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Some approaches to a conjecture on short cycles in digraphs

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Abstract

We consider the following special case of a conjecture due to Caccetta and Häggkvist: Let D be a digraph on n vertices that all have in-degree and out-degree at least $n/3$. Then, D contains a directed cycle of length 2 or 3. We discuss several necessary conditions for possible counterexamples to this conjecture, in terms of cycle structure, diameter, maximum degree, clique number, toughness, and local structure. These conditions have not enabled us to prove or refute the conjecture, but they lead to proofs of special instances of the conjecture.

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1. Preliminaries

Throughout we consider only digraphs without multiple arcs and without directed cycles of length 1 or 2. Such digraphs are sometimes called orientations.

Let $D=(V,A)$ be a digraph with vertex set V and arc set A. By n_D or just n we denote the number of vertices of D, i.e. $n=|V|$. Let $v\in V$ be a vertex of D. By $N^+(v)$ we denote the set of out-neighbors of v in D, i.e. $N^+(v)=\{u\in V \mid vu\in A\}$; by $N^-(v)$ we denote the set of in-neighbors of v, i.e. $N^-(v)=\{u\in V \mid uv\in A\}$. The out-degree and in-degree of v are $d^+(v)=|N^+(v)|$ and $d^-(v)=|N^-(v)|$, respectively. We set

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$\delta^+ = \min_{v\in V} d^+(v)$ and $\delta^- = \min_{v\in V} d^-(v)$, and $\delta = \min\{\delta^+, \delta^-\}$. For a nonempty subset $S \subseteq V$, we set $N^+(S) = \bigcup_{v\in S} N^+(v)$ and

$$N^{++}(v) = N^+(N^+(v)) \setminus N^+(v),$$

$$N^{--}(v) = N^-(N^-(v)) \setminus N^-(v).$$

The girth of D, denoted by $g(D)$, is the length of a shortest directed cycle of D. D is d-regular if each vertex of D has in-degree and out-degree d.

Behzad et al. [1] conjectured the following.

Conjecture 1. *Let D be a d-regular digraph. Then $g(D) \leqslant \lceil n/d \rceil$.*

Caccetta and Häggkvist [3] proposed a generalization of Conjecture 1, requiring merely a lower bound on the out-degrees of D.

Conjecture 2. *Let D be a digraph with $\delta^+ \geqslant d$. Then $g(D) \leqslant \lceil n/d \rceil$.*

In fact, as is easily seen, Conjecture 2 could be equivalently stated for digraphs in which all out-degrees are exactly d.

Conjecture 2 has been verified for values of d up to 5 [3,6,7], while in [4] Chvátal and Szemerédi established the bound $n/d + 2500$ for arbitrary values of d. Nishimura [9] refined their proof, reducing the additive constant from 2500 to 304.

Here, we focus on the special case $d = n/3$ of Conjectures 1 and 2 that are still open. We denote by $\vec{\Delta}$ a directed cycle of length 3. In fact, we consider the following two related conjectures.

Conjecture 3. *Let D be a digraph with $\delta^+ \geqslant n/3$. Then D contains a $\vec{\Delta}$.*

Conjecture 4. *Let D be a digraph with $\delta \geqslant n/3$. Then D contains a $\vec{\Delta}$.*

For several years, the best known results along these lines were due to Bondy [2] and De Graaf et al. [5], respectively.

Theorem 5. *Let D be a digraph with $\delta^+ \geqslant [(2\sqrt{6} - 3)/5]n$. Then D contains a $\vec{\Delta}$.*

Theorem 6. *Let D be a digraph with $\delta \geqslant [4/(20 - 4\sqrt{5} + 7\sqrt{2} - 3\sqrt{10})]n$. Then D contains a $\vec{\Delta}$.*

These results were recently improved by Shen [10]. A Master's student of the second author further improved the bound in Theorem 5 to $\delta^+ \geqslant 0.34096n$ [8].

