

where  $K'$  is a constant depending on  $k_\theta$  and  $k_\alpha$ .

Substituting (18) and (A-27) into (A-29) gives

$$\begin{aligned} & \|\tilde{\theta}_i(t+1)^T \bar{\alpha}_{i,k}(t+1) - \tilde{\theta}_i(t)^T \bar{\alpha}_{i,k+1}(t)\| \\ & \leq c_{1,i} \|e_{[k-i+1,k]}(t+1)\| + c_{2,i} \|\chi_{[k-i+1,k]}(t+1)\| \\ & \quad + c_{3,i} \|z_{[k-i+2,k+1]}(t)\| \|\tilde{e}_{[1,i]}(t+1)\| \end{aligned} \quad (\text{A-30})$$

where  $c_{m,i}$ , ( $m = 1, 2, 3$ ) are constants combining  $c_{m,p}$ , ( $m = 1, 2, 3; 1 \leq p \leq i-1$ ),  $k_\alpha$  and  $k_\theta$ . Thus  $c_{m,i}$  ( $m = 1, 2, 3$ ) are dependent of  $k_\theta$  and  $k_\alpha$  only. So far we have proved the inequality (A-22).

Using (A-22), it follows immediately from the definition of  $\chi_i(t+1)$  that

$$\begin{aligned} \|\chi_i(t+1)\| & \leq c'_{1,i} \|e_{[1,i-1]}(t+1)\| + c'_{2,i} \|\chi_{[1,i-1]}(t+1)\| \\ & \quad + c'_{3,i} \|z_{[1,i]}(t)\| \|\tilde{e}_{[1,i-1]}(t+1)\| \end{aligned} \quad (\text{A-31})$$

where  $c'_{m,i}$ , ( $m = 1, 2, 3$ ) are constants.

Since  $\chi_1(t+1) = 0$  and  $\chi_2(t+1) \leq k_\theta k_\alpha |e_1(t+1)| + k_\alpha |z_2^t| |\tilde{e}_1(t+1)|$ , it can be shown from (A-31) that

$$\begin{aligned} \|\chi_i(t+1)\| & \leq c''_{1,i} \|e_{[1,i-1]}(t+1)\| \\ & \quad + c''_{2,i} \|z_{[1,i]}(t)\| \|\tilde{e}_{[1,i-1]}(t+1)\| \end{aligned} \quad (\text{A-32})$$

where  $c''_{1,i}$  and  $c''_{2,i}$  are constants combining  $k_\theta$  and  $k_\alpha$ .

Taking  $c_1 = \max_{1 \leq i \leq n} \{c''_{1,i}\}$  and  $c_2 = \max_{1 \leq i \leq n} \{c''_{2,i}\}$ , (A-8) follows.

Using (A-2) and inequality

$$\begin{aligned} \|z(t_0-1)\| \|\tilde{e}_i(t_0)\| & \leq \frac{\|z(t_0-1)\| \|e_i(t_0)\|}{(1 + \|\alpha_i(t_0-1)\|^2)^{1/2}} \\ & \leq \frac{b_u \|x(t_0-1)\| \|e_i(t_0)\|}{(1 + k'_\alpha \|x(t_0-1)\|^2)^{1/2}} \\ & \leq \frac{b_u}{k'_\alpha} ((k_\theta k_\alpha + a_1)M_0 + a_1) \end{aligned} \quad (\text{A-33})$$

(A-9) follows.  $\square$

*Remark A.1:* Note that  $M_0$  is not a design parameter. For any bounded  $x(0)$  and  $y_m(t)$ , such a constant  $M_0$  always exists.

*Remark A.2:* In Lemma A.2, it is noted that the update law has the same properties as those given in [8] if the nonparametric uncertainties are removed and all the system parameters are considered to be constants. Moreover, the constants  $a_1$ ,  $a'_1$ , and  $a_2$  are functions of  $\epsilon$  and  $\epsilon_\theta$ . They can be made sufficiently small by specifying sufficiently small  $\epsilon$  and  $\epsilon_\theta$ .

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## When Is $(D, G)$ -Scaling Both Necessary and Sufficient

Gjerrit Meinsma, Tetsuya Iwasaki, and Minyue Fu

**Abstract**—It is shown that the well-known  $(D, G)$ -scaling upper bound of the structured singular value is a nonconservative test for robust stability with respect to certain linear time-varying uncertainties.

**Index Terms**—Duality, IQC, linear matrix inequalities, mixed structured singular values, robustness, time-varying systems.

## I. INTRODUCTION

Is the closed-loop stable in Fig. 1 for all  $\Delta$ 's in a given set of stable operators  $\mathcal{B}$ ? That, roughly, is the fundamental robust stability problem.

