nu \nabla^2 \omega, \quad (5.12)$$

and the local balance law for W is easily shown to be

$$\partial_t W + \text{div}[W\mathbf{v}] = \nu \omega \nabla^2 \omega. \quad (5.13)$$

For the total enstrophy we thus obtain

$$\partial_t \langle W \rangle = -\nu \gamma \langle W \rangle, \quad \text{with } \gamma = 2\langle |\nabla \omega|^2 \rangle / \langle \omega^2 \rangle \quad (5.14)$$

and the centro velocity for W for inviscid fluids is given by

$$\mathbf{V}_W = \langle W\mathbf{v} \rangle / \langle W \rangle, \quad (5.15)$$

while for viscous fluids an additional term should be added.

In general the dissipation rate $\alpha(\mathbf{v})$ of the energy density will not coincide with the dissipation rate $\gamma(\mathbf{v})$ of the enstrophy density (They coincide only for so-called planar Taylor vortices, see Ref. 15). This fact shows that viscosity acts different on different densities. This *selective dissipation* is well known and is responsible for the self-organization process in Navier-Stokes equations (see Ref. 15).

VI. CONCLUDING REMARKS

The motivation for and results of this paper can be explained by starting with some balance law for a certain continuous system. If E is a specified density that has a clear physical meaning (the energy, say), a balance law for E is of the form (3.1):

$$\partial_t E(\mathbf{u}) + \text{div } \mathbf{F}(\mathbf{u}) = -\nu S(\mathbf{u}).$$

This relation does not define F or S in a unique way. In fact, even if S is the result of adding dissipation to the the governing equation (symbolized by the factor ν), so that E satisfies the local conservation law (2.3),

$$\partial_t E(\mathbf{u}) + \text{div } \mathbf{F}(\mathbf{u}) = 0,$$

when dissipation is ignored, the flux density \mathbf{F} is not uniquely defined since any function $\text{curl } \mathbf{a}$ can be added to \mathbf{F} without changing (2.3). This nonuniqueness is quite cumbersome if one looks for a physical meaning of the flux \mathbf{F} or quantities expressed in \mathbf{F} . A particular example is the local energy velocity. Upon integrating (2.3) over arbitrary subdomains, it is quite natural to define a *local* energy velocity \mathbf{v}_E by

$$E \cdot \mathbf{v}_E = \mathbf{F}.$$

However, the velocity defined in this way clearly changes when a nonvanishing function $\text{curl } \mathbf{a}$ is added to \mathbf{F} . The same problem is encountered if dissipation is present.

This paper shows that this problem can be partly solved if it is known that, provided dissipation is ignored, the total energy $\langle E \rangle$ is conserved. Then, although \mathbf{F} is not unique, the

total flux $\langle F \rangle$ can be uniquely defined by taking pointwise no-flux boundary conditions for F (the conservation of $\langle E \rangle$ implies that the integration over the boundary of the flux component normal to the boundary vanishes). In the particular case of one-dimensional problems, the scalar function F is defined uniquely.

The uniqueness of $\langle F \rangle$ turns out to be of vital importance if one considers instead of the local energy velocity, an averagedlike velocity. In particular, in conservative systems, the velocity of the center of gravity—the centro velocity—of the energy is shown to be the energy-flux velocity (4.8), i.e., the quotient of $\langle F \rangle$ and $\langle E \rangle$, and is therefore uniquely defined. The same has been shown to be true if dissipation is added to the equations. Then the contribution S has been split into a part proportional to E and a divergence term. In the centro velocity appears the integrated expression of this divergence term; this term has also been chosen in a unique way, leading to a unique expression for the centro velocity.

Concerning the use of the centro velocity as a physical quantity to measure propagation speed, the following remarks are in order. First of all, by its definition, the centro velocity has some physical meaning. It will be clear, however, that when dissipation is the dominating feature, and not just acts as a perturbation of a conserved system, this concept, although well defined, will have little practical importance.

On the other hand, the centro velocity can be defined for (highly) nonlinear equations, and for all kind of solutions (provided the boundary conditions are satisfied). This is different for two other velocity concepts that are often used, the group and phase velocity, which are only well suited for linear (or weakly nonlinear) equations and for (quasi)monochromatic solutions.

A final remark is about the effect of dissipation on the centro velocity, i.e., about the contribution of the term $\langle \Theta \rangle$ in (4.7);

$$V = \frac{\langle F + v\Theta \rangle}{\langle E \rangle} = \frac{\langle F \rangle}{\langle E \rangle} + v \frac{\langle \Theta \rangle}{\langle E \rangle}.$$

In a somewhat different way, this has been investigated for equations like the uniformly damped KdV, and the KdV–Burgers equation (5.1) in a recent paper.¹⁶ To summarize these results here, it must first be noted that the KdV equation itself (so with $v = 0$) has a family of travelling waves (cnoidal waves) that can be parameterized with the total energy $e \equiv \langle E \rangle = \langle \frac{1}{2}u^2 \rangle$ and propagate with a certain velocity $\lambda = \lambda(e)$ depending on e . We denote such a wave by $U(e; x - \lambda t)$. Taking such a waveform with a specific value of e as an initial condition for the dissipative KdV equation, the resulting decaying evolution was written like

$$u(x, t) = U(e(t); x - \phi(t)), \quad (6.1)$$

for some functions $e(t)$ and $\phi(t)$, so as if the evolution follows an adiabatic path along the family of travelling waves. Equations for the functions $e(t)$ and $\phi(t)$ were derived. The equation for e in lowest order of v turns out to be an ordinary

differential equation for e itself, and can be solved (numerically for the KdV–Burgers equation; for the uniformly damped case, e decreases exponentially). In the same order, the position of the wave $\phi(t)$ follows from the equation

$$\partial_t \phi(t) = \lambda(e(t)), \quad (6.2)$$

which can be integrated once $e(t)$ has been determined. Numerical calculations of these equations, and a comparison with numerical calculations of the initial value problem of the continuous dissipative KdV equation itself showed that these simple, lowest-order equations are in fact very accurate. This holds true even when highly nonlinear effects are dominant initially, and up to rather large values of v .

Of relevance for the present paper is particularly the expression (6.2) that depicts the instantaneous propagation speed of the decaying wave as the speed of the exact travelling wave to which it is compared at that time. Since for a function u given by (6.1) the energy-flux velocity equals the propagation speed $\lambda(e)$:

$$\langle F \rangle / \langle E \rangle = \lambda(e),$$

$$\text{for } u = U(e; x - \phi), \quad \text{any value of } \phi, \quad (6.3)$$

these results show that the contribution $\langle \Theta \rangle$ is negligible in this case. Another way to see this is to note that the cnoidal wave $U(e; x - \phi)$ is an even function about $x = \phi$. This causes the expression in (5.6) to be even, and therefore the total flux $\langle \Theta \rangle$ given by (5.7) to be zero. Of course, the actual solution will not be even, but the numerical calculations show that the unevenness has only small effect on the propagation speed.

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