As observed by Seymour (see [2]), Conjecture 4 would be implied if every digraph (or every digraph with $\delta \geqslant n/3$) has a vertex v such that $|N^{++}(v)| \geqslant |N^+(v)|$. Hence, a necessary condition for a possible counterexample to Conjecture 4 is that every vertex

v satisfies $|N^{++}(v)| < |N^+(v)|$. In this paper, we discuss several other necessary conditions for possible counterexamples to Conjecture 4. These conditions lead to proofs of special instances of the conjecture, but have not (yet) enabled us to prove or refute the conjecture.

2. Conditions on cycles and diameter

In this section, we show by simple counting arguments that in a possible counterexample to Conjecture 4, every vertex is in a (large number of) directed four-cycle(s), and that the diameter of such a digraph is at most 4. We also give an upper bound for the maximum degree of such a digraph.

Theorem 7. *Let D be a digraph with $\delta \geqslant n/3$ and suppose D contains no $\vec{\Delta}$. Then every vertex of D is in a directed four-cycle.*

Proof. Consider an arbitrary vertex $v \in V(D)$.

Then we have

$$|N^+(v)|\delta \leqslant \sum_{u \in N^+(v)} d^+(u) \leqslant \binom{|N^+(v)|}{2} + |N^+(v)|\,|N^{++}(v)|,$$

yielding $|N^{++}(v)| \geqslant \delta - \frac{1}{2}(|N^+(v)| - 1)$. Similarly, we get $|N^{--}(v)| \geqslant \delta - \frac{1}{2}(|N^-(v)| - 1)$. If v is not in a directed four-cycle, then $N^+(v), N^{++}(v), N^-(v)$ and $N^{--}(v)$ are mutually disjoint. But then $3\delta \geqslant n \geqslant 1 + |N^+(v)| + |N^{++}(v)| + |N^-(v)| + |N^{--}(v)| \geqslant 2 + 2\delta + \frac{1}{2}|N^+(v)| + \frac{1}{2}|N^-(v)| \geqslant 2 + 3\delta$, a contradiction. $\square$

In fact, using Theorem 5 or the improved bounds we can obtain more: If we consider the subgraph H of D induced by $N^+(v)$, then the absence of a directed triangle in H implies

$$\delta_H^+ < \left(\frac{2\sqrt{6} - 3}{5}\right) |N^+(v)|.$$

But this implies

$$|N^{++}(v)| \geqslant \delta^+ - \delta_H^+ > \delta^+ - \left(\frac{2\sqrt{6} - 3}{5}\right) |N^+(v)|,$$

and similarly,

$$|N^{--}(v)| > \delta^- - \left(\frac{2\sqrt{6} - 3}{5}\right) |N^-(v)|.$$

If we denote $|N^{++}(v) \cap N^{--}(v)|$ by I_v, as in the proof of Theorem 7, we get

$$3\delta \geqslant n > 1 + |N^+(v)| + \delta^+ - \left(\frac{2\sqrt{6}-3}{5}\right)|N^+(v)| + |N^-(v)|$$

$$+\delta^- - \left(\frac{2\sqrt{6}-3}{5}\right)|N^-(v)| - |I_v|,$$

yielding

$$|I_v| > 1 - \delta + \left(1 - \frac{2\sqrt{6}-3}{5}\right)(|N^+(v)| + (|N^-(v)|)$$

$$\geqslant 1 + \left(1 - 2\left(\frac{2\sqrt{6}-3}{5}\right)\right)\cdot\delta \geqslant 1 + (n/15)(11 - 4\sqrt{6}).$$

So, in any possible counterexample to Conjecture 4, every vertex is in more than $1 + (n/15)(11 - 4\sqrt{6})$ directed four-cycles.

Using the improved bound $\delta^+ \geqslant 0.34096n$ instead, similar calculations yield more than $1 + 0.106n$ such cycles.

