AN ALGORITHMIC CHARACTERIZATION OF ANTIMATROIDS

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In an article entitled "Optimal sequencing of a single machine subject to precedence constraints" E.L. Lawler presented a now classical minmax result for job scheduling. In essence, Lawler's proof demonstrated that the properties of partially ordered sets were sufficient to solve the posed scheduling problem. These properties are, in fact, common to a more general class of combinatorial structures known as *antimatroids*, which have recently received considerable attention in the literature. It is demonstrated that the properties of antimatroids are not only sufficient but necessary to solve the scheduling problem posed by Lawler, thus yielding an algorithmic characterization of antimatroids. Examples of problems solvable by the general result are provided.

1. Introduction

In an article entitled "Optimal sequencing of a single machine subject to precedence constraints" Lawler showed how to solve the $1|prec|f_{\max}$ scheduling problem using a variant of the greedy algorithm. In his proof, Lawler took advantage of the fact that the underlying constraints of the problem were the constraints imposed by a partial order on the jobs to be scheduled. In actuality, the properties used by Lawler in his proof are common to a more general class of combinatorial structures known as *antimatroids.*

Antimatroids were apparently first considered by Edelman [4] and by Jamison-Waldner [9]. Korte and Lovász [11] studied these structures from a different perspective under the names *alternative precedence structures* and *upper interval greedoids.* Antimatroids have recently received considerable attention in the literature (e.g., Korte and Lovász [13, 14, 16]). There are a variety of reasons for this attention. As pointed out by Korte and Lovász [11], the class of antimatroids includes many interesting combinatorial structures within its realm. In addition, antimatroids are closely related to matroids in that both can be defined by a very similar set of axioms, the only difference being an exchange axiom for matroids but

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an anti-exchange axiom for antimatroids [13]. This close relationship to matroids has provided a fruitful combinatorial structure that is general enough to be interesting while still maintaining sufficient structure to be amenable to proofs.

This paper provides an algorithmic characterization of antimatroids that helps to provide further insight into the structure and algorithmic relevance of antimatroids. Section 2 introduces basic information about antimatroids that will be needed for the following sections. In Section 3 the $1|prec|f_{\max}$ scheduling problem is presented in a more abstract form that allows it to be defined on an arbitrary combinatorial structure and in Section 4 necessary properties of *truncated antimatroids* are introduced. The main result is proved in Section 5, where truncated antimatroids are characterized as exactly those combinatorial structures for which the associated scheduling problem is certain to succumb to the greedy algorithm. The paper is concluded with some examples of special cases of the scheduling problem.

2. Preliminaries

Let $\mathscr{F} \subseteq 2^E$ be a collection of sets on the finite ground set E. One of many equivalent axiomatizations of antimatroids is the following [2].

Definition. An *antimatroid* is a nonempty set system $(E, \mathscr{F})$ satisfying the following two properties:

Property (1) If $A \in \mathscr{F}$, then there exists an $x \in A$ such that $A-\{x\} \in \mathscr{F}$.

Property (2) If $A, B \in \mathscr{F}$, and $A \nsubseteq B$, then there exists an $x \in A-B$ such that $B \cup \{x\} \in \mathscr{F}$.

Any set system satisfying Property (1) will be called *accessible.*

An alternative definition makes use of the well-developed theory of formal languages. Given a finite alphabet E, a language $\mathscr{L}$ is a nonempty collection of words consisting of letters in E. Words will be denoted by the lower case greek letters α, β, and γ, or by a specified sequence of elements in E such as $\alpha = x_1 \ldots x_k$. The concatenation of two words α and β will be denoted $\alpha\beta$, and α_k will be used to denote a word of length k or the kth subword of α. The set of distinct elements in a word α will be denoted α^*.

A language $\mathscr{L}$ is *accessible* if $\alpha x \in \mathscr{L}$ implies $\alpha \in \mathscr{L}$. Words contained in a language $\mathscr{L}$ are *feasible.* A word α is *simple* if no letter appears more than once in any word, and a language $\mathscr{L}$ is called simple if every feasible word is simple. The finite set of all simple words defined by an alphabet E will be denoted by E^0. Using formal languages, an *antimatroid language* can be defined as follows.

Definition. An *antimatroid language* is a nonempty, simple language $(E, \mathscr{L})$ satisfying the following two properties:

Property (1) If $\alpha x \in \mathscr{L}$, then $\alpha \in \mathscr{L}$.
Property (2) If $\alpha, \beta \in \mathscr{L}$ and $\alpha^* \not\subseteq \beta^*$, then there exists an $x \in \alpha^*$ such that $\beta x \in \mathscr{L}$.

