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On the Classification of Plane Graphs Representing Structurally Stable Rational Newton Flows

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We study certain plane graphs, called Newton graphs, representing a special class of dynamical systems which are closely related to Newton's iteration method for finding zeros of (rational) functions defined on the complex plane. These Newton graphs are defined in terms of nonvanishing angles between edges at the same vertex. We derive necessary and sufficient conditions—of purely combinatorial nature—for an arbitrary plane graph in order to be topologically equivalent with a Newton graph. Finally, we analyse the structure of Newton graphs and prove the existence of a polynomial algorithm to recognize such graphs.

1. INTRODUCTION AND MOTIVATION

In this paper we study certain plane graphs (Newton graphs) which are closely related to the class of so-called rational Newton flows. Firstly, we briefly explain the concepts of Newton flow and Newton graph as well as their interrelationship.

Let f be a non-constant rational function of a complex variable z. So, f may be represented as $f = p_n/q_m$, where p_n and q_m are polynomials (of degree n and m, respectively) which are relatively prime. Let us consider the autonomous differential equation of the form

$$\frac{dz}{dt} = -\frac{f(z)}{f'(z)}, \tag{1.1}$$

where f' stands for the derivative. This equation (or, more precisely, the flow associated with it) will be called rational Newton flow. Note that Euler's discretization to (1.1) just yields the well-known Newton iteration method for finding the zeros of f; this explains the terminology. The right hand side of (1.1), and sometimes also the flow given by this equation, will be denoted by $\mathbf{N}(f)$.

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Roughly speaking, the rational Newton flow $\mathbf{N}(f)$ is called structurally stable, if under sufficiently small perturbations of the coefficients of p_n and q_m the topological features of the phase portraits of the resulting Newton flows remain the same.

In the case where $n > m$, it is possible to associate with each structurally stable rational Newton flow $\mathbf{N}(f)$ a connected plane graph, which will be denoted by $G(f)$. Such a connected plane graph either consists of only one vertex and no edge, or it exhibits the following two properties:

N_1: At any vertex of $G(f)$, the angle between (different) edges with this vertex in common is well-defined and never vanishes.

N_2: For any *finite* face of $G(f)$, the sum of all angles spanning a sector of this face at the vertices is equal to 2π. (In this paper, an angle is always counted as a positive number.)

A connected plane graph, for which the above Conditions N_1, N_2 hold, is called a Newton graph. As usual, two plane graphs are called equivalent ($\sim$) if a homeomorphism from the plane onto itself exists which maps the edges and vertices of one plane graph onto those of the other. On the other hand, two rational Newton flows are called equivalent ($\sim$) if there is a homeomorphism from the plane onto itself mapping the maximal trajectories of one flow onto those of the other. Up to equivalency, the set of all structurally stable Newton flows $\mathbf{N}(f)$, $n > m$, is just described by the set of all Newton graphs, or, more precisely (cf. [5]):

THEOREM 1.1. (a) *Let G be a Newton graph. Then, a structurally stable Newton flow $\mathbf{N}(f)$, $n > m$, exists such that $G \sim G(f)$.*

(b) *Let f_1 and f_2 be two rational functions with degree(numerator) > degree(denominator), the associated Newton flows being structurally stable. Then, $\mathbf{N}(f_1) \sim \mathbf{N}(f_2)$ iff $G(f_1) \sim G(f_2)$.*

From a graph theoretical point of view, the definition of Newton graph is not very satisfactory. However, one easily derives a necessary condition, of purely combinatorial nature, for G to be equivalent with a Newton graph. To this aim, let C be an arbitrary cycle in G, and introduce the integers: $n(C)$ = number of G-vertices inside C, but not on C; $l(C)$ = number of G-vertices on C; $r(C)$ = number of G-faces inside C.

The inward, resp. outward, angle at a G-vertex of C is the angle between two consecutive edges of C, spanning a sector of the interior resp. of the exterior of C.

One easily verifies that, for a Newton graph G, we have

$$\sum \text{(all inward angles at the } G\text{-vertices on } C) = 2\pi(r(C) - n(C)) > 0,$$

$$\sum \text{(all outward angles at the } G\text{-vertices on } C) = 2\pi(l(C) + n(C) - r(C)) > 0.$$

Consequently, a *necessary* condition for G in order to be equivalent with a Newton graph is that the following inequalities do hold for *all* cycles C:

$$n(C) < r(C) < n(C) + l(C).$$

The latter condition turns out to be also *sufficient*. In fact, this is our *main result*.

For an overall reference on the subject of Newton systems, we refer to Chapter 9 of our treatise [4], and for an introduction to the polynomial case to M. Shub, D. Tischler, and R. F. Williams [8] and to S. Smale [9].

2. The Main Results

Throughout this paper, let G be a connected plane graph of order (η) greater than two. A priori, we do not exclude either the occurrence of loops or multiple edges.