Denote by $\mathrm{diam}(D)$ the diameter of D, i.e. the maximum of the lengths $d(u,v)$ of all shortest directed u,v-paths taken over all ordered pairs $\{u,v\}$ of vertices of D.

Theorem 8. *Let D be a digraph with $\delta \geqslant n/3$ and suppose D contains no $\vec{\Delta}$. Then* $\mathrm{diam}(D) \leqslant 4$.

Proof. Suppose $\mathrm{diam}(D) \geqslant 5$, and consider a vertex v of D such that there exists a vertex u of D with $d(v,u) \geqslant 5$. Then, the four sets $N^+(v), N^{++}(v), N^-(u)$ and $N^{--}(u)$ are mutually disjoint. So, we have $n \geqslant 2 + |N^+(v)| + |N^{++}(v)| + |N^-(u)| + |N^{--}(u)|$. On the other hand, as in the proof of Theorem 7 we get $|N^{++}(v)| \geqslant \delta - \frac{1}{2}(|N^+(v)| - 1)$ and $|N^{--}(u)| \geqslant \delta - \frac{1}{2}(|N^-(u)| - 1)$. Combining the three inequalities, we obtain $3\delta \geqslant n \geqslant 3 + 2\delta + \frac{1}{2}(|N^+(v)| + |N^-(u)|) \geqslant 3 + 3\delta$, a contradiction. □

Theorem 8 suggests a way to attack Conjecture 4 by considering diameters. Note that for any vertex v of a digraph D without a $\vec{\Delta}$ and with $\delta \geqslant 1$, there are at least $d^+(v)$ vertices u such that $d(u,v) \geqslant 3$, and at least $d^-(v)$ vertices w such that $d(v,w) \geqslant 3$. This implies that for any possible counterexample D to Conjecture 4, we know that either $\mathrm{diam}(D) = 3$ or $\mathrm{diam}(D) = 4$.

Our last result of this section involves upper bounds on the maximum out- and in-degree, denoted by Δ^+ and Δ^-, respectively, of a possible counterexample to Conjecture 4.

Theorem 9. *Let D be a digraph with $\delta \geqslant n/3$ and suppose D contains no $\vec{\Delta}$. Then* $\Delta^+, \Delta^- < n - 1 - (\delta/5)(13 - 2\sqrt{6})$.

Proof. As before, for any $v \in V(D)$ we have

$$|N^{--}(v)| > \delta^{-} - \left(\frac{2\sqrt{6}-3}{5}\right)|N^{-}(v)|.$$

Using $n \geqslant 1 + |N^{+}(v)| + |N^{-}(v)| + |N^{--}(v)|$, we obtain $|N^{+}(v)| < n - 1 - (\delta/5)(13 - 2\sqrt{6})$. This gives the result for Δ^{+}, and the result for Δ^{-} can be obtained in a similar way. □

Also here, improvements are possible according to improvements in the bound of Theorem 5. Using $\delta^{+} \geqslant 0.34096n$, we get $\Delta^{+}, \Delta^{-} < n - 1 - 1.65904\delta$.

3. Conditions on subgraphs

We start this section with the simple observation that any tournament, i.e. any orientation of a complete graph, that contains a directed cycle, also contains a $\vec{\Delta}$. So, if D is a digraph without a $\vec{\Delta}$, then every complete subgraph of D is acyclic, and hence contains a source (a vertex with in-degree zero) and a sink (a vertex with out-degree zero). A subset of $V(D)$ that induces a complete subgraph of D we refer to as a clique. We denote by $\omega(D)$ the maximum cardinality among all cliques of D.

