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Riccati equations and normalized coprime factorizations for strongly stabilizable infinite-dimensional systems

Ruth F. Curtain[a,*], Hans Zwart[b]

[a] Department of Mathematics, University of Groningen, P.O. Box 800, 9700 AV Groningen, Netherlands
[b] Faculty of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands

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Abstract

The first part of the paper concerns the existence of strongly stabilizing solutions to the standard algebraic Riccati equation for a class of infinite-dimensional systems of the form $\Sigma(A,B,S^{-1/2}B^*,D)$, where A is dissipative and all the other operators are bounded. These systems are not exponentially stabilizable and so the standard theory is not applicable. The second part uses the Riccati equation results to give formulas for normalized coprime factorizations over $\mathbf{H}_\infty$ for positive real transfer functions of the form $D+S^{-1/2}B^*(sI-A)^{-1}B$.

Keywords: Normalized coprime factorizations; Strong stability; Positive real; Dissipative; Riccati equations; Infinite-dimensional systems; Colocated systems

1. Introduction

Most of the theory for Riccati equations concentrates on the existence of solutions which are exponentially stabilizing. Here we consider systems for which this theory is inapplicable, because they are not exponentially stabilizable by a bounded feedback. Specifically, we consider systems $\Sigma(A,B,S^{-1/2}B^*,B,D)$, where A is dissipative on a Hilbert space Z, $B\in\mathscr{L}(U,Z)$, where U is a Hilbert space and S, $D\in\mathscr{L}(U)$ with $S=S^*$ is coercive. The case $S=I$, $D=0$ has been considered in [2, 10, 11]. If $S=I$, these are usually termed "colocated" systems and there exists a considerable literature on such systems, see [3].

The colocated configuration is often preferred in designing controllers for large-scale flexible systems, see [8], and there is considerable interest in their properties, especially robustness properties. In this direction, we deduce formulas for normalized coprime factorizations of the transfer function $G(s)=D+S^{-1/2}B^*(sI-A)^{-1}B$, under the extra assumption that $G(s)$ is positive-real. Since positive-real systems have good robustness properties (see [5]), this is not so surprising. The formulas for the normalized coprime factorizations are not surprising either; they have the same form as for the matrix case, see [12]. The surprising feature is that we can only do this under the extra assumption that $(I+D)^{-1}S^{-1/2}$ is strictly positive. In order to establish the formulas for the normalized coprime factorizations we need to establish some results on Riccati equations. We give conditions for the existence of unique, strongly stabilizing solutions of the

* Corresponding author.

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following algebraic Riccati equation:

$$(A_Q)^*Qz + QA_Qz + QB(R_D)^{-1}B^*Qz + BS^{-1/2}S_DS^{-1/2}B^*z = 0 \tag{1}$$

for $z \in D(A)$, where $A_Q = A - B(R_D)^{-1}D^*S^{-1/2}B^* - B(R_D)^{-1}B^*Q, R_D = R + D^*D, S_D = I + DR^{-1}D^*$ and $R \in \mathscr{L}(U)$ is a coercive operator.

By strongly stabilizing we mean that the semigroup $T_Q(t)$ generated by A_Q has the property that $T_Q(t)z \to 0$ as $t \to \infty$ for all $z \in Z$. Similar results were given for the case $D = 0$ in [2], but he needs to assume that $A - BB^*Q$ was dissipative; it is not at all clear when this is satisfied. The basic assumption we need is that the output operator has the form $S^{-1/2}B^*$; this is how we show that the system is optimizable and so (1) has at least one self-adjoint solution. Our motivation for considering the general form of (1) with the feedthrough terms D is that we need this form to obtain the formulas for the normalized coprime factorization of $G(s)$. Doubly coprime factorizations may be used in parameterizing all controllers that stabilize the system in the input–output sense (see [14]), and normalized ones yield formulas for controllers that do this robustly (see [7]).

2. Riccati equations

We begin by considering the existence of strongly stabilizing, self-adjoint solutions $Q \in \mathscr{L}(Z)$ to the algebraic Riccati equation.

$$A^*Qz + QAz - QBR^{-1}B^*Qz + BS^{-1}B^*z = 0, \tag{2}$$

for $z \in D(A)$, under the first six of the following assumptions:

A1. A is the infinitesimal generator of a strongly continuous contraction semigroup $T(t)$ on the separable Hilbert space Z;

A2. U is a separable Hilbert space and $B \in \mathscr{L}(U,Z)$;

A3. $S = S^* \in \mathscr{L}(U)$ and is coercive, i.e., $\langle Su,u\rangle \geqslant \varepsilon\|u\|^2$ for some $\varepsilon > 0$;

A4. $R = R^* \in \mathscr{L}(U)$ and is coercive;

A5. $\Sigma(A,-,B^*)$ is approximately observable;

A6. A has compact resolvent;

A7. $\Sigma(A,B,-)$ is approximately controllable.

First we recall several well-known results from the literature, but for completeness we also supply short proofs.