There is an intriguing result by Megretski and Treil [4] and Shamma [8] which says, loosely speaking, that if  $M$  is a stable LTI operator and the set of  $\Delta$ 's is the set of contractive linear time-varying operators of some fixed block diagonal structure

$$\Delta = \text{diag}(\Delta_1, \Delta_2, \dots, \Delta_{m_F}) \quad (1)$$

that then the closed loop is robustly stable—that is, stable for all such  $\Delta$ 's—if and only if the  $\mathcal{H}_\infty$ -norm of  $DM D^{-1}$  is less than one for some constant diagonal matrix  $D$  that commutes with the  $\Delta$ 's. The problem can be decided in polynomial time, and it is a problem that has since long been associated with an *upper bound* of the structured singular value. The intriguing part is that the result holds for any number of LTV blocks  $\Delta_i$ , which is in stark contrast with the case that the  $\Delta_i$ 's are assumed time-invariant.

Paganini [6] extended this result by allowing for the more general block diagonal structure

$$\Delta = \text{diag}(\delta_1 I_{n_1}, \dots, \delta_{m_c} I_{n_{m_c}}, \Delta_1, \dots, \Delta_{m_F}). \quad (2)$$

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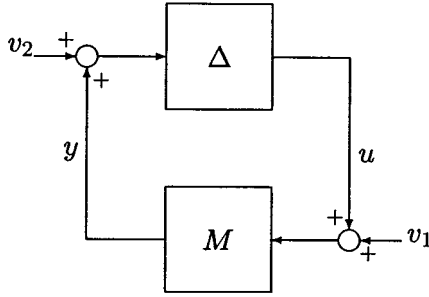


Fig. 1. The closed loop.

A precise definition is given in Section II. Paganini's result is an exact generalization and leads, again, to a convex optimization problem over the constant matrices  $D$  that commute with  $\Delta$ .

In view of the connection of these results with the upper bounds of the structured singular it is natural to ask if the well known  $(D, G)$ -scaling upper bound of the *mixed* structured singular value also has a similar interpretation. In this paper we show that that is indeed the case.

The  $(D, G)$ -scaling upper bound of the structured singular value was originally defined as a means to provide an easy-to-verify condition that guarantees robust stability with respect to the contractive linear *time-invariant* operators  $\Delta$  of the form

$$\Delta = \text{diag} \left( \tilde{\delta}_1 I_{\tilde{n}_1}, \dots, \tilde{\delta}_{m_r} I_{\tilde{n}_{m_r}}, \delta_1 I_{n_1}, \dots, \delta_{m_c} I_{n_{m_c}}, \Delta_1, \dots, \Delta_{m_F} \right) \quad (3)$$

with  $\tilde{\delta}_i$  denoting real-valued constants [1]. It is known that for general LTI plants  $M$  this sufficient condition is *necessary* as well if and only if

$$2(m_r + m_c) + m_F \leq 3.$$

(See [5].) It is natural to expect that the  $(D, G)$ -scaling condition becomes less conservative if the parameters  $\tilde{\delta}_i$  are relaxed to be real-valued LTV operators. However, no quantitative results about this are available in the literature.

In this paper we show that the  $(D, G)$ -scaling condition is in fact both necessary and sufficient for robust stability with respect to the contractive LTV operators  $\Delta$  of the form (3) with now  $\tilde{\delta}_i$  denoting linear *time-varying self-adjoint* operators on  $\ell_2$ . A precise definition follows. Paganini [7] has gone through considerable trouble to show that for his structure (2) one may assume causality of  $\Delta$  without changing the condition. In the extended structure (3) with self-adjoint  $\tilde{\delta}_i$  this is no longer possible.

## II. NOTATION AND PRELIMINARIES

$\ell_2 := \{x : \mathbb{Z} \mapsto \mathbb{R} : \sum_{k \in \mathbb{Z}} x^2(k) < \infty\}$ . The norm  $\|v\|_2$  of  $v \in \ell_2$  is the usual norm on  $\ell_2$  and for vector-valued signals  $v \in \ell_2^n$  the norm  $\|v\|_2$  is defined as  $(\|v_1\|_2^2 + \dots + \|v_n\|_2^2)^{1/2}$ . The induced norm is denoted by  $\|\cdot\|$ . So, for  $F : \ell_2^n \mapsto \ell_2^m$  it is defined as  $\|F\| := \sup_{u \in \ell_2^n} \|Fu\|_2 / \|u\|_2$ . For matrices  $F \in \mathbb{C}^{n \times m}$  the induced norm will be the spectral norm, and for vectors this reduces to the Euclidean norm.

$F^H$  is the complex conjugate transpose of  $F$ , and  $\text{He } F$  is the Hermitian part  $F$  defined as  $\text{He } F = (1/2)(F + F^H)$ .