Theorem 10. *Let D be a digraph with $\delta \geqslant n/3$ and suppose D contains no $\vec{\Delta}$. Then $\omega(D) < n/3$.*

Proof. Let S be a clique of D with $|S| = \omega(D)$, and denote by K the subgraph of D induced by S. By the above observation, K is an acyclic tournament. Let u and v denote the unique source and sink of K, respectively. Since $uv \in A(K)$, $N^{-}(u) \cap N^{+}(v) = \emptyset$. Suppose $\omega(D) \geqslant n/3$. Then $|N^{-}(u)| \geqslant n/3$ and $|N^{+}(v)| \geqslant n/3$ and $|V(K)| \geqslant n/3$ together imply $|N^{-}(u)| = |N^{+}(v)| = |V(K)| = n/3$, and $V(D) = N^{-}(u) \cup N^{+}(v) \cup V(K)$. Since $d^{+}_{K}(u) = n/3 - 1$, there is a vertex $w \in N^{+}(u) \cap N^{+}(v)$. Since $d^{+}(w) \geqslant n/3$ and $|N^{+}(v) \setminus \{w\}| = n/3 - 1$, we conclude that either $N^{+}(w) \cap V(K) \neq \emptyset$ or $N^{+}(w) \cap N^{-}(u) \neq \emptyset$, yielding a $\vec{\Delta}$, a contradiction. Hence $\omega(D) < n/3$. □

Theorem 11. *Let D be a digraph with $\delta \geqslant n/3$ and suppose D contains no $\vec{\Delta}$.* Then $\omega(D) \geqslant 4$.

Proof. We use a special case of a well-known theorem of Turán. Denote by $T_{3,n}$ the complete 3-partite graph on n vertices in which all parts are as equal as possible. It is known that any undirected graph G which contains no K_4 has at most $|E(T_{3,n})|$ edges, with equality only if $G = T_{3,n}$. It is clear that our digraph D has at least $n^2/3$ arcs, so $\omega(D) \leqslant 3$ would imply D is some $n/3$-regular orientation of $T_{3,n}$. In this case consider a shortest directed cycle C of D. Since C is a shortest cycle, C is an induced cycle of D. The structure of $T_{3,n}$ then implies C is a $\vec{\Delta}$, a contradiction. □

We next show that a possible counterexample to Conjecture 4 has a large toughness, where the toughness $\tau(D)$ of a digraph D is defined as the toughness of its underlying undirected graph, i.e.

$$\tau(D) = \min \frac{|S|}{c(D-S)},$$

where c denotes the number of components of the underlying graph, and the minimum is taken over all vertex cuts S of the underlying graph (assuming D is not complete).

Theorem 12. *Let D be a digraph with $\delta \geqslant n/3$ and suppose D contains no $\vec{\Delta}$. Then $\tau(D) \geqslant 2$.*

Proof. Suppose $\tau(D) < 2$. Clearly, D is not a complete graph by Theorem 10. For any vertex cut S, we have $|S|/c(D-S) < 2$. Let $D_1, D_2, \ldots, D_r$ denote the components of $D-S$.

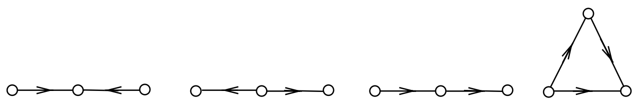
We distinguish the following cases:

Case 1: For every $i = 1, 2, \ldots, r$, $|V(D_i)| \geqslant 4$. Since $c(D-S) > \frac{1}{2}|S|$, we have $n \geqslant |S| + 4c(D-S) > 3(S)$, i.e. $|S| < n/3$, which means that the underlying graph of D has connectivity less than $n/3$. On the other hand, from the fact that the minimum degree of the underlying graph of D is at least $\frac{2}{3}n$, we know that the connectivity is at least $2\delta - n + 2 \geqslant n/3 + 2$, a contradiction.

Case 2: There is a $j \in \{1, 2, \ldots, r\}$ such that $|V(D_j)| \leqslant 3$.

Subcase 2.1. There is a $j \in \{1, 2, \ldots, r\}$ such that $|V(D_j)| \leqslant 2$. Then D_j consists of a single arc or a single vertex. We can easily deduce that $|S| \geqslant \frac{2}{3}n$. Since $n \geqslant |S| + c(D-S)$ and $c(D-S) > |S|/2$, we have $n > \frac{3}{2}|S| \geqslant n$, a contradiction.