Lemma 1. *Suppose that $T(t)$ is a weakly stable C_0-semigroup on the Hilbert space Z, i.e, $\langle z,T(t)y\rangle \to 0$ as $t \to \infty$ for all $z,y \in Z$. If its infinitesimal generator A has compact resolvent, then $T(t)$ is strongly stable, i.e., $T(t)z \to 0$ as $t \to \infty$ for all $z \in Z$.*

Proof. (a) We show that $T(t)$ is uniformly bounded in norm for $t \geqslant 0$. Since $T(t)$ is weakly stable, we have that $\|T(n)z\| \leqslant M_1$ uniformly for $n = 1,2,\ldots$ and applying the uniform boundedness theorem twice, we obtain $\|T(n)\| \leqslant M$ for all $n = 1,2,\ldots$ Any $t > 0$ may be written as $t = n + \delta$ for some $0 \leqslant \delta < 1$ and so

$$\|T(t)\| = \|T(n+\delta)\| \leqslant \|T(n)\|\|T(\delta)\| \leqslant MM_1\max(1,e^{\delta\omega}) = M_2 < \infty,$$

where $\|T(t)\| \leqslant M_1e^{\omega t}$.

(b) There exists a $\lambda \in R$ such that $(\lambda I - A)^{-1}$ is compact. Since $T(t)$ is weakly convergent to 0 as $n \to \infty$, we must have $(\lambda I - A)^{-1}T(n_r)y \to 0$ as $r \to \infty$ for a subsequence n_r. The uniform boundedness of $T(t)$ shows that in fact $(\lambda I - A)^{-1}T(t)y \to 0$ as $t \to \infty$. Suppose now that $x \in D(A)$, i.e., there exists $y \in Z$ such that $x = (\lambda I - A)^{-1}y$. Then

$$T(t)x = T(t)(\lambda I - A)^{-1}y = (\lambda I - A)^{-1}T(t)y \to 0 \quad \text{as } t \to \infty.$$

Finally, since $D(A)$ is dense in Z, for $z \in Z$ and a given $\varepsilon > 0$, there exists an $x \in D(A) : \|z-x\| < \varepsilon$. Thus,

$$\begin{aligned}\|T(t)z\| &\leqslant \|T(t)z - T(t)x\| + \|T(t)x\| \leqslant \|T(t)\|\|z-x\| + \|T(t)x\| \\ &\leqslant M_2\varepsilon + \|T(t)x\| \quad \text{(by part (a))}\end{aligned}$$

and $\|T(t)x\| \to 0$ as $t \to \infty$ proves the result. □

Lemma 2. *Suppose that assumptions* A1 *and* A2 *hold. The semigroup $T_B(t)$ generated by $A - BB^*$ is a contraction semigroup and for $x \in Z$ there hold*:

$$\int_0^\infty \|B^* T_B(t)x\|^2 \, \mathrm{d}t \leqslant \tfrac{1}{2}\|x\|^2, \tag{3}$$

$$\int_0^\infty \|B^* T_B^*(t)x\|^2 \, \mathrm{d}t \leqslant \tfrac{1}{2}\|x\|^2. \tag{4}$$

Proof. (a) $T_B(t)$ is a contraction semigroup since $A^* - BB^*$ and $A - BB^*$ are dissipative ($-BB^* \leqslant 0$).

(b) We establish (3) by differentiating $\|T_B(t)x\|^2$ with respect to t for $x \in D(A)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|T_B(t)x\|^2 &= \langle (A - BB^*)T_B(t)x, T_B(t)x\rangle + \langle T_B(t)x, (A - BB^*)T_B(t)x\rangle \\ &= \langle AT_B(t)x, T_B(t)x\rangle + \langle T_B(t)x, AT_B(t)x\rangle - 2\|B^*T_B(t)x\|^2\end{aligned}$$

and since $T(t)$ generates a contraction semigroup, A is dissipative and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|T_B(t)x\|^2 + 2\|B^*T_B(t)x\|^2 \leqslant 0.$$

On integrating, we obtain

$$\|T_B(t)x\|^2 + 2\int_0^t \|B^*T_B(t)x\|^2 \, \mathrm{d}s \leqslant \|x\|^2$$

which establishes (3) for $x \in D(A)$. Since $D(A)$ is dense in Z, it extends to all $x \in Z$.

(c) (4) is proved similarly to (3) by differentiating $\|T_B^*(t)x\|^2$ for $x \in D(A^*)$, and noting that since Z is a Hilbert space, $T^*(t)$ is also a contraction semigroup. □

We remark that under extra assumptions A5 and A6, $T_B(t)$ is strongly stable, see [4, 13, 1]. However, we do not need this result in our application.

Lemma 3. *Let A generate the strongly continuous C_0 semigroup $T(t)$ on the Hilbert space Z and let B, R and S satisfy the assumptions* A2–A4. *If the Riccati equation* (2) *has a strongly stabilizing solution, then it is the only one with this property.*

Proof. Suppose that Q_1 and Q_2 are both strongly stabilizing solutions of (2), i.e., $T_{Q_i}(t)z \to 0$ as $t \to \infty$ for $z \in Z$, $i = 1,2$, where $A - BB^*Q_i$ generates $T_{Q_i}(t)$. On rearranging (2) we obtain

$$(A - BR^{-1}B^*Q_1)^*Q_1 z + Q_1(A - BR^{-1}B^*Q_2)z = -Q_1BR^{-1}B^*Q_2 z - BS^{-1}B^*z,$$

and

$$(A - BR^{-1}B^*Q_1)^*Q_2 z + Q_2(A - BR^{-1}B^*Q_2)z = -Q_1BR^{-1}B^*Q_2 z - BS^{-1}B^*z.$$

Subtracting gives

$$(A - BR^{-1}B^*Q_1)^*(Q_1 - Q_2)z + (Q_1 - Q_2)(A - BR^{-1}B^*Q_2)z = 0$$

which implies that for $x, y \in D(A)$ there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle T_{Q_1}(t)y, (Q_1 - Q_2)T_{Q_2}(t)x\rangle = 0.$$

Thus,

$$\langle T_{Q_1}(t)y, (Q_1 - Q_2)T_{Q_2}(t)x\rangle = \text{constant},$$

and this constant must be zero, since both Q_1 and Q_2 are strongly stabilizing. Substituting $t = 0$ and noting that $D(A)$ is dense in Z shows that $Q_1 = Q_2$. □

As we already remarked, the above results have been well known for decades, see [2]. Strangely enough, the next result does not seem to be known.