Let J be a nonempty subset of the set of all G-faces. The subgraph of G consisting of all edges and vertices which are incident with a face in J is called $G(J)$. An *interior* vertex of $G(J)$ is a vertex which is only incident with faces in J, whereas a vertex is called *exterior* if it is incident with both a face in J and a face not in J. The number of interior and exterior vertices of $G(J)$ is denoted by $n(J)$ and $l(J)$, respectively; by $r(J)$ we denote the cardinality of J.

In Section 2, we shall prove:

THEOREM A. *The plane graph G is equivalent with a Newton graph if and only if the following inequalities hold for all nonempty sets J, not containing the infinite face*:

$$n(J) < r(J) < n(J) + l(J). \qquad (*)$$

Apparently, the above Inequalities $(*)$ imply

$$n(C) < r(C) < n(C) + l(C), \qquad (**)$$

where C is a cycle in G and the integers $n(C)$, $l(C)$, and $r(C)$ are defined as in Section 1. (Take for J all faces inside C.) In Section 4, we show that the converse is also true:

THEOREM B. *Inequalities $(*)$ hold for all nonempty subsets J of finite faces if and only if the Inequalities $(**)$ are fulfilled for all cycles C.*

As a consequence of Theorem A we prove the existence of a polynomial algorithm to recognize the class of Newton graphs (Section 5). In this latter section we also investigate the structure of Newton graphs, including some results on the construction of new Newton graphs from a given one.

3. The Proof of Theorem A

Let G be a plane graph as in Section 2. The vertices of G will be denoted by Ω_i, whereas $\deg \Omega_i$ stands for the number of edges incident with Ω_i, $i = 1, ..., \eta$. (A loop at Ω_i contributes twice to $\deg \Omega_i$.) The infinite face of G is denoted by r_0, and the finite faces by $r_1, ..., r_\mu$ (μ = cyclomatic number). The labeling of the vertices is organized in such a way that the exterior G-vertices (i.e., those which are indicent with r_0) are denoted by $\Omega_1, ..., \Omega_{\eta_0}$. The vertices $\Omega_{\eta_0+1}, ..., \Omega_\eta$ are referred to as interior G-vertices.

Due to the embedding of G, there is a cyclic, anticlockwise order on those edges which are incident with a particular vertex, say Ω_i. In accordance with this order, we label the edges at Ω_i by $i(1), ..., i(\delta)$, where $\delta = \deg(\Omega_i)$, and $i(\delta+1) = i(1)$. (Again, a loop contributes twice to this set of labels.)

In view of Fary's theorem (cf. [3]), there always exists a plane graph which is equivalent to G and for which all angles are well-defined. The angle between two edges at Ω_i with labels $i(k)$ and $i(k+1)$, respectively, is given by $2\pi\omega_{i(k)}$. The set $(A(G))$ of all reals $\omega_{i(k)}$, with $i = 1, ..., \eta$, $k = 1, ..., \deg \Omega_i$, is called the set of angles for G. The set of all angles spanning a sector of r_j is denoted by $a(r_j)$. Finally, for fixed i, the set of all angles $\omega_{i(k)}$ is called the set of angles at Ω_i, denoted $a(\Omega_i)$.

With the above notations we reformulate the definition of a Newton graph more precisely:

Definition 3.1. A connected plane graph G is called *Newton graph* if

(i) $\omega_{i(k)} > 0$, for all angles $\omega_{i(k)}$ in $A(G)$;

(ii) $\sum_{a(r_j)} \omega_{i(k)} = 1$, for $j = 1, ..., \mu$;

(iii) $\sum_{a(\Omega_i)} \omega_{i(k)} = 1$, for $i = 1, ..., \eta$.

One part of the assertion in Theorem A can be proved easily:

Lemma 3.1. *Let G be a Newton graph. Then, for any nonempty subset J of $\{0, 1, ..., \mu\}$ with $0 \notin J$, the Inequalities $(*)$ hold.*

Proof. In the case where $\mu = 0$, there is nothing to prove. So, we assume that $\mu \geq 1$. Definition 3.1 yields

$$\sum_{j \in J} \sum_{a(r_j)} \omega_{i(k)} = r(J).$$

The contribution of any *interior* vertex of $G(J)$ to the sum in the left-hand side is equal to 1, whereas each *exterior* vertex contributes with a number which is strictly between 0 and 1. Moreover, the set of exterior $G(J)$-vertices is nonempty. From this the assertion follows directly. ▮

The following corollary is an easy consequence of the Inequalities $(*)$ (put $J = \{1, ..., \mu\}$, resp. J = index set corresponding with the faces inside an eventual loop).

COROLLARY 3.1. *If G fulfils the Inequalities* $(*)$, *then* $\mu < \eta$ *and G does not exhibit loops.*

Throughout the rest of this section, we assume that the Inequalities $(*)$ hold for all nonempty subsets J of finite G-faces. We define the integers $p_0, p_1, ..., p_\mu$ as

$$p_0 = \eta - \mu; \qquad p_j = 1, \qquad \text{if} \quad j = 1, ..., \mu.$$

In view of Corollary 3.1, the integer p_0 is positive.