Subcase 2.2. For every $i = 1, 2, \ldots, r$, $|V(D_i)| \geqslant 3$. Then $|V(D_j)| = 3$. D_j could be any of the following four digraphs.



If D_j is one of (a), (b), (d), then we can easily deduce from the in- and out-neighbors of two suitable vertices of D_j that $|S| \geqslant \frac{2}{3}n$. Otherwise, D_j is (c) and we can deduce that $|S| \geqslant \frac{2}{3}n - 1$. In any case, $|S| \geqslant \frac{2}{3}n - 1$. So we have that $n \geqslant |S| + 3c(D-S) > |S| + \frac{3}{2}|S| \geqslant \frac{5}{2}(\frac{2}{3}n - 1)$, i.e. $4n < 15$, contradicting the fact that $n \geqslant 4$. $\square$

We complete this section with a result on the local structure of a possible counterexample to Conjecture 4.

Theorem 13. *Let D be a digraph with $\delta \geqslant n/3$ and suppose D contains no $\vec{\Delta}$. Then there is a vertex v such that $|A(D[N^+(v)])| \geqslant n^2/27$ or $|A(D[N^-(v)])| \geqslant n^2/27$.*

Proof. Let $N(u) = N^+(u) \cup N^-(u)$ for $u \in V(D)$ and consider $\sum_{uv \in A(D)} |N(u) \cap N(v)|$. It has three parts $\sum_{uv \in A(D)} |N^+(u) \cap N^+(v)|$, $\sum_{uv \in A(D)} |N^+(u) \cap N^-(v)|$ and

$\sum_{uv\in A(D)} |N^-(u)\cap N^-(v)|$, since $\sum_{uv\in A(D)} |N^-(u)\cap N^+(v)|=0$ (no $\vec{\Delta}$). It is not difficult to see that

$$\sum_{uv\in A(D)} |N^+(u)\cap N^+(v)| = \sum_{u\in V(D)} |A(D[N^+(u)])|,$$

$$\sum_{uv\in A(D)} |N^+(u)\cap N^-(v)| = \sum_{u\in V(D)} |A(D[N^+(u)])|,$$

$$\sum_{uv\in A(D)} |N^-(u)\cap N^-(v)| = \sum_{u\in V(D)} |A(D[N^-(u)])|.$$

So, we have that

$$\sum_{uv\in A(D)} |N(u)\cap N(v)| = 2\sum_{u\in V(D)} |A(D[N^+(u)])| + \sum_{u\in V(D)} |A(D[N^-(u)])|.$$

If for all $u\in V(D)$ we have $|A(D[N^+(u)])| < n^2/27$ and $|A(D[N^-(u)])| < n^2/27$, then

$$\sum_{uv\in A(D)} |N(u)\cap N(v)| < 3n\frac{n^2}{27}.$$

On the other hand, it is not difficult to see that

$$|N(u)\cap N(v)| \geqslant \frac{n}{3} \quad \text{for any } uv\in A(D).$$

So, we have $(n/3)|A(D)| < n^3/9$, i.e. $|A(D)| < n^2/3$, a contradiction. □

4. Maximal counterexamples

In this section, we determine some structural features of a maximal counterexample D to Conjecture 4, i.e. a digraph D such that D has no $\vec{\Delta}$, while $D+uv$ has a $\vec{\Delta}$ for each $uv\notin A(D)$.