The relationship between antimatroids and antimatroid languages is captured in a theorem proved by Korte and Lovász [11] about a more general class of simple languages called *greedoids*. We present the theorem here only as it relates to antimatroids.

Theorem 2.1. *If* $(E, \mathscr{L})$ *is an antimatroid language, then*

$$F(\mathscr{L}) = \{\alpha^*: \alpha \in \mathscr{L}\}$$

is an antimatroid $(E, F(\mathscr{L}))$. *Conversely, if* $(E, \mathscr{F})$ *is an antimatroid, then*

$$L(\mathscr{F}) = \{x_1 \dots x_k \in E^0: \{x_1, \dots, x_j\} \in \mathscr{F} \textit{ for } j = 0, \dots, k\}$$

is an antimatroid language $(E, L(\mathscr{F}))$. *Further,* $L(F(\mathscr{L})) = \mathscr{L}$ *and* $F(L(\mathscr{F})) = \mathscr{F}$.

The immediate implication of Theorem 2.1 is that antimatroids can be considered equivalently as set systems or as simple languages. Henceforth, the term antimatroid will be used for both antimatroids and antimatroid languages and the definition used will depend upon the development at hand.

The *k-truncation* or simply *truncation* of a simple language $(E, \mathscr{L})$ is the simple language defined by

$$\mathscr{L}_k = \{\alpha \in \mathscr{L}: |\alpha^*| \leq k\}.$$

The truncation of a set system is defined similarly.

The *rank* of a set $A \subseteq E$ is defined as

$$\varrho(A) = \max\{|\alpha^*|: \alpha \in \mathscr{L}, \alpha^* \subseteq A\}.$$

The rank of a simple language $(E, \mathscr{L})$, while properly denoted $\varrho(E)$, will be denoted $\varrho(\mathscr{L})$. The rank of a set system is defined similarly.

The greedy algorithm has a natural definition for simple languages. The following formal definition is provided for reference.

Definition. *The greedy algorithm.*

Let $(E, \mathscr{L})$ be a simple language with an associated function $W: \mathscr{L} \to \mathbb{R}$.
Let α initially be the empty word.
  Choose $x \in E - \alpha^*$ such that
    (1) $\alpha x \in \mathscr{L}$,
    (2) $W(\alpha x) \leq W(\alpha y)$ for all y such that $\alpha y \in \mathscr{L}$.
  Let $\alpha = \alpha x$ and repeat until α can no longer be augmented.

3. Problem

In order to extend Lawler's result and characterize antimatroids, it is necessary to generalize the $1|prec|f_{\max}$ scheduling problem to include arbitrary combinatorial structures. The following definitions provide the necessary generalization.

Definition. Let E be a finite ground set and let $f: E \times 2^E \to \mathbb{R}$. A *maximum nesting function* is a function of the form

$$W(x_1,\dots,x_k) = \max\{f(x_1,\{x_1\}),\dots,f(x_k,\{x_1,\dots,x_k\})\}.$$

A maximum nesting function will be called *f-monotone* if $f(x,A) \le f(x,B)$ whenever $B \subseteq A$.

The optimization problem of interest can now be defined as follows.

Definition. *The minmax nesting problem.* Given a simple language $(E,\mathscr{L})$ with an f-monotone maximum nesting function W and a nonnegative integer $k \le \varrho(\mathscr{L})$, find $\alpha_k \in \mathscr{L}$ such that

$$W(\alpha_k) = \min\{W(\beta_k)\colon \beta_k \in \mathscr{L}\}.$$

A very natural instance of a minmax nesting function is the following.

Definition. Let $t: E \to \mathbb{R}_+$ and for every $x \in E$ let $c_x: \mathbb{R} \to \mathbb{R}$ be a nonincreasing function. The f-monotone maximum nesting function defined by

$$f(x,A) = c_x\Bigg(\sum_{y \in A \cup \{x\}} t(y)\Bigg)$$

is called a *time dependent bottleneck function.*

In a scheduling context, for example, $t(x)$ might represent the time to complete task x and $c_x(A)$ might represent the time-sensitive cost of completing task x after first completing tasks A.