LEMMA 3.2. *For all nonempty subsets J of* $\{0, 1, ..., \mu\}$,

$$n(J) + l(J) \geqslant \sum_{j \in J} p_j,$$

whereas the equality holds iff $J = \{0, 1, ..., \mu\}$.

Proof. If $0 \notin J$, then the lemma follows directly from the Inequalities $(*)$. So, we assume that $0 \notin J^c$ (= complement of J).

In the case where $J^c \neq \varnothing$, the Inequalities $(*)$ applied to J^c, yield

$$n(J^c) < r(J^c) \qquad \left(= \eta - \sum_J p_j\right),$$

and, since $\eta = n(J) + l(J) + n(J^c)$, the assertion follows immediately. In the case where $J^c = \varnothing$, we obviously have $n(J) + l(J) = \sum_J p_j$. ▮

Let Ω_{i_0} be a vertex of G and let j_0 be an arbitrary index in $\{0, 1, ..., \mu\}$. We define the non-negative integers $p_0^0, p_1^0, ..., p_\mu^0$ as

$$p_j^0 = p_j, \qquad \text{if} \quad j \neq j_0 \qquad \text{and} \qquad p_{j_0}^0 := p_{j_0} - 1.$$

If $V(J)$ denotes the set of G-vertices, incident with r_j, $j \in J$, and $|\cdot|$ stands for cardinality, then the precedent lemma yields:

COROLLARY 3.2. *For all nonempty subsets J of* $\{0, 1, ..., \mu\}$ *we have*

$$|V(J) \backslash \{\Omega_{i_0}\}| \geqslant \sum_{j \in J} p_j^0.$$

Moreover, if $J = \{0, 1, ..., \mu\}$, *then the equality holds.*

DEFINITION 3.2. A transversal T of the collection $\{V(\{j\}) \mid j=1, ..., \mu\}$ is a set of μ *different* G-vertices $\Omega_{i_1}, ..., \Omega_{i_\mu}$, such that for all $j=1, ..., \mu$, $\Omega_{i_j} \in V(\{j\})$. Given such a transversal T, then Ω_{i_j} is said to represent the face r_j w.r.t. T.

LEMMA 3.3. *Let $i_0 \in \{1, ..., \eta\}$, and let $j_0 \in \{0, 1, ..., \mu\}$ be two indices, such that Ω_{i_0} is a vertex in r_{j_0}.*

If $j_0 = 0$, then a transversal (T) of $\{V(\{j\}) \mid j=1, ..., \mu\}$ exists such that:

(1) *the transversal T contains exterior vertices of G;*

(2) *the transversal T contains all interior vertices of G;*

(3) *the vertex Ω_{i_0} is not contained in T.*

If $j_0 \neq 0$, then a transversal (also denoted by T) of $\{V(\{j\}) \mid j=1, ..., \mu\}$ exists which has the Properties (1) *and* (2), *but which fulfils (in contradistinction with Property* (3)*):*

(3′) *the vertex Ω_{i_0} represents r_{j_0} w.r.t. T.*

Proof. In accordance with a slight generalization of Hall's theorem on distinct representatives (cf. L. Mirsky [7, Theorem 3.3.1]) the inequality in Corollary 3.2 is necessary and sufficient for the existence of pairwise *disjoint* sets $X_0, ..., X_\mu$ such that

$$X_j \subset V(\{j\}) \backslash \{\Omega_{i_0}\} \qquad \text{with} \quad |X_j| = p_j^0,\ j=0, 1, ..., \mu.$$

If $j_0 = 0$ (and thus $p_j^0 = 1$, for all $j=1, ..., \mu$), then the union $\bigcup_{j=1}^{\mu} X_j$ provides a transversal (T) of $\{V(\{j\}) \mid j=1, ..., \mu\}$, not containing Ω_{i_0}.

If $j_0 \neq 0$ (and thus $p_{j_0}^0 = 0$, $p_j^0 = 1$ for $j \neq j_0$ and $j \neq 0$), then the union $(\bigcup_{j=1}^{\mu} X_j) \cup \{\Omega_{i_0}\}$ provides a transversal (T) of $\{V(\{j\}) \mid j=1, ..., \mu\}$ such that Ω_{i_0} represents r_{j_0}.

In both cases, application of the Inequalities $(*)$ w.r.t. $J = \{1, ..., \mu\}$ yields $\#$(interior vertices of $G) < \mu$, and thus $T \cap V(\{0\}) \neq \emptyset$. So, the first assertion of the lemma holds for T. Next, we observe that, if we choose $J = \{0, 1, ..., \mu\}$ in Corollary 3.2, then

$$\eta - 1 = |V(J) \backslash \{\Omega_{i_0}\}| = \sum_{j=0}^{\mu} p_j^0 = \sum_{j=0}^{\mu} |X_j| = \left| \bigcup_{j=0}^{\mu} X_j \right|.$$

From this, the second assertion follows immediately. (Use the already proved Properties (3) or (3′).)