Theorem 4. *Under the assumptions* A1–A6, *the algebraic Riccati equation* (2) *has a unique strongly stabilizing solution.*

Proof. (a) Consider the following control problem that is associated with (2) and the quadratic cost functional:

$$J(u) = \int_0^\infty (\|R^{1/2}u(t)\|^2 + \|S^{-1/2}B^*z(t)\|^2)\,\mathrm{d}t \tag{5}$$

subject to the dynamics

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0. \tag{6}$$

Note that with $\tilde{u}(t) = -B^*z(t)$, we obtain $\dot{z}(t) = (A - BB^*)z(t)$ and thus $z(t) = T_B(t)z_0$, whence

$$\begin{aligned} J(\tilde{u}) &= \int_0^\infty \|R^{1/2}B^*T_B(t)z_0\|^2 + \|S^{-1/2}B^*T_B(t)z_0\|^2\,\mathrm{d}t \\ &\leqslant \tfrac{1}{2}\left[\|R^{1/2}\|^2 + \|S^{-1/2}\|^2\right]\|z_0\|^2 \qquad \text{(by (3))}. \end{aligned}$$

In other words, $\Sigma(A, B, S^{-1/2}B^*)$ is optimizable and from Curtain and Zwart [6, Theorem 6.2.4] we conclude that (2) has a solution $Q_0 = Q_0^* \geqslant 0$ which satisfies

$$\langle Q_0x_0, x_0\rangle = \int_0^\infty (\|R^{-1/2}B^*Q_0T_{Q_0}(t)x_0\|^2 + \|S^{-1/2}B^*T_{Q_0}(t)x_0\|^2)\,\mathrm{d}t, \tag{7}$$

where $A - BR^{-1}B^*Q_0$ generates $T_{Q_0}(t)$.

(b) We show that $T_{Q_0}(t)$ is uniformly bounded in norm. Now for $x \in Z$, there holds

$$T_{Q_0}(t)x = T_B(t)x + \int_0^t T_B(t-s)B\left[B^* - R^{-1}B^*Q_0\right]T_{Q_0}(s)x\,\mathrm{d}s$$

and so for all $x, y \in Z$ we have that

$$\begin{aligned} |\langle y, T_{Q_0}(t)x\rangle| &\leqslant \|y\|\|T_B(t)x\| + \left|\int_0^t \langle B^*T_B^*(t-s)y, \left[B^* - R^{-1}B^*Q_0\right]T_{Q_0}(s)x\rangle\,\mathrm{d}s\right| \\ &\leqslant \|y\|\|x\| + \left[\int_0^t \|B^*T_B^*(t-s)y\|^2\,\mathrm{d}s\right]^{1/2}\left[\int_0^t \|(B^* - R^{-1}B^*Q_0)T_{Q_0}(s)x\|^2\,\mathrm{d}s\right]^{1/2} \\ &\qquad \text{(since } T_B(t) \text{ is a contraction)} \end{aligned}$$

$$\leqslant \|y\|\|x\| + \frac{1}{\sqrt{2}}\|y\| \left[\left[\int_0^\infty \|B^* T_{Q_0}(s)x\|^2 \, ds \right]^{1/2} + \left[\int_0^\infty \|R^{-1}B^* Q_0 T_{Q_0}(s)x\|^2 \, ds \right]^{1/2} \right]$$

(by (4) and the Minkowski inequality)

$$\leqslant \|y\| \left[\|x\| + \|Q_0^{1/2}x\| \max \left[\|R^{-1/2}\|, \|S^{1/2}\| \right] \right] \quad \text{(by (7))}$$

$$\leqslant \left[1 + \|Q_0^{1/2}\| \max \left[\|R^{-1/2}\|, \|S^{1/2}\| \right] \right] \|x\|\|y\|.$$

(c) It remains to show that $T_{Q_0}(t)$ is weakly stable. Substituting $x = T_{Q_0}(t)z$ in (7) we obtain

$$\langle Q_0 T_{Q_0}(t)z, T_{Q_0}(t)z \rangle = \int_t^\infty \langle (Q_0 BR^{-1}B^*Q_0 + BS^{-1}B^*)T_{Q_0}(s)z, T_{Q_0}(s)z \rangle \, ds$$

which converges to 0 as $t \to \infty$.

(d) Next we show that $\ker Q_0 = 0$. Suppose that there exists a nonzero x such that $Q_0 x = 0$. Then from (7) we deduce

$$0 = \int_0^\infty (\|R^{-1/2}B^*Q_0 T_{Q_0}(s)x\|^2 + \|S^{-1/2}B^* T_{Q_0}(s)x\|^2) \, ds$$

and so

$$B^*Q_0 T_{Q_0}(s)x = B^* T_{Q_0}(s)x = 0 \quad \text{for } s \geqslant 0.$$

But

$$T_{Q_0}(t)x = T(t)x - \int_0^t T(t-s)BR^{-1}B^*Q_0 T_{Q_0}(s)x \, ds = T(t)x$$

and this implies that $B^*T(t)x = 0$ for $t \geqslant 0$, violating the approximate observability assumption A5. So $Q_0 > 0$.