The name *time dependent bottleneck function* arises from the close relationship to *generalized bottleneck functions* introduced by Korte and Lovász [11]. Generalized bottleneck functions can be viewed as the special case of a time dependent bottleneck function with $t(x) = 1$ for all $x \in E$. Also motivated in part by the work of Lawler, Korte and Lovász [11] demonstrated that the greedy algorithm solves the minmax nesting problem defined by generalized bottleneck functions if and only if the underlying simple language is a greedoid. The following section introduces the class of simple languages that plays a similar role for time dependent bottleneck functions and more generally for f-monotone maximum nesting functions.

4. Truncated antimatroids

The f-monotone maximum nesting problem presented in the previous section is intimately related with antimatroids or, more specifically, with *truncated antimatroids*. We formally make the connection by introducing the following property for a set system $(E, \mathscr{F})$.

Property (2′) If $A, B \in \mathscr{F}$ with $|B| < \varrho(\mathscr{F})$ and $A \nsubseteq B$, then there exists an $x \in A$ such that $B \cup \{x\} \in \mathscr{F}$.

Note that Property (2′) is a relaxation of Property (2) in the definition of an antimatroid in that it allows an arbitrary rank to be imposed, whereas the very definition of an antimatroid implies that its rank is equal to $|\bigcup_{A \in \mathscr{F}} A|$. In fact, the specific relationship between these two properties suggests that the operation of truncation provides the formal link between antimatroids and accessible set systems with the weaker Property (2′). Certainly, the truncation of any antimatroid satisfies Property (2′). But even further, any accessible set system satisfying Property (2′) defines in a natural way an antimatroid of which it is a truncation, as the following proof demonstrates.

Proposition 4.1. *Let $(E, \mathscr{F})$ be an accessible set system of rank m. If $(E, \mathscr{F})$ satisfies Property* (2′), *then the set system $(E, \mathscr{F}')$ defined by*

$$\mathscr{F}' = \{B \subseteq E\colon B = A_1 \cup \cdots \cup A_k \textit{ for some } A_1, \ldots, A_k \in \mathscr{F}\}$$

is an antimatroid. Further, $(E, \mathscr{F})$ is the m-truncation of $(E, \mathscr{F}')$.

Proof. Let A' and B' be two sets in $\mathscr{F}'$. We first demonstrate that $\mathscr{F}'$ is an antimatroid. To show that $\mathscr{F}'$ satisfies the first property defining antimatroids, let $A_1, \ldots, A_j \in \mathscr{F}$ be such that $A' = A_1 \cup \cdots \cup A_j$. Let $A_j - \{x\} \in \mathscr{F}$ and assume without loss of generality that $x \notin A_1, \ldots, A_{j-1}$ for otherwise we could let $A_j = A_j - \{x\}$. But then $A' - \{x\} = A_1 \cup \cdots \cup (A_j - \{x\}) \in \mathscr{F}'$. To show that $\mathscr{F}'$ satisfies the second property defining antimatroids, suppose $A' \nsubseteq B'$. Let $A_i = \{x_1, \ldots, x_k\}$ be indexed so that $\{x_1, \ldots, x_p\} \in \mathscr{F}$ for $p = 1, \ldots, k$, and let i and q be chosen such that $A_1 \cup \cdots \cup A_{i-1} \cup \{x_1, \ldots, x_{q-1}\} \subseteq B'$ but $A_1 \cup \cdots \cup A_{i-1} \cup \{x_1, \ldots, x_q\} \nsubseteq B'$. Then $x_q \in A'$ and $B' \cup \{x_q\} \in \mathscr{F}'$, completing the proof.

To show that the m-truncation of $(E, \mathscr{F}')$ is $(E, \mathscr{F})$ it is sufficient to demonstrate that $A \in \mathscr{F}$ if and only if $A \in \mathscr{F}'$ and $|A| \leq m$. It is clear by the construction of $\mathscr{F}'$ that $A \in \mathscr{F}$ implies $A \in \mathscr{F}'$ and obviously $|A| \leq m$, completing half of the proof. To complete the proof, assume $A \in \mathscr{F}'$ and $|A| \leq m$ and let $A_1, \ldots, A_j \in \mathscr{F}$ be such that $A = A_1 \cup \cdots \cup A_j$. Let $A_i = \{x_1, \ldots, x_k\}$ be indexed so that $\{x_1, \ldots, x_p\} \in \mathscr{F}$ for $p = 1, \ldots, k$. Clearly, if $A_1 \cup \cdots \cup A_{i-1} \cup \{x_1, \ldots, x_{q-1}\} \in \mathscr{F}$, then since $\{x_1, \ldots, x_q\} \in \mathscr{F}$ it follows by Property (2′) that $A_1 \cup \cdots \cup A_{i-1} \cup \{x_1, \ldots, x_q\} \in \mathscr{F}$. As $A_1 \in \mathscr{F}$, it follows by induction that $A = A_1 \cup \cdots \cup A_j \in \mathscr{F}$, completing the proof. □

Proposition 4.1 thus demonstrates that the class of accessible set systems satisfying Property (2′) is exactly the class of truncated antimatroids, so that henceforth we refer to such set systems as truncated antimatroids. We state this result in the following proposition for completeness.