An operator  $\Delta : \ell_2^n \mapsto \ell_2^n$  is said to be *contractive* if  $\|\Delta v\|_2 \leq \|v\|_2$  for every  $v \in \ell_2^n$ . Lower case  $\delta$ 's always denote operators from  $\ell_2^1$  to  $\ell_2^1$ . Then for  $u, y \in \ell_2^1$  the expression  $y = \delta I_n u$  is defined to mean that the entries  $y_k$  of  $y$  satisfy  $y_k = \delta u_k$ . An operator  $\delta : \ell_2 \mapsto \ell_2$

is *self-adjoint* if  $\langle u, \delta v \rangle = \langle \delta u, v \rangle$  for all  $u, v \in \ell_2$ . Note that all real-valued static LTV operators are self-adjoint.

The  $M$  and  $\Delta$  throughout denote bounded operators from  $\ell_2^n$  to  $\ell_2^n$  and  $M$  is assumed linear time invariant (LTI). Bounded operators on  $\ell_2^n$  are also called *stable*. Note that this notion of stability does not require the operator to be causal, thus is less restrictive than the standard notion of stability for linear control systems. The reader should keep this in mind in interpreting the results of this paper.

Hats will denote  $Z$ -transforms, so if  $y \in \ell_2$  then  $\hat{y}(z)$  is defined as  $\hat{y}(z) = \sum_{k \in \mathbb{Z}} y(k) z^{-k}$ . To avoid clutter we shall use for functions  $\hat{f}$  of frequency the notation

$$\hat{f}_\omega := \hat{f}(e^{i\omega}).$$

### A. Stability

The closed loop depicted in Fig. 1 is called *internally stable* (or simply *stable*) if the map from  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  to  $\begin{bmatrix} u \\ y \end{bmatrix}$  is bounded as a map from  $\ell_2^{2n}$  to  $\ell_2^{2n}$ . Because of stability of  $M$  and  $\Delta$ , we claim that the closed loop is stable iff  $(I - \Delta M)^{-1}$  is bounded. This property is well-known for the case where  $M$ ,  $\Delta$  and  $(I - \Delta M)^{-1}$  are causal. To see that this property holds without the causality requirement, we define

$$\tilde{u} = u + v_1; \quad \tilde{y} = y + v_2.$$

Obviously, the map from  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  to  $\begin{bmatrix} u \\ y \end{bmatrix}$  is stable iff the map from  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  to  $\begin{bmatrix} \tilde{u} \\ \tilde{y} \end{bmatrix}$  is stable. From Fig. 1, the latter map is given by

$$\begin{bmatrix} I & -\Delta \\ -M & I \end{bmatrix}^{-1}$$

which is bounded iff  $(I - \Delta M)^{-1}$  is bounded.

The closed loop in Fig. 1 will be called *uniformly robustly stable* with respect to some set  $\mathcal{B}$  of stable LTV operators if there exists  $\gamma > 0$  such that

$$\left\| \begin{bmatrix} u \\ y \end{bmatrix} \right\|_2 \leq \gamma \left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_2 \quad \forall \Delta \in \mathcal{B}, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \ell_2^{2n}. \quad (4)$$

In another word, uniform robust stability means that the maximum  $l_2$  gain from  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  to  $\begin{bmatrix} u \\ y \end{bmatrix}$  is bounded by a constant which is independent of the uncertainties.

We only consider  $\Delta$ 's with norm at most one and stable  $M$ . In that case (4) holds if and only if there is an  $\epsilon > 0$  such that

$$\|(I - \Delta M)u\|_2 \geq \epsilon \|u\|_2 \quad \forall \Delta \in \mathcal{B}, \quad u \in \ell_2^n.$$

### B. The $\Delta$ 's and the $(D, G)$ -Scaling Matrices

Throughout we assume that  $\Delta : \ell_2^n \mapsto \ell_2^n$  and that  $\Delta$  is of the form

$$\Delta = \text{diag} \left( \tilde{\delta}_1 I_{\tilde{n}_1}, \dots, \tilde{\delta}_{m_r} I_{\tilde{n}_{m_r}}, \delta_1 I_{n_1}, \dots, \delta_{m_c} I_{n_{m_c}}, \Delta_1, \dots, \Delta_{m_F} \right) \quad (5)$$

with

$$\begin{cases} \tilde{\delta}_i : \ell_2 \mapsto \ell_2 & \text{LTV, self-adjoint and } \|\tilde{\delta}_i\| \leq 1 \\ \delta_i : \ell_2 \mapsto \ell_2 & \text{LTV and } \|\delta_i\| \leq 1 \\ \Delta_i : \ell_2^{q_i} \mapsto \ell_2^{q_i} & \text{LTV and } \|\Delta_i\| \leq 1. \end{cases} \quad (6)$$