(e) From part (c) we have that

$$|\langle T_{Q_0}(t)x, Q_0 y \rangle| \leqslant \|Q_0^{1/2}\| \|Q_0^{1/2} T_{Q_0}(t)x\| \|y\| \to 0 \quad \text{as } t \to \infty$$

for all $x, y \in Z$. Since Q_0 is self-adjoint and positive we have that the range of Q_0 is dense in Z. Hence, for every $z \in Z$ and every $\varepsilon > 0$ there exists an $Q_0 y$ such that $\|z - Q_0 y\| \leqslant \varepsilon$. Since from part (b), $\|T_{Q_0}(t)\| \leqslant M$ for $t \geqslant 0$, we may conclude that

$$|\langle T_{Q_0}(t)x, z \rangle| \leqslant |\langle T_{Q_0}(t)x, (z - Q_0 y) \rangle| + |\langle T_{Q_0}(t)x, Q_0 y \rangle| \leqslant M\varepsilon + |\langle T_{Q_0}(t)x, Q_0 y \rangle|$$

and so $T_{Q_0}(t)$ is weakly stable.

(f) Lemmas 1 and 3 complete the proof. □

We remark that we do not need to assume that $T_{Q_0}(t)$ is a contraction semigroup, as was done in [2, p. 339]. Indeed, this need not be the case in general. We give a matrix counterexample. Choose $A = 0, B = I$, and

$$R^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 5 & -16 \\ -16 & 52 \end{pmatrix}.$$

Then

$$Q_0 = \begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix}$$

solves (2), but

$$A_{Q_0} = A - BR^{-1}B^*Q_0 = \begin{pmatrix} 1 & -4 \\ 3 & -10 \end{pmatrix}$$

does not generate a contraction semigroup: $A_{Q_0} + A^*_{Q_0}$ has a negative and a positive eigenvalue.

As a corollary of this theorem we have an existence result for the control problem defined by (5), (6).

Corollary 5. *Under the assumptions* A1–A6 *the control problem with the quadratic cost functional* (5) *and subject to the dynamics* (6) *has the unique minimizing feedback control given by*

$$\tilde{u}(t) = -R^{-1}B^*Qz(t),$$

where Q is the unique, strongly stabilizing solution to (2). *The minimum cost equals $\langle Qz_0, z_0\rangle$ and the closed-loop system operator $A - BR^{-1}B^*Q$ generates a strongly stable semigroup.*

In a similar manner, it is possible to obtain existence results for the more general algebraic Riccati equation for $z \in D(A)$

$$\begin{aligned} &(A - BR_D^{-1}D^*S^{-1/2}B^*)^*Qz + Q(A - BR_D^{-1}D^*S^{-1/2}B^*)z \\ &\quad -QBR_D^{-1}B^*Qz + BS^{-1/2}(I + DR^{-1}D^*)^{-1}S^{-1/2}B^*z = 0, \end{aligned} \tag{8}$$

where $R_D = R + D^*D$.

Theorem 6. *Under assumptions* A1–A6, *the algebraic Riccati equation* (8) *has a unique, strongly stabilizing solution, i.e., $A - BR_D^{-1}D^*S^{-1/2}B^* - BR_D^{-1}B^*Q$ generates a strongly stable semigroup.*

Proof. (a) As before, we consider the associated control problem with the quadratic cost functional

$$J_1(u) = \int_0^\infty (\|y(t)\|^2 + \langle Ru(t), u(t)\rangle)\,\mathrm{d}t \tag{9}$$

subject to the dynamics

$$\dot{z}(t) = Az(t) + Bu(t), \tag{10}$$

$$y(t) = S^{-1/2}B^*z(t) + Du(t). \tag{11}$$

Notice that

$$J_1(u) = \int_0^\infty \left\langle Q_1 \begin{pmatrix} z(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} z(s) \\ u(s) \end{pmatrix} \right\rangle \mathrm{d}s,$$

where

$$Q_1 = \begin{pmatrix} BS^{-1}B^* & BS^{-1/2}D \\ D^*S^{-1/2}B^* & R + D^*D \end{pmatrix} = \begin{pmatrix} I & -F^* \\ 0 & I \end{pmatrix} \begin{pmatrix} BS_1^{-1}B^* & 0 \\ 0 & R_D \end{pmatrix} \begin{pmatrix} I & 0 \\ -F & I \end{pmatrix},$$

$S_1 = S^{1/2}(I + DR^{-1}D^*)S^{1/2}$, $R_D = R + D^*D$, $F = -R_D^{-1}D^*S^{-1/2}B^*$. Thus,

$$J_1(u) = J(u_1) = \int_0^\infty \|S_1^{-1/2}B^*z(t)\|^2 + \|R_D^{1/2}u_1(t)\|^2\,\mathrm{d}t, \tag{12}$$

where $u_1(t) = u(t) - Fz(t) = u(t) + R_D^{-1}D^*S^{-1/2}B^*z(t)$. So, minimizing (9) is equivalent to minimizing $J(u_1)$ given by (12) subject to the dynamics

$$\dot{z}(t) = (A - BR_D^{-1}D^*S^{-1/2}B^*)z(t) + Bu_1(t), \quad z(0) = z_0. \tag{13}$$

This is similar to the control problem considered in the proof of Theorem 4, except that $\tilde{A} := A - BR_D^{-1}D^*S^{-1/2}B^*$ need not be dissipative. On examining the proof of Theorem 4 we see that all of the arguments extend to this new situation.