Proposition 4.2. *An accessible set system $(E, \mathcal{F})$ satisfies Property* (2′) *if and only if it is a truncated antimatroid.*

5. Results

With all of the preliminary results presented, we are now in a position to complete the main theorem.

Theorem 5.1. *Let $(E, \mathcal{L})$ be a simple language. The greedy algorithm solves the minmax nesting problem for every f-monotone maximum nesting function W if and only if $(E, \mathcal{L})$ is a truncated antimatroid.*

Proof. (if). Clearly the empty word is the minimum cost word of length 0. To complete the inductive argument, let $x_1 \ldots x_k$ be such that for $i = 1, \ldots, k$, $W(x_1 \ldots x_k) = \min\{W(\alpha_k)\colon \alpha_k \in \mathcal{L}\}$, and let x_{k+1} be a greedy choice for $x_1 \ldots x_k$. If $x_1 \ldots x_{k+1}$ is not optimal among words of length $k+1$, then it follows from the definition of a maximum nesting function that there exists a solution $y_1 \ldots y_{k+1}$ such that

$$\max\{f(y_i, (y_1 \ldots y_i)^*)\} \geq \max\{f(x_i, (x_1 \ldots x_i)^*)\}, \quad i = 1, \ldots, k,$$

and in addition

$$\max\{f(y_i, (y_1 \ldots y_i)^*)\} < \max\{f(x_i, (x_1 \ldots x_i)^*)\}, \quad i = 1, \ldots, k+1.$$

Together, these two conditions imply

$$f(y_i, (y_1 \ldots y_i)^*) < f(x_{k+1}, (x_1 \ldots x_{k+1})^*), \quad i = 1, \ldots, k+1. \tag{a}$$

Certainly, $(x_1 \ldots x_k)^* \neq (y_1 \ldots y_k)^*$. If so, then since $y_1 \ldots y_k y_{k+1} \in \mathcal{L}$ and $x_1 \ldots x_k \in \mathcal{L}$ it follows that $x_1 \ldots x_k y_{k+1} \in \mathcal{L}$ and clearly $f(y_{k+1}, (y_1 \ldots y_k y_{k+1})^*) = f(y_{k+1}, (x_1 \ldots x_k y_{k+1})^*)$. However, by (a) this implies y_{k+1} is a strictly better choice than x_{k+1}, contradicting the fact that x_{k+1} is a greedy choice.

Thus, let j be the smallest index such that $y_j \notin (x_1 \ldots x_k)^*$ so that $x_1 \ldots x_k y_j \in \mathcal{L}$. Since $(y_1 \ldots y_j)^* \subseteq (x_1 \ldots x_k y_j)^*$, it follows by the monotonicity of f that $f(y_j, (x_1 \ldots x_k y_j)^*) \leq f(y_j, (y_1 \ldots y_j)^*)$. However, again by (a) this implies y_j is a strictly better choice than x_{k+1}, contradicting the fact that x_{k+1} is a greedy choice.

(only if). Clearly, $(E, \mathcal{L})$ must be accessible since a maximum nesting function W can be easily defined for which an inaccessible word α uniquely minimizes W over all words of length $|\alpha^*|$ while the greedy algorithm can never generate an inaccessible word. Thus, suppose $(E, \mathcal{L})$ is accessible but is not a truncated antimatroid so that there exist words $\alpha,\beta\in\mathscr{L}$ with $|\beta^*|<\varrho(\mathscr{L})$ such that $\alpha^*\not\subseteq\beta^*$ and $x\in\alpha^*-\beta^*$ implies $\beta x\notin\mathscr{L}$. We consider a time dependent bottleneck function W defined as follows. Let $t(x)=1$ for $x\notin\alpha^*\cup\beta^*$. Let $t(x_1\dots x_k)$ denote $\sum_{i=1}^{k} t(x_i)$ and define $t(x)$ for $x\in\alpha^*\cup\beta^*$ implicitly by $t(\alpha)=t(\beta)+1$. Finally, let

$$c_x(t)=0, \qquad x\in\alpha^*\cup\beta^*,$$

$$c_x(t)=\begin{cases}1, & t<t(\beta)+2,\\ 0, & t\geq t(\beta)+2,\end{cases} \qquad x\notin\alpha^*\cup\beta^*.$$

Suppose there exist no words γ beginning with the word α such that $|\gamma^*|=\varrho(\mathscr{L})$. Since α can be generated by the greedy algorithm the greedy algorithm could fail to generate a minimum cost word of length $\varrho(\mathscr{L})$.