The dimensions and numbers  $\tilde{n}_i, n_i, q_i, m_r, m_c, m_F$  of the various identity matrices and  $\Delta_i$  blocks are fixed, but otherwise  $\Delta$  may vary

over all possible  $n \times n$  LTV operators of the form (5), (6). Given that, the sets  $\mathcal{D}$  and  $\mathcal{G}$  of  $D$  and  $G$ -scales are defined accordingly as

$$\begin{aligned} \mathcal{D} &= \left\{ D = \text{diag} \left( \tilde{D}_1, \dots, \tilde{D}_{m_r}, D_1, \dots, D_{m_c}, \right. \right. \\ &\quad \left. \left. d_1 I_{q_1}, \dots, d_{m_F} I_{q_{m_F}} \right) : 0 < D = D^T \in \mathbb{R}^{n \times n} \right\} \\ \mathcal{G} &= \left\{ G = \text{diag} \left( \tilde{G}_1, \dots, \tilde{G}_{m_r}, 0, \dots, 0, \right. \right. \\ &\quad \left. \left. 0, \dots, 0 \right) : G = G^H \in j\mathbb{R}^{n \times n} \right\}. \end{aligned}$$

Note that the  $D$ -scales are assumed real-valued and that the  $G$ -scales are taken to be purely imaginary. As it turns out there is no need to consider a wider class of  $D$  and  $G$ -scales.

### III. THE DISCRETE-TIME RESULT

*Theorem 3.1:* The discrete-time closed-loop in Fig. 1 with stable LTI plant with transfer matrix  $M$  is uniformly robustly stable with respect to  $\Delta$ 's of the form (5), (6) if and only if there is a constant matrix  $D \in \mathcal{D}$  and a constant matrix  $G \in \mathcal{G}$  such that

$$M_\omega^H D M_\omega + j \left( G M_\omega - M_\omega^H G \right) - D < 0 \quad \forall \omega \in [0, 2\pi]. \quad (7)$$

□

The existence of such  $D$  and  $G$  can be tested in polynomial time. The remainder of this paper is devoted to a proof of this result. Megretski [3] showed this for the full blocks case (1); Paganini [6] derived this result for the case that the  $\Delta$ 's are of the form (2). The proof of the general case (5) follows the same lines as that of [6] and [5]. A key idea is to replace the condition of the contractive  $\Delta$ -blocks with an integral quadratic condition independent of  $\Delta$ :

*Lemma 3.2:* Let  $u, y \in \ell_2^q$  and consider the quadratic integral

$$\Sigma(u, y) := \int_0^{2\pi} (\hat{y}_\omega - \hat{u}_\omega)(\hat{y}_\omega + \hat{u}_\omega)^H d\omega \in \mathbb{R}^{q \times q}. \quad (8)$$

The following hold.

1. There is a contractive self-adjoint LTV  $\tilde{\delta} : \ell_2 \mapsto \ell_2$  such that  $u = \tilde{\delta} I_q y$  if and only if  $\Sigma(u, y)$  is Hermitian and nonnegative definite.
2. There is a contractive LTV  $\delta : \ell_2 \mapsto \ell_2$  such that  $u = \delta I_q y$  if and only if the Hermitian part of  $\Sigma(u, y)$  is nonnegative definite.
3. There is a contractive LTV  $\Delta : \ell_2^q \mapsto \ell_2^q$  such that  $u = \Delta y$  if and only if the trace of  $\Sigma(u, y)$  is nonnegative.

*Proof:* See the Appendix. ■

A consequence of this result is the following.

*Lemma 3.3:* Let  $u$  be a nonzero element of  $\ell_2^n$ . Then  $(I - \Delta M)u = 0$  for some  $\Delta$  of the form (5), (6) iff

$$\Sigma(u, M u) := \int_0^{2\pi} (M_\omega - I) \hat{u}_\omega \hat{u}_\omega^H (M_\omega + I)^H d\omega \quad (9)$$

is of the form, shown in (10) at the bottom of the page, with  $\tilde{Z}_i = \tilde{Z}_i^T \geq 0$ ,  $\text{He } \tilde{Z}_i \geq 0$ ,  $\text{Tr } Z_i \geq 0$ , and with “?” denoting an irrelevant entry. Here the partitioning of (10) is compatible with that of  $\Delta$ .

*Proof (Sketch):* The equation  $(I - \Delta M)u = 0$  is the same as

$$u = \Delta M u.$$

With appropriate partitionings, the expression  $u = \Delta M u$  can be written row-block by row-block as

$$\begin{aligned} u_1 &= \tilde{\delta}_1 M_1 u \\ u_2 &= \tilde{\delta}_2 M_2 u \\ &\vdots \\ u_K &= \Delta_{m_F} M_K u. \end{aligned}$$

By Lemma 3.2 there exist contractive  $\tilde{\delta}_i, \delta_i$  and  $\Delta_i$  of the form (6) for which the above equalities hold iff certain quadratic integrals  $\Sigma_i$  have certain properties. It is not too difficult to figure out that these quadratic integrals  $\Sigma_i$  are exactly the blocks on the diagonal of  $\Sigma(u, M u)$ , and that the conditions on these blocks are that they satisfy  $\Sigma_i = \Sigma_i^T \geq 0$ ,  $\text{He } \Sigma_i \geq 0$ , or  $\text{Tr } \Sigma_i \geq 0$ , corresponding to the three types of uncertainties. ■

*Proof of Theorem 3.1:* Suppose such  $D \in \mathcal{D}$  and  $G \in \mathcal{G}$  exist. Then a standard argument will show that there is an  $\epsilon > 0$  such that  $\|(I - \Delta M)u\|_2 \geq \epsilon \|u\|_2$  for all  $u$  and contractive  $\Delta$  of the form (5). This is the definition of uniformly robustly stable.