(a) With

$$\tilde{u}_1(t) = -B^*z(t) + R_D^{-1/2}D^*S^{-1/2}B^*z(t) = -B^*z(t) - FB^*z(t),$$

we obtain

$$J(\tilde{u}_1) = \int_0^\infty (\|S_1^{-1/2}B^*T_B(t)z_0\|^2 + \|R_D^{1/2}(I+F)B^*T_B(t)z_0\|^2)\,\mathrm{d}t < \infty$$

and $\Sigma(A, B, S_1^{-1/2}B^*)$ is optimizable.

(b) From Curtain and Zwart [6, Theorem 6.2.4], we conclude that (8) has a solution $Q_0 = Q_0^* \geqslant 0$ which satisfies

$$\langle Q_0x, x\rangle = \int_0^\infty \|R_D^{-1/2}B^*Q_0\tilde{T}_{Q_0}(s)x\|^2 + \|S_1^{-1/2}B^*\tilde{T}_{Q_0}(s)x\|^2\,\mathrm{d}s, \tag{14}$$

where $\tilde{A} - BR_D^{-1}B^*Q_0$ generates $\tilde{T}_{Q_0}(t)$.

(c) Writing

$$\begin{aligned}\tilde{A} - BR_D^{-1}B^*Q_0 &= A - BR_D^{-1}D^*S^{-1/2}B^* - BR_D^{-1}B^*Q_0\\ &= (A - BB^*) + BK_1B^* + BK_2B^*Q_0,\end{aligned}$$

we see that using the perturbation formula for the semigroup $\tilde{T}_{Q_0}(t)$ in terms of $T_B(t)$ and the estimates from (14), we can prove that $\tilde{T}_{Q_0}(t)$ is uniformly bounded in norm for $t \geqslant 0$. The final steps follow just as in the proof of Theorem 4 using the approximate observability of $\Sigma(A, -, B^*)$ and the fact that $A - BR_D^{-1}D^*S^{-1/2}B^* - BR_D^{-1}B^*Q_0$ has compact resolvent (as a bounded perturbation of A). □

Again, this theorem has implications for an optimal control problem.

Corollary 7. *Under the assumptions* A1–A6, *the control problem with the quadratic cost functional* (9) *subject to the dynamic constraints* (10), (11) *has the unique minimizing control given by*

$$\tilde{u}(t) = -(R_D^{-1}D^*S^{-1/2}B^* + R_D^{-1}B^*Q)z(t),$$

where Q is the unique, strongly stabilizing solution of (8). *Moreover, the minimum cost equals $\langle Qz_0, z_0\rangle$ and the closed-loop system has a strongly stable semigroup.*

In Section 3 we shall need results for dual Riccati equations. Of course, these are easily deduced from Theorem 6.

Theorem 8. *Under the assumptions* A1–A4, A6 *and* A7, *the following algebraic Riccati equation has a unique, strongly stabilizing solution* (*i.e.*, $A - (PBS^{-1/2} + BD^*)S_D^{-1}S^{-1/2}B^*$ *generates a strongly stable semigroup*):

$$\begin{aligned}&(A - BR_D^{-1}D^*S^{-1/2}B^*)Pz + P(A - BR_D^{-1}D^*S^{-1/2}B^*)^*z\\ &-PBS^{-1/2}S_D^{-1}S^{-1/2}B^*Pz + BR_D^{-1}B^*z = 0,\end{aligned} \tag{15}$$

for $z \in D(A^*)$, where $R_D = I + D^*D, S_D = (I + DR^{-1}D^*)^{-1}$.

3. Normalized coprime factors

In this section, we give formulas for normalized doubly coprime factorizations of a transfer matrix $G(s) = D + S^{-1/2}B^*(sI - A)^{-1}B$, where A, B, and S satisfy the assumptions A1–A3, A5–A7, $U = C^m, R = I$ and $D \in \mathscr{L}(U)$. First we recall some definitions of coprime factorizations over $\mathscr{M}\mathbf{H}_\infty$, the set of matrices of any size with all components in $\mathbf{H}_\infty$, the Hardy space of complex-valued functions that are holomorphic and bounded on $C_0^+ = \{s \in C : \mathrm{Re}(s) > 0\}$.

Definition 9. Suppose that there exist matrices M, N, $\tilde{X}$, $\tilde{Y} \in \mathscr{M}\mathbf{H}_\infty$ with M square and $\det(M) \not\equiv 0$ on C_0^+ such that

$$G(s) = N(s)M(s)^{-1} \quad \text{for } s \in C_0^+, \tag{16}$$

$$\tilde{X}(s)M(s) - \tilde{Y}(s)N(s) = I \quad \text{for } s \in C_0^+. \tag{17}$$

We say that $G = NM^{-1}$ is a right-coprime factorization of G over $\mathscr{M}\mathbf{H}_\infty$.