Let us denote the G-angles by x_k, with $k=1, ..., K$, where

$K=\sum_{i=1}^{\eta} \deg \Omega_i$. We associate with G an $(\eta+\mu)\times K$-matrix $M(G)$ with coefficients m_{lk}:

$$m_{lk}=\begin{cases}1, & \text{if } 1\leqslant l\leqslant \eta \text{ and } x_k \text{ is an angle at } \Omega_l, \text{ i.e., } x_k\in a(\Omega_l);\\ 1, & \text{if } \eta<l\leqslant \eta+\mu \text{ and } x_k \text{ is an angle in } a(r_{l-\eta});\\ 0, & \text{otherwise.}\end{cases}$$

The vertex Ω_{l_1} and the *finite* face $r_{l_2-\eta}$ are called *associated* if the intersection $a(\Omega_{l_1})\cap a(r_{l_2-\eta})$ is nonempty. In this case we write $(l_1, l_2)\in \mathbf{O}$. (Note that $1\leqslant l_1\leqslant \eta$ and $\eta<l_2\leqslant \eta+\mu$.)

Apparently, the plane graph G is equivalent with a Newton graph iff the following system of $(\eta+\mu)$-equalities and K inequalities has a solution:

$$\left[M(G)\left|\begin{matrix}-1\\ \vdots\\ -1\end{matrix}\right.\right]\cdot(x\mid 1)^{\mathrm{T}}=\underbrace{(0,...,0)}_{(\eta+\mu)\text{-times}}{}^{\mathrm{T}} \qquad \text{(I)}$$

$$x_k>0, \qquad k=1, ..., K.$$

Here,

$$\left[M(G)\left|\begin{matrix}-1\\ \vdots\\ -1\end{matrix}\right.\right]$$

stands for the matrix $M(G)$ augmented with a $(K+1)$st column, each element of which is equal to -1, and $(x\mid 1)=(x_1, ..., x_K, 1)$.

Basically due to Stiemke's theorem (cf. O. L. Mangasarian [6]), System (I) has a solution iff the following System (II) does not have a solution for which at least one of the inequalities is strict:

$$\left[\frac{M^{\mathrm{T}}(G)}{-1\cdots-1}\right]\cdot z^{\mathrm{T}}\geqslant \underbrace{(0,...,0)}_{(K+1)\text{-times}}{}^{\mathrm{T}}, \qquad \text{with} \quad z=(z_1, ..., z_{\eta+\mu}). \qquad \text{(II)}$$

Here,

$$\left[\frac{M^{\mathrm{T}}(G)}{-1\cdots-1}\right]$$

denotes the matrix $M^{\mathrm{T}}(G)$ augmented with a $(K+1)$st row, each coefficient of which being equal to -1.

Obviously, System (II) is equivalent with

$$\begin{aligned} z_{l_1}+z_{l_2}&\geqslant 0 \qquad \text{for all} \quad (l_1, l_2)\in \mathbf{O}\\ z_i&\geqslant 0 \qquad \text{for all} \quad i=1, ..., \eta_0 \qquad \text{(III)}\\ \sum_{l=1}^{\eta+\mu} z_l&\leqslant 0.\end{aligned}$$

Consequently, we have proved:

LEMMA 3.4. *G is equivalent with a Newton graph iff System* (III) *does not have a solution for which at least one of the inequalities is strict.*

We prove the non-trivial part of Theorem A:

LEMMA 3.5. *If the Inequalities* $(*)$ *hold for all nonempty subsets J of finite faces, then G is equivalent with a Newton graph.*

Proof. In the case where $\mu = 0$ (i.e., G is a plane tree) the proof is obvious. So, we may assume that $\mu \geqslant 1$. Let us suppose that the Inequalities $(*)$ hold, but G is not equivalent with a Newton graph. Then there must exist a solution, say $\tilde{z}_1, ..., \tilde{z}_{\eta+\mu}$, of System (III) such that at least one of the inequalities is strict. We lead this to a contradiction. Let $i_0 \in \{1, ..., \eta\}$ and $j_0 \in \{0, 1, ..., \mu\}$ with Ω_{i_0} in r_{j_0} be given. The transversal T of $\{V(\{j\}) \mid j = 1, ..., \mu\}$ w.r.t. the pair (Ω_{i_0}, r_{j_0}), induces an *injective* map ψ from $\{\eta+1, ..., \eta+\mu\}$ to $\{1, ..., \eta\}$: $\psi(\eta+j) = i$, where i corresponds to the vertex Ω_i representing r_j w.r.t. T; compare Lemma 3.3. From this lemma it is also clear that ψ has the following properties

$$(\psi(l), l) \in \mathbf{O}, \qquad \text{for all} \quad l = \eta+1, ..., \eta+\mu, \text{ and}$$
$$\operatorname{Im} \psi \supsetneqq \{\eta_0 + 1, ..., \eta\}, \tag{a}$$

where $\{\eta_0+1, ..., \eta\} = \varnothing$ if $\eta_0 = \eta$.