Theorem 14. *Let D be a digraph without $\vec{\Delta}$ such that for every vertex $v\in V(D)$ $N^{++}(v)\neq\emptyset$ and $N^{--}(v)\neq\emptyset$. If D has a maximal number of arcs, then we have*

(i) $N^{++}(v)=N^{--}(v)$ *for every* $v\in V(D)$.
(ii) *For every* $v\in V(D)$, $N^-(v)=V(D)\setminus(N^+(v)\cup N^{++}(v)\cup\{v\})$ *and* $N^+(v)=V(D)\setminus(N^-(v)\cup N^{--}(v)\cup\{v\})$.
(iii) $\mathrm{diam}(D)=3$.
(iv) $d(u,v)+d(v,u)\leqslant 4$ *for any two vertices u and v of D.*
(v) *Any two vertices of D are in a common directed four-cycle.*

Proof. (i) Since D has no $\vec{\Delta}$ and D is maximal with respect to the number of arcs, we can deduce that for any vertex v of D and for any $u\in N^{++}(v)$, $N^+(u)\cap N^-(v)\neq\emptyset$; otherwise $D+vu$ has no $\vec{\Delta}$ and more arcs than D. So, we have $N^{++}(v)\subseteq N^{--}(v)$.

Similarly, for any $u \in N^{--}(v)$, we have $N^{-}(u) \cap N^{+}(v) \neq \emptyset$ and therefore, $N^{--}(v) \subseteq N^{++}(v)$. This proves (i).

(ii) Let $Y(v) = V(D) \backslash (N^{+}(v) \cup N^{++}(v) \cup \{v\})$. Then for any $w \in Y(v) \backslash N^{-}(v)$ we have $N^{+}(w) \cap N^{-}(v) \neq \emptyset$; otherwise $D + vw$ has no $\vec{\Delta}$ and more arcs than D. So, $w \in N^{--}(v)$. Since $N^{++}(v) = N^{--}(v)$, we have $w \in N^{++}(v)$. Thus, $Y(v) \setminus N^{-}(v) = \emptyset$. Since D has no $\vec{\Delta}$, we have $N^{-}(v) \subseteq Y(v)$. Therefore, $Y(v) = N^{-}(v)$, i.e. $N^{-}(v) = V(D) \setminus (N^{+}(v) \cup N^{++}(v) \cup \{v\})$. Similarly, we can prove that $N^{+}(v) = V(D) \setminus (N^{-}(v) \cup N^{--}(v) \cup \{v\})$, completing the proof of (ii).

(iii) From the facts that $N^{++}(v) = N^{--}(v)$ and $N^{-}(v) = V(D) \setminus (N^{+}(v) \cup N^{++}(v) \cup \{v\})$, we can deduce that $\mathrm{diam}(D) \leqslant 3$. Since D has no $\vec{\Delta}$, we have $\mathrm{diam}(D) = 3$, proving (iii).

(iv) and (v) are easy consequences of (i) and (ii). □

5. Digraphs with few directed triangles

We next give examples of $n/3$-regular digraphs which have precisely $n/3$ mutually disjoint directed triangles. These examples somehow indicate that a possible proof of Conjecture 4 could be very complicated, because each vertex in the example graphs is in just one $\vec{\Delta}$, and the total number of $\vec{\Delta}$s is linear in n.

For any integer $k \geqslant 1$, let D_k be a digraph on $3k$ vertices labeled by $1^{(i)}, 2^{(i)}, 3^{(i)}$ for $i = 1, 2, \ldots, k$ and with the following arcs.

(i) For any $i \in \{1, 2, \ldots, k\}$ $1^{(i)} \to 2^{(i)} \to 3^{(i)} \to 1^{(i)}$ is a directed triangle.
(ii) For any $i, j \in \{1, 2, \ldots, k\}$ with $i < j$,

$$1^{(i)} \to 1^{(j)} \to 2^{(i)} \to 2^{(j)} \to 3^{(i)} \to 3^{(j)} \to 1^{(i)}$$

is a directed six-cycle.

Then D_k is a k-regular digraph with exactly k (mutually disjoint) directed triangles. In fact, D_k is the Cayley digraph of $\mathbb{Z}_{3k}$ with respect to the subset $\{1, 2, \ldots, k\}$, and had also appeared in [3].

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