Conversely suppose the closed loop is uniformly robustly stable. For some  $\epsilon > 0$ , then,  $\|(I - \Delta M)u\|_2 \geq \epsilon$  for every  $u$  of unit norm. Define

$$\mathcal{W} := \{ \Sigma(u, M u) : \|u\|_2 = 1 \} \subset \mathbb{R}^{n \times n}. \quad (11)$$

By application of Lemma 3.3, the set  $\mathcal{W}$  does not intersect the convex cone  $\mathcal{Z}$  defined as

$$\begin{aligned} \mathcal{Z} &:= \{ Z : Z \text{ is of the form (10) with} \\ &\quad \tilde{Z}_i = \tilde{Z}_i^T \geq 0, \text{He } \tilde{Z}_i \geq 0, \text{Tr } Z_i \geq 0 \}. \end{aligned}$$

In the Appendix we show that in fact  $\mathcal{W}$  is bounded away from  $\mathcal{Z}$ . Remarkably the closure  $\bar{\mathcal{W}}$  of  $\mathcal{W}$  is convex. This observation is from Megretski and Treil [4], and for completeness a proof is listed in the Appendix, Lemma 5.1. Because  $\mathcal{W}$  is bounded away from  $\mathcal{Z}$ , also the closure  $\bar{\mathcal{W}}$  is bounded away from  $\mathcal{Z}$ , so there is a  $\gamma > 0$  such that  $\bar{\mathcal{W}}$  also does not intersect

$$\mathcal{Z}_\gamma := \mathcal{Z} + \{ Z \in \mathbb{R}^{n \times n} : \|Z\| \leq \gamma \}.$$

Both  $\bar{\mathcal{W}}$  and  $\mathcal{Z}_\gamma$  are convex and have empty intersection, and therefore a hyper-plane exists that separates the two sets [2, p. 133]. In other

$$\left[ \begin{array}{ccc|ccc|ccc} \tilde{Z}_1 & ? & ? & ? & ? & ? & ? & ? & ? \\ ? & \tilde{Z}_2 & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & \ddots & ? & ? & ? & ? & ? & ? \\ \hline ? & ? & ? & \tilde{Z}_1 & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & \tilde{Z}_2 & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & \ddots & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? & Z_1 & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & Z_2 & ? \\ ? & ? & ? & ? & ? & ? & ? & ? & \ddots \end{array} \right] \in \mathbb{R}^{n \times n} \quad (10)$$

words there is a nonzero matrix  $E \in \mathbb{R}^{n \times n}$  (say of unit norm) such that<sup>1</sup>

$$\langle E, \bar{W} \rangle \leq \langle E, \mathcal{Z}_\gamma \rangle. \quad (12)$$

As an inner product take  $\langle X, Y \rangle = \text{Tr } X^T Y$ . In particular (12) says that  $\langle E, \mathcal{Z} \rangle$  is bounded from below. By Lemma 5.3 that is the case if and only if  $E$  is of the form

$$E = \text{diag} \left( \tilde{E}_1, \dots, \tilde{E}_{m_r}, E_1, \dots, E_{m_c}, e_1 I, \dots, e_{m_F} I \right)$$

with  $\tilde{E}_i + \tilde{E}_i^T \geq 0$ ,  $E_i = E_i^T \geq 0$  and  $0 \leq e_i \in \mathbb{R}$ , that is, if and only if  $E \in \mathcal{D} + j\mathcal{G}$ . In that case  $\inf \langle E, \mathcal{Z} \rangle = 0$ , and so

$$a_\gamma := \inf \langle E, \mathcal{Z}_\gamma \rangle < 0.$$

From (12) we thus see that  $\langle E, \bar{W} \rangle \leq a_\gamma < 0$ . If  $\|u\|_2 = 1$ , then

$$\begin{aligned} & \int_0^{2\pi} \hat{u}_\omega^H (\text{He}(M_\omega + I))^H E (M_\omega - I) \hat{u}_\omega d\omega \\ &= \text{Re} \text{Tr} \int_0^{2\pi} E (M_\omega - I) \hat{u}_\omega \hat{u}_\omega^H (M_\omega + I)^H d\omega \\ &= \langle E, \Sigma(u, Mu) \rangle \leq \sup \langle E, \mathcal{W} \rangle \leq a_\gamma < 0. \end{aligned} \quad (13)$$

This being at most  $a_\gamma < 0$  for every  $u \in \ell_2^n$ ,  $\|u\|_2 = 1$  implies that

$$\text{He}(M_\omega + I)^H (E + \epsilon I) (M_\omega - I) < 0 \quad \forall \omega \in [0, 2\pi] \quad (14)$$

for some small enough  $\epsilon > 0$ . Express  $E + \epsilon I$  as  $E + \epsilon I = D + jG$  for some  $D \in \mathcal{D}$  and  $G \in \mathcal{G}$ . Then (14) becomes (7). ■

#### IV. THE CONTINUOUS-TIME RESULT

Analogous to the discrete-time case we say that a continuous-time system is *uniformly robustly stable* if there is a  $\gamma > 0$  such that (4) holds for all  $v_1, v_2 \in L_2$ . Completely analogous to the discrete-time case it can be shown that the following holds.