Thus, suppose there exists a word γ beginning with the word α such that $|\gamma^*|=\varrho(\mathscr{L})$. For the defined time dependent bottleneck function W, $W(\gamma_{|\beta^*|+1})=0$ while for any word βx, if such a word exists, $W(\beta x)=1$. Since β can be generated by the greedy algorithm, the greedy algorithm could fail to generate a minimum cost word of length $|\beta^*|+1\leq\varrho(\mathscr{L})$, completing the proof. □

6. Examples

Theorem 5.1 not only provides an algorithmic characterization of antimatroids but extends Lawler's result to this more general class of combinatorial structures. The following is a small set of examples of problems captured by Theorem 5.1.

6.1. Job scheduling under precedence constraints

As an initial example let us indicate how Lawler's [17] result can be viewed as a special case of Theorem 5.1. Consider a set of jobs E where each job $x\in E$ has a processing time $t(x)\geq 0$. The jobs have to be sequenced so that precedence constraints, given by a partial order $(E,\leq)$, are observed. Note that for any feasible schedule $x_1,\dots,x_{|E|}$ the completion time of x_i is $t(x_1)+\cdots+t(x_i)$. A penalty function $p_x(t)$ is given for each job $x\in E$ that is nondecreasing with respect to the completion time of x. The problem is to find a feasible schedule for E that minimizes the maximum penalty incurred.

As stated, the problem is not a time dependent bottleneck function since the $p_x(t)$ are nondecreasing functions of t rather than nonincreasing functions. However, noting that the feasible schedules of $(E,\leq)$ read in reverse order are exactly the feasible schedules relative to the dual order $(E,\geq)$, the problem is seen to be equivalent to the scheduling problem on $(E,\geq)$ relative to the penalties $c_x(t)=p_x(T+t(x)-t)$ where $T=\sum_{x\in E} t(x)$. Clearly, the accessible language induced by the feasible schedules of $(E,\geq)$ is an antimatroid language. Moreover, the functions

$c_x(t)$ are nonincreasing functions of t. Hence, Theorem 5.1 guarantees the optimality of the greedy algorithm when applied to the associated time dependent bottleneck function.

A simplified version of Lawler's result is obtained by assuming the functions $p_x(t)$ are nonincreasing and letting $c_x(t) = p_x(t)$. In this case, the greedy algorithm can be applied directly to the feasible schedules of $(E, \leq)$. Such a scenario might arise if the jobs were being scheduled in a deflationary environment. Alternatively, it is possible to consider a period of constant costs but with discounting included since the discounted job costs are nonincreasing.

It is valuable to note that in order to apply the greedy algorithm when $c_x(t)$ is nonincreasing the $c_x(t)$ do not need to be known in advance. When a job is finished at time $\bar{t}$, $c_x(t)$ can be calculated for all jobs that can commence at time $\bar{t}$ and the job with the minimum cost chosen. Having such a weak requirement on knowledge of the $c_x(t)$ is extremely important since it implies that only qualitative rather than quantitative cost projections are necessary. It is also valuable to note that the proof of Theorem 5.1 demonstrates that any set of k jobs chosen by the greedy algorithm minimizes the maximum job cost over all choices and orderings of k jobs. Finally, note that the problem also can be stated in terms of a company seeking to maximize the minimum profit among a set of contracts in a inflationary period.

6.2. *Road construction in a deflationary period*

Consider a construction company charged with the task of constructing a road network E connecting a set of locations V. The construction equipment is initially located at location $r \in V$. Since the equipment needs a road on which to travel when it is relocated, construction cannot begin on the road connecting locations x and y until roads have been constructed linking r to x or y. Each road x has a fixed completion time $t(x)$, and the work is being scheduled in a deflationary period so that the later construction of a road is commenced the less it will cost. The problem is to find a feasible construction sequence that minimizes the maximum of the incurred road construction costs. This is an example of a time dependent bottleneck function on a class of antimatroids originally called *line search greedoids* by Korte and Lovász [11] and so by Theorem 5.1 it is solvable by the greedy algorithm.

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