Moreover, we have (cf. Properties (3) and (3′) in Lemma 3.3)

$$\text{if } j_0 = 0, \text{ then } i_0 \notin \operatorname{Im} \psi;$$
$$\text{if } j_0 \neq 0, \text{ then } i_0 = \psi(\eta + j_0). \tag{b}$$

In view of (iii), (a), and the injectivity of ψ, the solution $\tilde{z}_l$ fulfils

$$0 \leqslant \sum_{l=\eta+1}^{\eta+\mu} (\tilde{z}_{\psi(l)} + \tilde{z}_l) + \sum_{\substack{l=1 \\ l \notin \operatorname{Im} \psi}}^{\eta} \tilde{z}_l = \sum_{l=1}^{\eta+\mu} \tilde{z}_l \leqslant 0,$$

and thus, again by (III), we have

$$\tilde{z}_{\psi(l)} + \tilde{z}_l = 0, \qquad l = \eta+1, ..., \eta+\mu$$
$$\tilde{z}_l = 0, \qquad l \in \{1, ..., \eta\} \backslash \operatorname{Im} \psi. \tag{IV}$$

We emphasize that (IV) holds for any map ψ corresponding to any choice of the indices i_0 and j_0 with $\Omega_{i_0} \in r_{j_0}$. From (IV) we derive (by aid of (a) and (b)) the following conclusions:

$$\tilde{z}_l = 0, \qquad \text{for all} \quad l \in \{1, ..., \eta_0\}. \tag{c}$$

Let $i_0 \in \{1, ..., \eta\}$, and $j_0 \in \{1, ..., \mu\}$ be arbitrary, then

$$\begin{aligned} &\tilde{z}_{i_0} = 0 \text{ (and thus } \tilde{z}_{\eta + j_0} = 0) \text{ implies} \\ &\tilde{z}_l = 0 \text{ for all } l \in \{1, ..., \eta\} \text{ with } \Omega_l \in r_{j_0}. \end{aligned} \tag{d}$$

We define $J_1 := \{j \in \{0, 1, ..., \mu\} \mid \text{ an } \Omega_l \text{ in } r_j \text{ exists with } \tilde{z}_l = 0\}$. Obviously (use (c)) we have $0 \in J_1$. Moreover, from the very definition of J_1 as well as from (d), it follows that $V(J_1) \cap V(J_1^c) = \varnothing$. Using the connectedness of G it follows that $V(J_1^c) = \varnothing$ and thus $J_1 = \{0, 1, ..., \mu\}$. Consequently, we find, using (d), that $\tilde{z}_l = 0$ for all $l \in \{1, ..., \eta + \mu\}$. However, the latter conclusion violates the assumption that System (III) has a solution for which at least one of the inequalities is strict.

4. The Proof of Theorem B

Let G be a plane graph and consider the blocks of the boundary of the infinite face of G. Such a block is either a cycle or a tree of order two. In case of a cycle (tree), the graph consisting of all G-vertices and edges inside or on this cycle (tree) is called a "cycle block" ("tree block"). Both a cycle and tree block will be called plane blocks of G. From the very construction of these cycle blocks it will be clear that a connected plane graph fulfils the Inequalities (∗) iff this holds for each of its cycle blocks. The same assertion is true w.r.t. the Inequalities (∗∗).

Another observation is the following. Let $m(J)$ be the number of connected components of $G(J)$. Then, the Inequalities (∗) are equivalent with the (apparently stronger) statement that the inequalities

$$n(J) + m(J) \leqslant r(J) \leqslant n(J) + l(J) - m(J)$$

hold for all nonempty subsets J of finite faces of G.

Now, we prove the "if part" of Theorem B by induction on the cyclomatic number μ of G. We assume $\mu \geqslant 2$ since, if $\mu = 1$, the assertion follows directly by applying the Inequalities (∗∗) to the unique cycle in G.

Moreover, in view of the first observation above we restrict ourselves to the case where G consists of only one cycle block with cycle C.

Let J' be the set of all faces except those in J and the infinite face. The case where $J' = \varnothing$ being trivial, we assume $J' \neq \varnothing$.