**Theorem 4.1:** The continuous-time closed-loop in Fig. 1 with stable LTI plant with transfer matrix  $M$  is uniformly robustly stable with respect to  $\Delta$ 's of the form (5) with

$$\begin{cases} \tilde{\delta}_i : L_2 \mapsto L_2 & \text{LTV, self-adjoint and } \|\tilde{\delta}_i\| \leq 1 \\ \delta_i : L_2 \mapsto L_2 & \text{LTV and } \|\delta_i\| \leq 1 \\ \Delta_i : L_2^{q_i} \mapsto L_2^{q_i} & \text{LTV and } \|\Delta_i\| \leq 1 \end{cases}$$

if and only if there is a constant matrix  $D \in \mathcal{D}$  and a constant matrix  $G \in \mathcal{G}$  such that

$$M(j\omega)^H D M(j\omega) + j(GM(j\omega) - M(j\omega)^H G) - D < 0$$

for all  $\omega \in \mathbb{R} \cup \infty$ . □

#### APPENDIX

**Proof of Lemma 3.2:** Items 2 and 3 are proved in [6] (note that the Hermitian part of (8) is  $\int_0^{2\pi} \hat{y}_\omega \hat{y}_\omega^H - \hat{u}_\omega \hat{u}_\omega^H d\omega$ , and its trace equals  $2\pi(\|y\|_2^2 - \|u\|_2^2)$ ).

If  $u := \tilde{\delta} I_q y$  with  $\tilde{\delta}$  self-adjoint and contractive then (8) is easily seen to be Hermitian and  $\geq 0$ . Conversely suppose (8) is Hermitian and

<sup>1</sup>In (12) the expression  $\langle E, \bar{W} \rangle$  denotes the set  $\{x : x = \langle E, Y \rangle, Y \in \bar{W}\}$  and the inequality in (12) is defined to mean that every element of the set on the left-hand side,  $\langle E, \bar{W} \rangle$ , is less than or equal to every element of the set on the right-hand side,  $\langle E, \mathcal{Z}_\gamma \rangle$ .

nonnegative. Now let  $\{f_i\}_{i=0,1,2,\dots}$  be an orthonormal basis of  $\ell_2$ , and expand  $y \in \ell_2^q$  in this basis

$$y = \sum_{j=0,1,\dots} \gamma(j) f_j, \quad \gamma(j) \in \mathbb{R}^q.$$

We may associate with this expansion the matrix  $Y \in \mathbb{R}^{\infty \times q}$  of coefficients

$$Y = \begin{bmatrix} \gamma_1(0) & \gamma_2(0) & \cdots & \gamma_q(0) \\ \gamma_1(1) & \gamma_2(1) & \cdots & \gamma_q(1) \\ \gamma_1(2) & \gamma_2(2) & \cdots & \gamma_q(2) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

The matrix  $U$  is likewise defined from  $u$ . In this matrix notation the expression  $u = \tilde{\delta} I_q y$  becomes  $U = \tilde{\Delta} Y$ , and the quadratic integral (8) becomes

$$\Sigma(u, y) = (Y^T - U^T)(Y + U).$$

By assumption the above is Hermitian and nonnegative definite, that is,

$$Y^T U = U^T Y \quad \text{and} \quad U^T U \leq Y^T Y. \quad (15)$$

We may assume without loss of generality that the orthonormal basis  $\{f_j\}$  was chosen such that the first, say  $p$ , elements  $\{f_1, \dots, f_p\}$  span the space spanned by the entries  $\{y_1, \dots, y_q\}$  of  $y$ . Then  $Y$  is of the form

$$Y = \begin{bmatrix} I_p \\ 0_{\infty \times p} \end{bmatrix} C \quad \text{for some full row rank } C \in \mathbb{R}^{p \times q}.$$

Then the second inequality of (15) is that  $U^T U \leq C^T C$ . This implies that  $U$  is of the form  $U = VC$  for some  $V$ . Partition  $V$  as  $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$  with  $V_1 \in \mathbb{R}^{p \times p}$ . The two formulas of (15) then become

$$\begin{aligned} C^T V_1 C &= C^T V_1^T C \quad \text{and} \\ C^T (V_1^T V_1 + V_2^T V_2) C &\leq C^T I_p C. \end{aligned} \quad (16)$$