Suppose that there exist matrices $\tilde{M}, \tilde{N}, X, Y \in \mathscr{M}\mathbf{H}_\infty$ with $\tilde{M}$ square and $\det(\tilde{M}) \not\equiv 0$ on C_0^+ such that

$$G(s) = \tilde{M}(s)^{-1}\tilde{N}(s) \quad \text{for } s \in C_0^+, \tag{18}$$

$$\tilde{M}(s)X(s) - \tilde{N}(s)Y(s) = I \quad \text{for } s \in C_0^+. \tag{19}$$

We say that $G = \tilde{M}^{-1}\tilde{N}$ is a left-coprime factorization of G over $\mathscr{M}\mathbf{H}_\infty$. If, in addition, the following identity holds, we say that $G = \tilde{M}^{-1}\tilde{N} = NM^{-1}$ is a doubly coprime factorization over $\mathscr{M}\mathbf{H}_\infty$

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \text{ on } C_0^+. \tag{20}$$

If (M, N) satisfy conditions (16), (17) and also

$$N(j\omega)^*N(j\omega) + M(j\omega)^*M(j\omega) = I \quad \text{for } \omega \in R, \tag{21}$$

we say that $G = NM^{-1}$ is a normalized right-coprime factorization. If $(\tilde{M}, \tilde{N})$ satisfy conditions (18) and (19) and also

$$\tilde{N}(j\omega)\tilde{N}(j\omega)^* + \tilde{M}(j\omega)\tilde{M}(j\omega)^* = I \quad \text{for } \omega \in R, \tag{22}$$

we say that $G = \tilde{M}^{-1}\tilde{N}$ is a normalized left-coprime factorization. If (16)–(22) all hold, we say that $G = \tilde{M}^{-1}\tilde{N} = NM^{-1}$ is a normalized doubly coprime factorization.

If $\Sigma(A, B, C, D)$ is an exponentially stabilizable and exponentially detectable state linear system, then formulas for normalized doubly coprime factorizations of $G(s) = D + C(sI - A)^{-1}B$ can be given in terms of solutions to two algebraic Riccati equations (see Theorem 7.3.11 and Exercise 7.2.9 in [6]). However, our class of systems is neither exponentially stabilizable nor detectable and so this result is not applicable to our situation. Although it is easy to show that for $G(s) = D + S^{-1/2}B^*(sI - A)^{-1}B$, under our assumptions A1–A7, the same formulas still yield the sought factorizations satisfying (16)–(22), it is not easy to show that $M, N, \tilde{M}, \tilde{N}, X, Y, \tilde{X}, \tilde{Y}$ are in $\mathscr{M}\mathbf{H}_\infty$. In fact, we do not believe that this is always the case. We have only been able to show this for the following special case.

Definition 10. Let $G(s)$ be an $m \times m$ matrix-valued complex function. G is positive real if it satisfies the following conditions:

(i) $G(s)$ has real coefficients;

(ii) $G(s)$ is holomorphic on $\mathrm{Re}(s) > 0$;
(iii) $G(s)^* + G(s) \geqslant 0$ on $\mathrm{Re}(s) \geqslant 0$.

An example of a positive real system is $G(s) = D + B^*(sI - A)^{-1}B$ under our assumptions and $D + D^* \geqslant 0$. In [5], it is proved that a positive real system has the following coprime factorizations:

$$G = \tilde{M}^{-1}\tilde{N} = NM^{-1},$$

where

$$N = \tilde{N} = G(I + G)^{-1} \text{ and } M = \tilde{M} = (I + G)^{-1}. \tag{23}$$

So if our system $G(s) = D + S^{-1/2}B^*(sI - A)^{-1}B$ is positive real, it has the coprime factorization:

$$M_0 = \tilde{M}_0 = (I + D)^{-1} - (I + D)^{-1}S^{-1/2}B^*(sI - A_0)^{-1}B(I + D)^{-1}, \tag{24}$$

$$N_0 = \tilde{N}_0 = D(I + D)^{-1} + (I + D)^{-1}S^{-1/2}B^*(sI - A_0)^{-1}B(I + D)^{-1}, \tag{25}$$

where

$$A_0 = A - B(I + D)^{-1}S^{-1/2}B^*. \tag{26}$$

We shall also use the following lemma (a simple proof is in [9]).

Lemma 11. *Suppose that $U = C^m, Y = C^n$. If $f \in \mathbf{L}_2((0,\infty); \mathscr{L}(U,Y))$, and its Fourier transform $\hat{f} \in \mathbf{L}_\infty((-j\infty, j\infty); \mathscr{L}(U,Y))$, then $\hat{f} \in \mathscr{M}\mathbf{H}_\infty$.*

Now if $f(t) = 0$ for $t \leqslant 0$ the Laplace and Fourier transforms of f are isomorphic and so Lemma 11 also holds for the Laplace transform of functions which are zero for $t \leqslant 0$.

Theorem 12. *Consider $G(s) = D + S^{-1/2}B^*(sI - A)^{-1}B$ under the assumptions* A1–A7 *with $U = C^m, R = I$ and $D \in \mathscr{L}(U)$.*

(a) *G has the factorizations $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$, satisfying* (16)–(22), *where*

$$N(s) = DR_D^{-1/2} + (S_D^{-1}S^{-1/2}B^* - DR_D^{-1}B^*Q)(sI - A_Q)^{-1}BR_D^{-1/2},$$

$$M(s) = R_D^{-1/2} - R_D^{-1}(D^*S^{-1/2}B^* + B^*Q)(sI - A_Q)^{-1}BR_D^{-1/2},$$

$$X(s) = [I + (S_D^{-1}S^{-1/2}B^* - DR_D^{-1}B^*Q)(sI - A_Q)^{-1}(PBS^{-1/2} + BD^*)S_D^{-1}]S_D^{1/2},$$

$$Y(s) = -R_D^{-1}(D^*S^{-1/2}B^* + B^*Q)(sI - A_Q)^{-1}(PBS^{-1/2} + BD^*)S_D^{-1/2},$$

$$\tilde{N}(s) = S_D^{-1/2}D + S_D^{-1/2}S^{-1/2}B^*(sI - A_P)^{-1}(BR_D^{-1} - PBS^{-1/2}S_D^{-1}D),$$