The subset J is called of Type C, if an edge x exists, incident both with C and with a face *not* in J. In this case, the induction hypothesis applies to $G - x$. Thus J satisfies the Inequalities (∗) in $G - x$ and therefore in G. So, we may assume J not to be of Type C. But then, J' must be of Type C and must satisfy the "extended" Inequalities (∗), i.e.,

$$n(J') + l(J') - m(J') \geqslant r(J') \geqslant n(J') + m(J'). \tag{1}$$

Moreover, the Inequalities (**) applied to cycle C yield

$$n(C)+1 \leqslant r(C) \leqslant n(C)+l(C)-1. \tag{2}$$

Now subtracting (1) from (2), and using $r(J)+r(J')=r(C)$, we get

$$\begin{aligned} &n(C)-n(J')-l(J')+m(J')+1 \\ &\qquad \leqslant r(J) \leqslant n(C)+l(C)-n(J')-m(J')-1. \end{aligned} \tag{3}$$

By assumption on J, no face of J' shares an edge with C. Hence, $G(J')\cap C$ consists of a set of k vertices, and $G(J)\cap C=C$. As a consequence we have $n(J)+n(J')+l(J')=n(C)+k$, and $n(J)+n(J')+l(J)=n(C)+l(C)$. Substituting this into (3) gives

$$n(J)+m(J')+1-k \leqslant r(J) \leqslant n(J)+l(J)-m(J')-1. \tag{4}$$

The right-hand inequality in (4) implies the right-hand side of (*).

We turn over to the left-hand part of Inequalities (*). Suppose that no connected component of $G(J')$ has more than one point in common with C and thus, $k \leqslant m(J')$. In this case, the left-hand inequality in (4) implies

$$n(J)+1 \leqslant r(J).$$

So, we assume $|K\cap C|>1$ for some connected component K of $G(J')$. Define a *partition* $J=\bigcup J_l$ by $j, \tilde{j}\in J_l$ iff $r_j \cup r_{\tilde{j}}$ is contained in one face of $G(J'\cup\{0\})$.

It is easily seen that: (i) Every J_l is of Type C, and (ii) $n(J)=\sum n(J_l)$. From (i) we get by induction hypothesis that $n(J_l)<r(J_l)$. Together with (ii), this gives us the required inequality.

5. The Structure of Newton Graphs; Balanced Graphs

As already mentioned, the concept of Newton graph arises from the study of certain dynamical systems. In fact, we proved [5] that a structurally stable system $\mathbf{N}(f)$, $n>m$, exhibits only *nondegenerate* equilibria, namely, n *attractors* (the zeros for f), m *repellors* (the poles for f), and $(n+m-1)$ *simple* saddles (the critical points for f, defined by $f'=0, f\neq 0$). The unstable manifolds at the saddles serve as the edges of $G(f)$, whereas the basins of repulsion of the repellors are (the interiors of) the finite $G(f)$-faces; the infinite face corresponds with $z=\infty$.

Hence, the graph $G(f)$ may be regarded as the "principal" part of the phase portrait of $\mathbf{N}(f)$. Moreover, the possible topological structures of the latter phase portrait are given by the possible structures of all Newton graphs of order n and cyclomatic number m. (Compare Theorem 1.1.) In the special case where $n = m + 1$, the Newton graph $G(f)$ turns out (cf. [5]) to fulfil the additional property

N_3: The sum of all angles spanning a sector of the *infinite* face equals 2π.

Such graphs (i.e., N_1, N_2, and N_3 hold) are called *balanced* and are of great importance for the description of the structure of Newton graphs.

The following lemma is almost evident.

LEMMA 5.1. *Let G be a Newton graph, then G is balanced iff* $\eta = \mu + 1$.

Proof. Both sides of the assertion imply that all plane blocks into which G can be subdivided are cycle blocks. Under this latter condition, one easily verifies that $\eta - \mu$ just equals the sum of all angles spanning a sector of r_0. Now, the assertion follows directly from the Property N_3. ∎

In the proof of Lemma 5.1, let $C_1, ..., C_k$ be the cycles enclosing the cycle blocks into which G is subdivided. We obviously have

$$\eta - \mu = \sum_{i=1}^{k} [n(C_i) + l(C_i) - 1 - r(C_i)] + 1,$$

where, due to Inequalities (**) we have $[n(C_i) + l(C_i) - 1 - r(C_i)] \geqslant 0$. Thus, if G is balanced, then Lemma 5.1 yields $n(C_i) + l(C_i) = r(C_i) + 1$. Using (the proof of) Lemma 5.1 once again, we arrive at:

COROLLARY 5.1. *All plane blocks into which a balanced graph is subdivided are cycle blocks, and each of these blocks is balanced.*

Now, we show that balanced graphs appear—in a natural way—as subgraphs of more general Newton graphs. To this aim, observe that $G(\{j\})$, $1 \leqslant j \leqslant \mu$, is just one cycle block. We denote the enclosing cycle by F_j, (the face cycle w.r.t. the finite face r_j).

LEMMA 5.2. *Let G be a Newton graph, then* $r(F_j) = n(F_j) + 1$, $j = 1, ..., \mu$.