As  $C$  has full row rank, (16) is equivalent to that

$$V_1 = V_1^T \quad \text{and} \quad V_1^T V_1 + V_2^T V_2 \leq I_p.$$

It is now immediate that  $U$  equals  $U = \tilde{\Delta} Y$  for  $\tilde{\Delta}$  defined as

$$\tilde{\Delta} := \begin{bmatrix} V_1 & V_2^T \\ V_2 & -V_2 V_1 (I - V_1^2)^{-1} V_2^T \end{bmatrix}. \quad (17)$$

It is easy to verify that  $\tilde{\Delta}$  is contractive. Furthermore  $\tilde{\Delta}$  is symmetric and so the corresponding operator  $\tilde{\delta}$  is self-adjoint.

(It may happen that  $I - V_1^2$  is singular. In that case the inverse in (17) may be replaced with the Moore–Penrose inverse.) ■

**Lemma 5.1:** The closure of (11) is convex.

**Proof:** The proof hinges on the fact that  $\lim_{N \rightarrow \infty} \langle u, \sigma^N v \rangle = 0$  for every pair  $u, v \in \ell_2^q$  and with  $\sigma^N$  denoting the  $N$ -step delay.

Let  $u, v \in \ell_2^q$  both have unit norm, i.e.,  $\Sigma(u, Mu), \Sigma(v, Mv) \in \mathcal{W}$ . Given  $N \in \mathbb{N}$  and  $\lambda \in [0, 1]$  define  $x$  as

$$x := \sqrt{\lambda} u + \sqrt{1 - \lambda} \sigma^N v.$$

Since  $\Sigma$  is linear in its two arguments, we have that

$$\begin{aligned} \Sigma(x, Mx) &= \lambda \Sigma(u, Mu) + \sqrt{1 - \lambda} \sqrt{\lambda} \Sigma(u, M \sigma^N v) \\ &\quad + \sqrt{1 - \lambda} \sqrt{\lambda} \Sigma(\sigma^N v, Mu) \\ &\quad + (1 - \lambda) \Sigma(v, Mv). \end{aligned}$$

As  $N \rightarrow \infty$  the contributions of  $\Sigma(u, M \sigma^N v)$  and  $\Sigma(\sigma^N v, Mu)$  tend to zero, so

$$\lim_{N \rightarrow \infty} \Sigma(x, Mx) = \lambda \Sigma(u, Mu) + (1 - \lambda) \Sigma(v, Mv).$$

That this is an element of the closure of (11) follows from the fact that  $\lim_{N \rightarrow \infty} \|x\|_2^2 = \lambda \|u\|_2^2 + (1 - \lambda) \|v\|_2^2 = 1$ . ■

*Lemma 5.2:* Uniform robust stability implies that  $\mathcal{W}$  is bounded away from  $\mathcal{Z}$ .

*Proof:* Suppose to the contrary that

$$\inf_{u \in \ell_2^n, \|u\|_2=1, Z \in \mathcal{Z}} \|\Sigma(u, Mu) - Z\| = 0.$$

This means that there is a sequence  $\{u^k, Q_k\}_{k \in \mathbb{N}} \subset \ell_2^n \times \mathbb{R}^{n \times n}$  such that

$$\Sigma(u^k, Mu^k) + Q_k \in \mathcal{Z}, \quad \|u^k\|_2 = 1, \quad \lim_{k \rightarrow \infty} \|Q_k\| = 0.$$

For each  $k$  define  $y^k := Mu^k \in \ell_2^n$  and take  $z^k$  to be any element of  $\ell_2^n$  whose entries are mutually orthogonal and have unit norm,  $\langle z_i^k, z_j^k \rangle = \delta_{ij}$ , and whose entries are also orthogonal to all entries of  $u^k$  and  $y^k$ . With it define

$$\begin{aligned} \bar{u}^k &:= u^k + \frac{1}{2} \left( \sqrt{\|Q_k\|} I_n - \frac{1}{\sqrt{\|Q_k\|}} Q_k \right) z^k \\ \bar{y}^k &:= y^k + \frac{1}{2} \left( \sqrt{\|Q_k\|} I_n + \frac{1}{\sqrt{\|Q_k\|}} Q_k \right) z^k. \end{aligned}$$

The reason for this definition is that now

$$\begin{aligned} \Sigma(\bar{u}^k, \bar{y}^k) &= \int_0^{2\pi} \left( \hat{y}_\omega^k - \hat{u}_\omega^k + \frac{1}{\sqrt{\|Q_k\|}} Q_k \hat{z}_\omega^k \right) \\ &\quad \times \left( \hat{y}_\omega^k + \hat{u}_\omega^k + \sqrt{\|Q_k\|} \hat{z}_\omega^k \right)^H d\omega \\ &= \Sigma(u^k, y^k) + Q_k \in \mathcal{Z}. \end{aligned}$$