$$\tilde{M}(s) = S_D^{-1/2} - S_D^{-1/2}S^{-1/2}B^*(sI - A_P)^{-1}(PBS^{-1/2} + BD^*)S_D^{-1},$$

$$\tilde{X}(s) = R_D^{1/2} - R_D^{-1/2}(D^*S_{-1/2}B^* + B^*Q)(sI - A_P)^{-1}(BR_D^{-1} - PBS^{-1/2}S_D^{-1}D),$$

$$\tilde{Y}(s) = -R_D^{-1/2}(D^*S^{-1/2}B^* + B^*Q)(sI - A_P)^{-1}(PBS^{-1/2} + BD^*)S_D^{-1},$$

and Q and P are the unique strongly stabilizing solutions to (8) *and* (15), *respectively, with*

$$R_D = I + D^*D, \qquad S_D = I + DD^*, \qquad R = I,$$

$$A_P = A - (PBS^{-1/2} + BD^*)S_D^{-1}S^{-1/2}B^*,$$

$$A_Q = A - BR_D^{-1}(D^*S^{-1/2}B^* + B^*Q).$$

$N, M, \tilde{N}, \tilde{M}$ are in $\mathscr{M}\mathbf{H}_\infty$, but, in general, $X, Y, \tilde{X}, \tilde{Y}$ will only be in $\mathscr{M}\mathbf{H}_2$.

(b) *If $G(s)$ is positive real, and $(I+D)^{-1}S^{-1/2}$ is strictly positive, then $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ forms a normalized doubly coprime factorization of G over $\mathscr{M}\mathbf{H}_\infty$.*

Proof. (a.1) That $M, N, \tilde{M}, \tilde{N}$, etc., satisfy (16)–(22) follows exactly as in [6, Theorem 7.3.11 and Exercise 7.29]; it is just linear algebra and it makes use of the two Riccati equations (8) and (15).

(a.2) We show that N, M are elements of $\mathscr{M}\mathbf{H}_\infty$. (21) shows that M and N are in $\mathscr{M}\mathbf{L}_\infty(-\mathrm{j}\omega, \mathrm{j}\omega)$. Now from (7) we deduce that

$$\int_0^\infty \|B^* T_Q(t)z\|^2 \,\mathrm{d}t < \infty, \int_0^\infty \|B^* Q T_Q(t) z\|^2 \,\mathrm{d}t < \infty,$$

and so the inverse Laplace transforms of N, M, X and Y without the constant term are in $\mathbf{L}_2((0,\infty); \mathscr{L}(U))$, $U = C^m$. Lemma 11 shows that $N, M \in \mathscr{M}\mathbf{H}_\infty$.

(a.3) (22) shows that $\tilde{N}$ and $\tilde{M} \in \mathscr{M}\mathbf{L}_\infty(-\mathrm{j}\omega, \mathrm{j}\omega)$. A dual argument shows that

$$\int_0^\infty \|B^* T_P^*(t) z\|^2 \,\mathrm{d}t < \infty, \int_0^\infty \|B^* P T_P^*(t) z\|^2 \,\mathrm{d}t < \infty,$$

and thus the inverse Laplace transform of $\tilde{N}$ and $\tilde{M}$ are in $\mathbf{L}_2((0,\infty); \mathscr{L}(U))$, $U = C^m$. Lemma 11 now shows that $\tilde{N}, \tilde{M} \in \mathscr{M}\mathbf{H}_\infty$.

(b.1) We show that $G = NM^{-1}$ is a coprime factorization by establishing that it is related to the known coprime factorization $G = N_0 M_0^{-1}$ by a factor which is invertible over $\mathscr{M}\mathbf{H}_\infty$. We define

$$K = M_0^{-1} M = (I+G)M = M + N \in \mathscr{M}\mathbf{H}_\infty.$$

We find an explicit expression for

$$\begin{aligned} KR_D^{1/2} &= (I + D + S^{-1/2}B^*(sI-A)^{-1}B)(I - R_D^{-1}(D^*S^{-1/2}B^* + B^*Q)(sI - A_Q)^{-1}B) \\ &= (I+D) - (I+D)R_D^{-1}(D^*S^{-1/2}B^* + B^*Q)(sI-A_Q)^{-1}B + S^{-1/2}B^*(sI-A)^{-1} \\ &\quad \times \left[sI - A_Q - BR_D^{-1}(D^*S^{-1/2}B^* + B^*Q)\right](sI - A_Q)^{-1}B \\ &= (I+D) - (I+D)R_D^{-1}(D^*S^{-1/2}B^* + B^*Q)(sI-A_Q)^{-1}B + S^{-1/2}B^*(sI-A_Q)^{-1}B. \end{aligned}$$

Now we invert $(I+D)^{-1}KR_D^{1/2}$ to obtain

$$R_D^{-1/2}K^{-1}(I+D) = I + \left[R_D^{-1}(D^*S^{-1/2}B^* + B^*Q) - (I+D)^{-1}S^{-1/2}B^*\right](sI - \tilde{A})^{-1}B,$$

where

$$\begin{aligned} \tilde{A} &= A_Q + BR_D^{-1}(D^*S^{-1/2}B^* + B^*Q) - B(I+D)^{-1}S^{-1/2}B^* \\ &= A - B(I+D)^{-1}S^{-1/2}B^*. \end{aligned}$$