Proof. The face cycles in $G(\{j\})$ are denoted by $F_{j1}, ..., F_{j\nu(j)}$, where $F_{j1} = F_j$. If $\nu(j) = 1$, then we are done. So, we assume that $\nu(j) > 1$. From the very definition of Newton graph, it will be clear that the subgraph $G(\{j\})$ has no bridges. Moreover, the face cycles $F_{j1}, ..., F_{j\nu(j)}$ are the only cycles in $G(\{j\})$, and any pair has at most one vertex in common. From this it follows that

$$\begin{aligned} n(F_{j1}) &= \sum_{\nu=2}^{\nu(j)} [l(F_{j\nu}) - 1] + \sum_{\nu=2}^{\nu(j)} n(F_{j\nu}) \\ &= \sum_{\nu=2}^{\nu(j)} [l(F_{j\nu}) + n(F_{j\nu})] - \nu(j) + 1. \end{aligned}$$

Application of the Inequalities (**) to each $F_{j\nu}$ yields

$$n(F_{j1}) < r(F_{j1}) = 1 + \sum_{\nu=2}^{\nu(j)} r(F_{j\nu}) \leqslant \sum_{\nu=2}^{\nu(j)} [l(F_{j\nu}) + n(F_{j\nu})] - \nu(j) + 2.$$

Now the assertion follows immediately. ∎

The (Newton) subgraph of G, generated by the edges and vertices of G inside and on the face cycle $F_{j\nu}$, is denoted by $G_{j\nu}$. The connected components of the union $\bigcup_\nu G_{j\nu}$, $\nu = 2, ..., \nu(j)$, are denoted by $G_j^1, ..., G_j^s$. Then, we have

$$n(F_{j1}) = \eta(G_j^1) + \cdots + \eta(G_j^s) - s.$$

From Lemma 5.2, it follows that

$$n(F_{j1}) = \mu(G_j^1) + \cdots + \mu(G_j^s).$$

Now, in view of Corollary 3.1, we may conclude that $\eta(G_j^l) = \mu(G_j^l) + 1$, for $l = 1, ..., s$. Thus (cf. Lemma 5.1 and Corollary 5.1) we have proved:

LEMMA 5.3. *Given a Newton graph G, then each $(G_{j\nu})$ enclosed by the cycle $F_{j\nu}$, $j = 1, ..., \mu$; $\nu = 2, ..., \nu(j)$, is* balanced.

As an application of the results obtained up till now, we have:

LEMMA 5.4. *Let C be a cycle in a Newton graph G. If C is contained in the boundaries of two different* finite *faces, say r_1 and r_2, then $l(C) = 2$.*

Proof. It is easily seen that C corresponds to the intersection $r_1 \cap r_2$. Moreover, one of the faces—say r_1—is inside C, whereas the other is situated outside. Then, it is not difficult to verify that $C = F_{11}$, and also $C = F_{2\nu_0}$, with $1 < \nu_0 \leqslant \nu(2)$. Now, from Lemmas 5.1, 5.2, and 5.3, it follows that

$$n(C) + 1 = r(C) = \mu(G_{2\nu_0}) = n(C) + l(C) - 1,$$

which proves the assertion. ∎

Remark 5.1. In the case where G is a balanced graph, the assertion of Lemma 5.4 also holds if C is contained in the boundaries of two faces, one of which being the infinite face.

The following constructions of Newton graphs by means of "graph-attachment" may be interpreted in terms of Newton flows $\mathbf{N}(f)$.

Let H be an *arbitrary* connected plane graph. Then, it is always possible to implant H (up to equivalency) into a finite face r_j of G, by identifying exactly one exterior H-vertex, with a vertex Ω from r_j. Such graphs are denoted by $G \cup_j^\Omega H$.

The "if part" of the following result follows from Theorem A and Lemma 5.1, whereas the "only if part" is a consequence of Lemmas 5.2 and 5.3.

LEMMA 5.5. *Let G be a Newton graph. Then, any $G \cup_j^\Omega H$, with $j \geqslant 1$ and $\Omega \in r_j$, is equivalent with a Newton graph if and only if H is equivalent with a balanced graph.*

Next we deal with the case where all exterior vertices of H (and not, as above, only one) are used for attachment. To this aim, let C be an "empty" cycle in G, i.e., $n(C) = 0$. Consider a cycle block H for which the enclosing cycle C' has the same length as C, i.e., $l(C) = l(C')$. Then, it is always possible to implant H (up to equivalency) into the interior of C by identifying (the edges and vertices of) the cycle C and C'. In general, this can be done in several ways. The resulting graphs are denoted by $G \cup_C H$. Clearly, when $G \cup_C H$ is equivalent with a Newton graph, then this is also true for H.

If $n((H)$ stands for # interior H-vertices $(= n(C'))$, then:

LEMMA 5.6. *Let G, C, and H be as introduced above. Moreover, let both G and H be equivalent with Newton graphs. Then, in order for $G \cup_C H$ to be equivalent with a Newton graph it is necessary that*

$$1 \leqslant \mu(H) - n(H) \leqslant \eta(G) - \mu(G),$$

whereas the following condition is sufficient

$$\mu(H) - n(H) = 1.$$

The necessity condition follows directly from the construction of $G \cup_C H$ (and application of Corollary 3.1 to G and H). The proof of the sufficiency part is based on a straightforward (but tough) verification of the Inequalities $(*)$, and will be deleted. Note that in case where G is balanced both conditions coincide (cf. Lemma 5.1).