So we see that  $\Sigma(\bar{u}^k, \bar{y}^k)$  is an element of  $\mathcal{Z}$  and, hence,  $\bar{u}^k = \Delta^k \bar{y}^k$  for some contractive  $\Delta^k$  of the form (5), (6). Finally consider

$$\begin{aligned} (I - \Delta^k M) \bar{u}^k &= \bar{u}^k - \Delta^k M(u^k + (\bar{u}^k - u^k)) \\ &= \bar{u}^k - \Delta^k (y^k + M(\bar{u}^k - u^k)) \\ &= \bar{u}^k - \Delta^k (\bar{y}^k + (y^k - \bar{y}^k)) - \Delta^k M(\bar{u}^k - u^k) \\ &= -\Delta^k (y^k - \bar{y}^k) - \Delta^k M(\bar{u}^k - u^k). \end{aligned} \quad (18)$$

Using the fact that  $\|\bar{u}^k - u^k\|_2 = \mathcal{O}(\sqrt{\|Q_k\|})$ ,  $\|\bar{y}^k - y^k\|_2 = \mathcal{O}(\sqrt{\|Q_k\|})$  and that  $\lim_{k \rightarrow \infty} \|Q_k\| = 0$ , we obtain from (18) that

$$\lim_{k \rightarrow \infty} (I - \Delta^k M) \bar{u}^k = 0, \quad \lim_{k \rightarrow \infty} \|\bar{u}^k\|_2 = 1.$$

This contradicts uniform robust stability.  $\blacksquare$

*Lemma 5.3:*  $\inf_{Z \in \mathcal{Z}} \text{Tr } E^T Z$  is bounded from below for some  $E \in \mathbb{R}^{n \times n}$  if and only if  $E$  is of the form

$$E = \text{diag} \left( \tilde{E}_1, \dots, \tilde{E}_{m_r}, E_1, \dots, E_{m_c}, e_1 I, \dots, e_{m_F} I \right)$$

with  $\tilde{E}_i + \tilde{E}_i^T \geq 0$ ,  $E_i = E_i^T \geq 0$  and  $e_i \geq 0$ .

*Proof:* Suppose that  $\inf_{Z \in \mathcal{Z}} \text{Tr } E^T Z$  is bounded from below. The off-diagonal blocks of  $E$  are then zero for the following reason: Let  $F$  be equal to  $E$  but with its blocks on the diagonal equal to zero. The off-diagonal blocks of  $Z \in \mathcal{Z}$  are not restricted in any way so  $Z := \lambda F$  is an element of  $\mathcal{Z}$  for every  $\lambda \in \mathbb{R}$ . If  $F$  is nonzero then  $\text{Tr } E^T Z = \text{Tr } E^T (\lambda F) = \lambda \text{Tr } F^T F$  and this is unbounded from below as a function of  $\lambda$ . Therefore  $F$  must be zero, i.e.,  $E$  is block-diagonal.

The general form of a block-diagonal  $E$  is

$$E = \text{diag} \left( \tilde{E}_1, \dots, \tilde{E}_{m_r}, E_1, \dots, E_{m_c}, \tilde{E}_1, \dots, \tilde{E}_{m_F} \right)$$

Express  $Z$  as in (10). Then

$$\text{Tr } E^T Z = \sum \text{Tr } \tilde{E}_i^T \tilde{Z}_i + \sum \text{Tr } E_i^T \bar{Z}_i + \sum \text{Tr } \tilde{E}_i^T Z_i.$$

Each block of  $Z \in \mathcal{Z}$  can vary independently of all other blocks of  $Z$ , so the only way that the above is bounded from below is that all

$$\begin{aligned} \inf_{\tilde{Z}_i = \tilde{Z}_i^T \geq 0} \text{Tr } \tilde{E}_i^T \tilde{Z}_i, \quad \inf_{\text{He } \bar{Z}_i \geq 0} \text{Tr } E_i^T \bar{Z}_i \quad \text{and} \\ \inf_{\text{Tr } Z_i \geq 0} \text{Tr } \tilde{E}_i^T Z_i \end{aligned}$$

are bounded from below. It is fairly easy to show that

$$\begin{aligned} \inf_{\tilde{Z}_i = \tilde{Z}_i^T \geq 0} \text{Tr } \tilde{E}_i^T \tilde{Z}_i > -\infty &\Leftrightarrow \text{He } \tilde{E}_i \geq 0 \\ \inf_{\text{He } \bar{Z}_i \geq 0} \text{Tr } E_i^T \bar{Z}_i > -\infty &\Leftrightarrow E_i = E_i^T \geq 0 \\ \inf_{\text{Tr } Z_i \geq 0} \text{Tr } \tilde{E}_i^T Z_i > -\infty &\Leftrightarrow \tilde{E}_i = e_i I, \quad 0 < e_i \in \mathbb{R}. \end{aligned}$$

(This is considered in more detail in [5].)  $\blacksquare$

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