To prove that $K^{-1} \in \mathscr{M}\mathbf{H}_\infty$ we note that, under our assumptions, $B(I+D)^{-1}S^{-1/2}B^* = BFF^*B^*$ for an invertible F. By Lemma 2 (applied to BF), $\int_0^\infty \|F^*B^*T_{BF}^*(t)x\|^2\,\mathrm{d}t < \infty$ and since F^* is invertible, this shows that the inverse Laplace transform of K^{-1} is in $\mathbf{L}_2((0,\infty); \mathscr{L}(C^m))$. Next we show that $K^{-1} \in \mathscr{M}\mathbf{L}_\infty(-\mathrm{j}\omega, \mathrm{j}\omega; \mathscr{L}(C^m))$ and apply Lemma 11 to deduce that $K^{-1} \in \mathscr{M}\mathbf{H}_\infty$. Now (N, M) are normalized and so

$$N(\mathrm{j}\omega)^* N(\mathrm{j}\omega) + M(\mathrm{j}\omega)^* M(\mathrm{j}\omega) = I \quad \text{for } \omega \in R$$

and

$$K(\mathrm{j}\omega)^{-*}N(\mathrm{j}\omega)^*N(\mathrm{j}\omega)K(\mathrm{j}\omega)^{-1} + K(\mathrm{j}\omega)^{-*}M(\mathrm{j}\omega)^*M(\mathrm{j}\omega)K(\mathrm{j}\omega)^{-1}$$

$$= K(\mathrm{j}\omega)^{-*}K(\mathrm{j}\omega)^{-1}.$$

Thus,

$$N_0(\mathrm{j}\omega)^*N_0(\mathrm{j}\omega) + M_0(\mathrm{j}\omega)^*M_0(\mathrm{j}\omega) = K(\mathrm{j}\omega)^{-*}K(\mathrm{j}\omega)^{-1},$$

and since $N_0, M_0 \in \mathscr{M}\mathbf{H}_\infty$, we see that $K^{-1} \in \mathscr{M}\mathbf{L}_\infty$. So by Lemma 11, K and $K^{-1} \in \mathscr{M}\mathbf{H}_\infty$. The proof for $(\tilde{M}, \tilde{N})$ follows using a similar argument. Define $L = \tilde{M} + \tilde{N} \in \mathscr{M}\mathbf{H}_\infty$, and show that $L^{-1} \in \mathscr{M}\mathbf{H}_\infty$. Thus, with

$$\tilde{X}_1 = -\tilde{Y}_1 = K^{-1} \quad \text{and} \quad X_1 = -Y_1 = L^{-1},$$

we obtain a normalized doubly coprime factorization. □

In fact, we have proved some interesting properties of positive real systems.

Corollary 13. *Suppose that $G(s) = B^*(sI - A)^{-1}B$ is positive real and that assumptions* A1–A7 *are satisfied with $U = C^m, S = I$ and $D = 0$. Then the following transfer matrices are in $\mathscr{M}\mathbf{H}_\infty$:*

$$B^*(sI - A_B)^{-1}B, \qquad B^*Q(sI - A_B)^{-1}B,$$

$$B^*(sI - A_B)^{-1}PB, \qquad B^*(sI - A_Q)^{-1}B,$$

$$B^*Q(sI - A_Q)^{-1}B, \qquad B^*(sI - A_P)^{-1}PB$$

and

$$B^*(sI - A_P)^{-1}B,$$

where Q and P are the solutions of the Riccati equations (8) *and* (15), *respectively and $A_B = A - BB^*, A_Q = A - BB^*Q, A_P = A - PBB^*$.*

We remark that the extra conditions in b of Theorem 12 hold in the following special cases:
(i) $D = 0$, $S^{-1/2}B^*(sI - A)^{-1}B + B^*(\bar{s}I - A^*)^{-1}BS^{-1/2} \geqslant 0$ in $\mathrm{Re}(s) > 0$;
(ii) $D = \delta^2 I$, $S^{-1/2}B^*(sI - A)^{-1}B + B^*(\bar{s}I - A^*)^{-1}BS^{-1/2} \geqslant 0$ in $\mathrm{Re}(s) > 0$;
(iii) $S = \gamma^2 I, D^* + D \geqslant 0$.
It is interesting to conclude with the remark that we have also found a spectral factorization without assuming the above extra conditions.

Corollary 14. *Consider the state linear system $\Sigma(A_B, B, S^{-1/2}B^*, 0)$ under the assumptions* A1–A6 *with $R = I$ and $A_B = A - BB^*$. Then*

$$\Gamma(\mathrm{j}\omega) = I - B^*(\mathrm{j}\omega I + A_B^*)^{-1}BS^{-1}B^*(\mathrm{j}\omega I - A_B)^{-1}B \tag{27}$$

has the spectral factorization

$$\Gamma(\mathrm{j}\omega) = W(\mathrm{j}\omega)^*W(\mathrm{j}\omega) \quad \text{for } \omega \in R, \tag{28}$$

where

$$W(s) = I + B^*Q(sI - A_B)^{-1}B \tag{29}$$

with W and W^{-1} holomorphic in C_0^+, and Q is the unique stabilizing solution of (2).

Proof. It is easily verified that (28) holds using the Riccati equation (2). Note that

$$W(s)^{-1} = I - B^*Q(sI - A + BB^* + BB^*Q)^{-1}B$$

and that $W(s)^{-1} \in \mathbf{H}_\infty(\mathscr{L}(U))$ follows from Theorem 12 with $D = 0, R_D = I$, for then $W(s)^{-1} = M(s)$. □

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