A cycle block H—as in the sufficiency part of Lemma 5.6—with $l(C')$ and $n(H)$ arbitrary is easily constructed: Start with an arbitrary "empty" cycle C', represented as a Newton graph; draw a differentiable chord of C', subdividing the interior of C' into two parts such that the sum of the inward angles (none of them vanishing) of each part equals π, and choose an interior vertex on this chord; apply this procedure to one of the "empty" cycles obtained in this way; etc.

On the other hand, let G be an arbitrary Newton graph, with finite face

r_j. Then, after deleting from G all (possible) subgraphs $G_{j\nu}$, $\nu = 2, ..., \nu(j)$, the resulting graph is again Newtonian (cf. Lemma 5.3 and Theorem A), and has F_j as an "empty" cycle. Now we may implant—by means of Lemma 5.6—a cycle block H, with $l(C') = l(F_j)$ and $n(H)$ arbitrary. Finally, we re-attach (cf. Lemma 5.5) the subgraphs $G_{j\nu}$ in order to obtain a new Newton graph which exhibits—up to equivalency—all G-faces, with the exception of r_j; instead of r_j it has $\mu(H)$ faces, structured as H.

In the above construction, the numbers of "added" vertices and faces are equal. In terms of structurally stable Newton flows $\mathbf{N}(f)$, this construction ensures the existence of systems $\mathbf{N}(\hat{f})$, again structurally stable, which can be obtained from $\mathbf{N}(f)$ by implanting k attractors and k repellors, $k > 0$, into one specific bounded basin of repulsion, and leaving the topological structure of all other basins unchanged. From this point of view, our construction is a first step in order to find (or even enumerate) for given n and m all different (up to equivalency) structurally stable systems $\mathbf{N}(f)$. (See also A. v.d. Tuin [10], and our paper [5], where this problem is solved in some simple cases.)

Remark 5.2. Given two Newton graphs G and H, as introduced in the sufficiency part of Lemma 5.6. Then, it is possible to draw G as a Newton graph, such that the inward angles of C and C' at corresponding vertices are equal. (The proof, running along the same lines as the proof of Theorem A will be deleted.) This observation clarifies the geometry behind Lemma 5.6. A similar comment is possible w.r.t. Lemma 5.5.

In order to prove that the detection of Newton graphs is *polynomial*, let $\hat{G}$ be a finite, bipartite graph with bipartition (X, Y). For $S \subset X$, we denote its neighbour set by $N(S)$. Then, the inequalities

$$|S| < |N(S)|, \qquad \text{all nonempty } S, \tag{***}$$

hold iff *each* bipartite graph, obtained from $(\hat{G}, X, Y)$ by adding one vertex (P) to X, and one edge which joins P to an Y-vertex, fulfils the inequalities $|\tilde{S}| \leqslant |N(\tilde{S})|$ for all $\tilde{S} \subset X \cup \{P\}$ (see also [2] where Condition (***) "Strong Hall Property" is discussed in more detail). As a consequence, the verification of the Inequalities (***) is polynomial (cf. [1]).

LEMMA 5.7. *There exists a polynomial algorithm, detecting whether an arbitrary connected plane graph is equivalent with a Newton graph or not.*

Proof. In view of Theorem A, we must prove that a polynomial verification of both inequalities in (*) is possible. In order to do so for the second ones, we specify the bipartite graph $(\hat{G}, X, Y)$ as follows: $X = \{$finite G-faces$\}$, and $Y = \{G$-vertices$\}$, whereas adjacency is defined by inclusion.

Apparently the Inequalities (***) are equivalent with the second inequalities in (*), and we are done.

Now, we turn over to the first inequalities in (*). The search of plane blocks being polynomial, we assume G to be a cycle block with C as enclosing cycle. We specify $(\hat{G}, X, Y)$ as follows: $X = \{G\text{-vertices } \textit{inside } C\}$, and $Y = \{\text{finite } G\text{-faces}\}$, whereas adjacency is defined by inclusion. We claim that the first inequalities in (*) are equivalent with Inequalities (***). To this aim, let $S \subset X$ and define $J = \{j \in \{1, ..., \mu\} \mid S \cap r_j \neq \emptyset\}$, i.e., $J = N(S)$. Since every vertex in S is interior to $G(J)$, from the first inequalities in (*) it follows that $|S| \leqslant n(J) < r(J) = |N(S)|$. Conversely, let $J \subset \{1, ..., \mu\}$, $J \neq \emptyset$, and assume $S = \{\text{interior vertices of } G(J)\}$. The case $S = \emptyset$ being trivial, we assume $S \neq \emptyset$. Then, we have $S \subset X$ and $N(S) \subset J$. Hence, by Inequalities (***), $r(J) \geqslant |N(S)| > |S|$ $(= n(J))$. ∎

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